

## ON THE ISOMETRIC PATH PARTITION PROBLEM

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### Abstract

The isometric path cover (partition) problem of a graph consists of finding a minimum set of isometric paths which cover (partition) the vertex set of the graph. The isometric path cover (partition) number of a graph is the cardinality of a minimum isometric path cover (partition). We prove that the isometric path partition problem and the isometric  $k$ -path partition problem for  $k \geq 3$  are NP-complete on general graphs. Fisher and Fitzpatrick in [*The isometric number of a graph*, J. Combin. Math. Combin. Comput. 38 (2001) 97–110] have shown that the isometric path cover number of the  $(r \times r)$ -dimensional grid is  $\lceil 2r/3 \rceil$ . We show that the isometric path cover (partition) number of the  $(r \times s)$ -dimensional grid is  $s$  when  $r \geq s - 1$ . We establish that the isometric path cover (partition) number of the  $(r \times r)$ -dimensional torus is  $r$  when  $r$  is even and is either  $r$  or  $r + 1$  when  $r$  is odd. Then, we demonstrate that the isometric path cover (partition) number of an  $r$ -dimensional Benes network is  $2^r$ . In addition, we provide partial solutions for the isometric path cover (partition) problems for cylinder and multi-dimensional grids. We apply two different techniques to achieve these results.

**Keywords:** path cover problem, isometric path partition problem, isometric path cover problem, multi-dimensional grids, cylinder, torus.

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### 1. INTRODUCTION

An undirected connected graph is represented by  $G = (V, E)$  where  $V$  is the vertex set and  $E$  is the edge set. A *path* means a simple path with distinct vertices. A path between two vertices is an *isometric path* if it induces the

shortest distance between the two points. Let us recall that *isometric path* and *geodesic* are other names for shortest path. While an *isometric path cover* is a set of isometric paths which cover the vertex set  $V$ , an *isometric path partition* is a set of isometric paths which partition  $V$ . The *isometric path cover (partition) number* which is denoted by  $\text{ip}_c(G)$  ( $\text{ip}_p(G)$ ) is the cardinality of a minimum isometric path cover (partition). The *isometric path cover (partition) problem* is to find a minimum isometric path cover (partition). Let  $\text{diam}(G)$  denote the diameter of graph  $G$ .

Given a graph  $G = (V, E)$ , a *isometric  $k$ -path* is an isometric path having at most  $k$  vertices. A set  $S$  of isometric  $k$ -paths is an *isometric  $k$ -path partition* if each vertex of  $V$  belongs to exactly one member of  $S$ . The *isometric  $k$ -path partition problem* is to find an isometric  $k$ -path partition of minimum cardinality in  $G$ .

Last few decades, the theory of isometric paths has been studied extensively. Aggarwal *et al.* [1] have illustrated the application of the isometric path cover problem in the design of VLSI layouts. The theory of isometric path problems is the backbone in the design of efficient algorithms in transport networks [20], computer networks [14, 24], parallel architectures [25], social networks [3, 12], VLSI layout design [1], wireless sensor networks [6], multimedia networks [5] and in other networks such as GIS networks [26], large network systems [2] and stochastic networks [23].

Since the Hamiltonian path problem is NP-complete [11], the path cover problem and the path partition problem are NP-complete. Since the Hamiltonian induced path problem is NP-complete [4, 11], the induced path cover problem and the induced path partition problem are NP-complete. However, the complexity status of the isometric path cover problem and the isometric path partition problem are unknown [16]. This fact has been highlighted and emphasized recently [16, 17]. In this paper, we settle the long-standing open problem [16] by proving that the isometric path partition problem is NP-complete on general graphs.

The isometric path cover number has been computed for trees, cycles, complete bipartite graphs, and the Cartesian product of paths (including hypercubes) under some restricted cases [7, 8, 9, 10]. Fisher and Fitzpatrick [7] have established the lower bound  $\text{ip}_c(G) \geq \lceil \frac{|V|}{\text{diam}(G)+1} \rceil$  and have shown that the isometric path cover number of the  $(r \times r)$  grid is  $\lceil 2r/3 \rceil$ . Fitzpatrick *et al.* [10] have shown that the isometric path cover number of the hypercube  $Q_r$  is at least  $2r/(r+1)$ . In addition, they have also shown that  $\text{ip}_c(Q_r) = 2^{r-\log_2(r+1)}$  when  $r+1$  is a power of 2. Pan and Chang have given a linear-time algorithm to solve the isometric path cover problem on block graphs [21], complete  $r$ -partite graphs, and the Cartesian products of 2 or 3 complete graphs [22]. There is no literature on the isometric path partition problem [16]. The readers are suggested to read the survey paper by Manuel [16] for detailed information.

In Section 2, we show that the isometric path partition problem is NP-complete. We also show that the isometric  $k$ -path partition problem is NP-complete on general graphs for  $k \geq 3$ . In Section 3, we demonstrate that the isometric path cover problem and the isometric path partition problem are two different combinatorial problems. In Section 4, we compute the exact values of the isometric path cover and isometric path partition number for cylinders and multi-dimensional grids under certain conditions. In Sections 5 and 6, we also derive the exact values of the isometric path cover number and isometric path partition number for square torus and Benes networks.

## 2. THE ISOMETRIC PATH PARTITION PROBLEM IS NP-COMPLETE FOR GENERAL GRAPHS

The main result of this section is that the isometric path partition problem is NP-complete on general graphs. In order to prove this result, we need the following theorem.

**Theorem 1** [19]. *The 3-path partition problem is NP-complete on bipartite graphs.*

As a first step, we provide a polynomial reduction from the 3-path partition problem on bipartite graphs to the isometric path partition problem on general graphs. The key fact is that each 3-path in a bipartite graph is an isometric path. Given a graph  $G = (V, E)$  where  $V = \{1, 2, \dots, n\}$ , the transformed graph is denoted by  $G' = (V', E')$ . The vertex set  $V'$  is  $V \cup \{x, y, z\}$ . The edge set  $E'$  is  $E \cup \{xz, zy\} \cup \{iz / i \in V\}$ , see Figure 1.

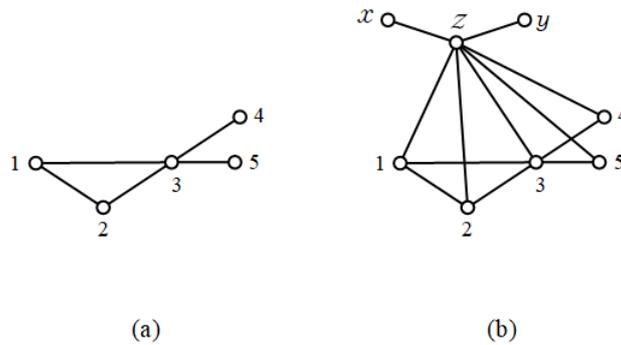


Figure 1. (a)  $G = (V, E)$  and (b)  $G' = (V', E')$ .

Next, we will identify some basic structural properties of  $G' = (V', E')$ .

**Property 2.** *If  $G$  is a bipartite graph, then  $\text{diam}(G') = 2$ . Moreover, an isometric path of length  $\text{diam}(G')$  in  $G'$  is of the form  $xzy$ ,  $izx$ ,  $izy$ ,  $ikj$  or  $izj$  for some  $i, j, k \in V$ .*

**Property 3.** *A bipartite graph  $G$  has a 3-path partition  $S$  of cardinality  $k$  if and only if  $G'$  has an isometric path partition  $S'$  of cardinality  $k + 1$ .*

**Proof.** Let  $S$  be a 3-path partition of  $G$ . Then  $S' = S \cup \{xzy\}$  is an isometric path partition of  $G'$  and  $|S'| = |S| + 1$ . Now, let us prove the converse.

Let  $S'$  be an isometric path partition of  $G'$ . Suppose that  $xzy$  is a member of  $S'$ . Then, define  $S = S' \setminus \{xzy\}$ . Since  $\text{diam}(G') = 2$ ,  $S'$  is 3-path partition of  $G'$ . Hence,  $S$  is 3-path partition of  $G$  and  $|S| = |S'| - 1$ .

Suppose that  $izj$  is a member of  $S'$  for some  $i$  and  $j$  of  $V$ . In such a situation, isometric paths  $x$  and  $y$  are members of  $S'$ . Let us construct  $S'' = S' \setminus \{x, y, izj\} \cup \{i, j, xzy\}$ . Now,  $S''$  is an isometric path partition of  $G'$  and  $|S''| = |S'|$ . Then, let us construct  $S$  from  $S''$  such that  $S = S'' \setminus \{xzy\}$ . As argued before,  $S$  is 3-path partition of  $G$  and  $|S| = |S''| - 1 = |S'| - 1$ .

Suppose that  $izx$  is a member of  $S'$  for some  $i$  of  $V$ . In such a situation, isometric path  $y$  is a member of  $S'$ . Let us construct  $S''' = S' \setminus \{y, izx\} \cup \{i, xzy\}$ . Now,  $S'''$  is an isometric path partition of  $G'$  and  $|S'''| = |S'|$ . Then, let us construct  $S$  from  $S'''$  such that  $S = S''' \setminus \{xzy\}$ . As argued before,  $S$  is 3-path partition of  $G$  and  $|S| = |S'''| - 1 = |S'| - 1$ .

The last case is that  $izy$  is a member of  $S'$  for some  $i$  of  $V$ . It is similar to the previous case. ■

Applying Property 2 and 3, we state one of the main results of this paper.

**Theorem 4.** *The isometric path partition problem is NP-complete on general graphs.*

The isometric  $k$ -path partition problem is a generalization of the isometric path partition problem. Using the same logic, one can also prove that the following.

**Theorem 5.** *The isometric  $k$ -path partition problem is NP-complete on general graphs for  $k \geq 3$ .*

### 3. ISOMETRIC PATH PARTITION VERSUS ISOMETRIC PATH COVER

Our point of discussion in this section is to emphasize that the isometric path cover problem and the isometric path partition problem are two different combinatorial problems. From the perspective of computational complexity, let us see how these two problems differ even on simple architectures such as trees and

grids. For the star graph  $K_{1,\ell}$ , while  $\text{ip}_c(K_{1,\ell}) = \lceil \ell/2 \rceil$ ,  $\text{ip}_p(K_{1,\ell}) = \ell - 1$ . Pan and Chang [21] have proved that the isometric path cover number of trees with  $\ell$  leaves is  $\lceil \ell/2 \rceil$ . For the isometric path partition number, this is not true even in complete binary trees. In fact, it is a challenge to find the isometric path partition number for trees and the isometric path partition number for trees is unknown [16].

Fisher and Fitzpatrick [7] have shown that the isometric path cover number of the grid  $P_r \square P_r$  is  $\lceil 2r/3 \rceil$ . However, the isometric path partition number of the grid  $P_r \square P_r$  remains an open problem [16]. It seems that the isometric path partition number of the grid  $P_r \square P_r$  is  $r$ . However, it requires a mathematical proof which seems to be a challenge.

There are some features which are common to both problems. One straightforward lower bound that is common to both problems is as follows.

**Theorem 6** [7]. *If  $\text{diam}(G)$  denotes the diameter of a graph  $G$ , then  $\text{ip}_p(G) \geq \text{ip}_c(G) \geq \left\lceil \frac{|V(G)|}{\text{diam}(G)+1} \right\rceil$ .*

The bound in Theorem 6 is very effective and useful. This paper truly exploits this lower bound to prove that the isometric path cover number and the isometric path partition number are equal for some networks such as  $(r \times r)$ -dimensional torus and Benes networks.

Monnot and Toulouse [19] have studied an NP-complete problem that is whether a graph on  $nk$  vertices can be partitioned into  $n$  paths of length  $k$ . An isometric path version of this problem is to decide if a graph  $G = (V, E)$  can be partitioned into isometric paths of length equal to  $\text{diam}(G)$ . Any fixed interconnection network which possesses this feature is considered as a “good” architecture [15, 25] because it is easy to design and implement efficient data communication, message broadcasting and other routing algorithms. In the following sections, we point out that a few fixed interconnection networks such as torus and Benes networks inherit this nice feature.

#### 4. THE ISOMETRIC PATH COVER/PARTITION PROBLEM ON THE CARTESIAN PRODUCT $P_r \square G$

Given a graph  $G$ , let  $\text{diam}(G)$  denote the diameter of  $G$ . In this section, we consider the Cartesian product  $P_r \square G$  where  $G$  is any graph and  $P_r$  is a path graph on  $r$  vertices. The vertex set  $V(P_r)$  of  $P_r$  is  $\{1, 2, \dots, r\}$  and the vertex set  $V(P_r \square G)$  of  $P_r \square G$  is  $\{(j, v) \mid j \in V(P_r) \text{ and } v \in V(G)\}$ . For each  $j \in V(P_r)$ , the subgraph induced by the vertices  $\{(j, v) \mid v \in V(G)\}$  of  $P_r \square G$  is denoted by  $G^j$ . In other words, there are  $r$  copies of  $G$  in  $P_r \square G$  which are represented

by  $G^1, G^2, \dots, G^r$ , respectively. An edge of  $G^j$  in  $P_r \square G$  is called  $G^j$ -edge. A  $G$ -edge in  $P_r \square G$  is a  $G^j$ -edge for some  $j = 1, 2, \dots, r$ .

**Lemma 7** [13]. *Given a graph  $G$  with  $\text{diam}(G) = d$  and a path graph  $P_r$ , an isometric path of the Cartesian product  $P_r \square G$  can have at most  $d$   $G$ -edges.*

**Lemma 8.** *Given a graph  $G$  and a path graph  $P_r$ ,  $\text{ip}_c(P_r \square G) \leq \text{ip}_p(P_r \square G) \leq |V(G)|$ .*

**Proof.** It is enough to construct an isometric path partition of cardinality  $|V(G)|$ . Corresponding to each  $v \in V(G)$ , consider the path  $P^v = \{(1, v), (2, v), \dots, (r, v)\}$ . Each path  $P^v$ ,  $v \in V(G)$ , is an isometric path in  $P_r \square G$  [13]. Since the collection  $\{P^v \mid v \in V(G)\}$  of isometric paths partitions  $V(P_r \square G)$ ,  $\{P^v \mid v \in V(G)\}$  is an isometric path partition of  $P_r \square G$ . Therefore,  $\text{ip}_p(P_r \square G) \leq |V(G)|$ . ■

The following lemma is the key to derive a lower bound on  $\text{ip}_c(P_r \square G)$  and  $\text{ip}_p(P_r \square G)$ .

**Lemma 9.** *Given a graph  $G$  and a path graph  $P_r$ , let  $S$  be an isometric path cover (partition) of Cartesian product  $P_r \square G$ . If there exists  $j_0$ ,  $1 \leq j_0 \leq r$ , such that no isometric path of  $S$  contains an edge of  $G^{j_0}$ , then  $|S| \geq |V(G)|$ .*

**Proof.** An isometric path of the Cartesian product  $G \square H$  of graphs  $G$  and  $H$  does not pass through the same edge of  $G$  (respectively,  $H$ ) in two different copies of  $H$  (respectively  $G$ ) [13]. Since no edge of  $G^{j_0}$  is covered by any isometric path of  $S$ , each vertex of  $G^{j_0}$  requires a distinct isometric path of  $S$  to be covered. Thus,  $|S| \geq |V(G^{j_0})| = |V(G)|$ . ■

**Lemma 10.** *Let  $G$  be a graph and let  $P_r$  be a path satisfying  $r \geq \text{diam}(G)|V(G)|$ , then  $\text{ip}_p(P_r \square G) \geq \text{ip}_c(P_r \square G) \geq |V(G)|$ .*

**Proof.** Suppose there exists an isometric path cover  $S$  such that  $|S| < |V(G)|$ . Then by Lemma 9, for every  $G^j$ ,  $1 \leq j \leq r$ , there exists at least one  $G^j$ -edge  $e^j$  of  $G^j$  such that  $e^j$  is covered by some isometric path of  $S$ . Thus, the number of  $G$ -edges of  $P_r \square G$  which are covered by the isometric paths of  $S$  is greater than or equal to  $r$ . Since each isometric path of  $S$  can cover at most  $\text{diam}(G)$  number of  $G$ -edges, the number of  $G$ -edges which are covered by the isometric paths of  $S$  is less than or equal to  $\text{diam}(G)|S|$ . Thus,  $r \leq \text{diam}(G)|S|$ . Since  $|S| < |V(G)|$ ,  $r \leq \text{diam}(G)|S| < \text{diam}(G)|V(G)|$ . This is a contradiction to the hypothesis. ■

Following Lemmas 10 and 8, we state the following result.

**Theorem 11.** *Let  $G$  be a graph and let  $P_r$  be a path satisfying  $r \geq \text{diam}(G)|V(G)|$ , then  $\text{ip}_p(P_r \square G) = \text{ip}_c(P_r \square G) = |V(G)|$ .*

Now, we consider multi-dimensional grids  $P_{d_1} \square P_{d_2} \square \cdots \square P_{d_r}$ , which are the Cartesian product of paths  $P_{d_1}, P_{d_2}, \dots, P_{d_r}$ . Fisher and Fitzpatrick [7] have shown that the isometric path cover number of  $P_r \square P_r$  is  $\lceil 2r/3 \rceil$ . Fitzpatrick *et al.* [10] have shown that the lower bound on the isometric path cover number of the hypercube  $Q_r$  is  $2r/(r+1)$ . In addition, they have also shown that  $\text{ip}_c(Q_r) = 2^{r-\log_2(r+1)}$  when  $r+1$  is a power of 2. Though there are some results available for the isometric path cover number on grids, there is no literature for the isometric path partition problem on multi-dimensional grids, including 2-dimensional grids and cylinders [16].

**Theorem 12.** *Given an  $r$ -dimensional grid  $P_{d_1} \square P_{d_2} \square \cdots \square P_{d_r}$ ,  $\text{ip}_p(P_{d_1} \square P_{d_2} \square \cdots \square P_{d_r}) = \text{ip}_c(P_{d_1} \square P_{d_2} \square \cdots \square P_{d_r}) = d_2 d_3 \cdots d_r$  when  $d_1 \geq ((d_2 - 1) + (d_3 - 1) \cdots (d_r - 1)) \times (d_2 d_3 \cdots d_r)$ .*

**Proof.** The proof follows from Theorem 11 where  $G = P_{d_2} \square P_{d_3} \square \cdots \square P_{d_r}$ . The diameter of  $P_{d_2} \square P_{d_3} \square \cdots \square P_{d_r}$  is  $(d_2 - 1) + (d_3 - 1) + \cdots + (d_r - 1)$  and the number of vertices of  $P_{d_2} \square P_{d_3} \square \cdots \square P_{d_r}$  is  $d_2 d_3 \cdots d_r$ . ■

**Corollary 13.** *The isometric path cover (partition) number of the  $(r \times s)$ -dimensional grid is  $s$  when  $r \geq s(s - 1)$ .*

**Theorem 14.** *Given a cylinder  $P_r \square C_s$ ,  $\text{ip}_p(P_r \square C_s) = \text{ip}_c(P_r \square C_s) = s$  when  $r \geq \lfloor s/2 \rfloor s$ .*

**Proof.** The proof of the theorem follows from Theorem 11 where  $G = C_s$ . The diameter of  $C_s$  is  $\lfloor s/2 \rfloor$  and the number of vertices of  $C_s$  is  $s$ . ■

## 5. THE ISOMETRIC PATH PARTITION PROBLEM ON TORI

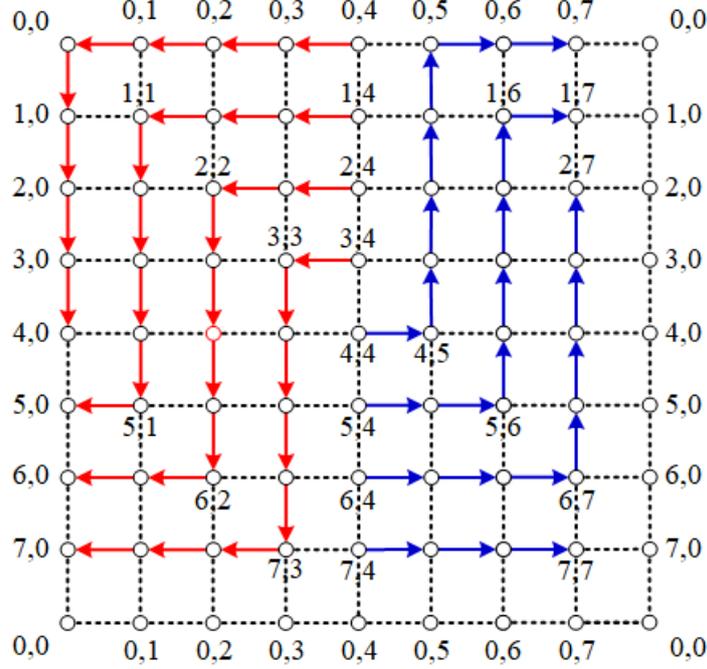
In this section, we study the exact value of the isometric path partition number of  $(r \times r)$ -dimensional torus. To our knowledge, there is no literature on the isometric path partition problem on tori. In this section, we will show that the isometric path cover (partition) number of the  $(r \times r)$ -dimensional torus is  $r$  when  $r$  is even and is either  $r$  or  $r + 1$  when  $r$  is odd. An  $(8 \times 8)$ -dimensional torus is given in Figure 2 and a  $(9 \times 9)$ -dimensional torus is given in Figure 3.

**Theorem 15.** *The isometric path cover (partition) number of the  $(r \times r)$ -dimensional torus is  $r$ , when  $r$  is even, and is either  $r$  or  $r + 1$ , when  $r$  is odd.*

**Proof.** Let  $G$  be an  $(r \times r)$ -dimensional torus network. The proof is by construction. When  $r$  is even, let us define the following isometric paths:

$$P_0 = (0, \frac{r}{2}) \cdots (0, 0) \cdots (\frac{r}{2}, 0);$$

$$P_i = (i, \frac{r}{2}) \cdots (i, i) \cdots (i + \frac{r}{2}, i) \cdots (i + \frac{r}{2}, 0) \text{ for } i = 1, 2, \dots, \frac{r}{2} - 1;$$

Figure 2. An  $(8 \times 8)$ -dimensional torus.

$P_i = (i, \frac{r}{2}) \cdots (i, i+1) \cdots (i - \frac{r}{2}, i+1) \cdots (i - \frac{r}{2}, r-1)$  for  $i = \frac{r}{2}, \frac{r}{2} + 1, \dots, r-2$ ;  
 $P_{r-1} = (r-1, \frac{r}{2})(r-1, \frac{r}{2} + 1) \cdots (r-1, r-1)$ ,

see Figure 2. Let us consider the collection  $\mathcal{C} = \{P_i \mid i = 0, 1, \dots, r-1\}$  of isometric paths. It is straightforward to verify that the collection  $\mathcal{C}$  is an isometric path partition of the  $(r \times r)$ -dimensional torus when  $r$  is even. Thus,  $\text{ip}_p(G) \leq |\mathcal{C}| = r = \left\lceil \frac{|V(G)|}{\text{diam}(G)+1} \right\rceil$ . By Theorem 6,  $\text{ip}_p(G) \geq \text{ip}_c(G) \geq \left\lceil \frac{|V(G)|}{\text{diam}(G)+1} \right\rceil$ . Therefore,  $\text{ip}_p(G) = \text{ip}_c(G) = r$ .

When  $r$  is odd, let us consider the following paths:

$$P_0 = (0, \frac{r-1}{2}) \cdots (0, 0) \cdots (\frac{r-1}{2}, 0);$$

$$P_i = (i, \frac{r-1}{2}) \cdots (i, i) \cdots (i + \frac{r-1}{2}, i) \cdots (i + \frac{r-1}{2}, 0) \text{ for } i = 1, 2, \dots, \frac{r-1}{2} - 1;$$

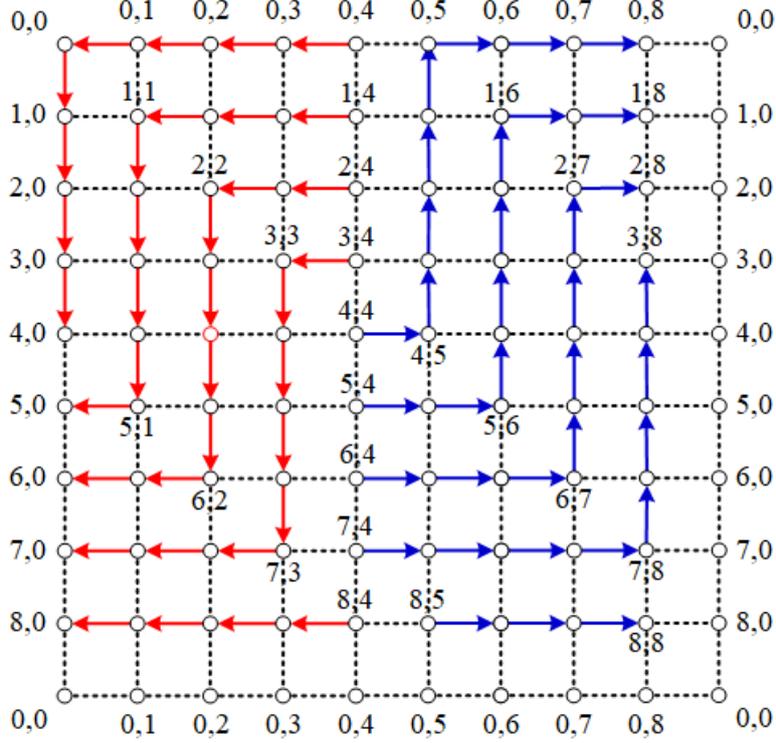
$$P_i = (i, \frac{r-1}{2}) \cdots (i, i+1) \cdots (i - \frac{r-1}{2}, i+1) \cdots (i - \frac{r-1}{2}, r-1) \text{ for } i = \frac{r-1}{2}, \frac{r-1}{2} + 1 \cdots r-3;$$

$$P_{r-2} = (r-2, \frac{r-1}{2}) \cdots (r-2, r-1) \cdots (\frac{r-3}{2}, r-1);$$

$$P_{r-1,1} = (r-1, \frac{r-1}{2})(r-1, \frac{r-1}{2} - 1) \cdots (r-1, 0);$$

$$P_{r-1,2} = (r-1, \frac{r-1}{2} + 1)(r-1, \frac{r-1}{2} + 2) \cdots (r-1, r-1),$$

see Figure 3. Let us consider the collection  $\mathcal{C} = \{P_i \mid i = 1, 2, \dots, r-3\} \cup$


 Figure 3. A  $(9 \times 9)$ -dimensional torus.

$\{P_0, P_{r-2}, P_{r-1,1}, P_{r-1,2}\}$  of isometric paths. It is straightforward to verify that the above collection  $\mathcal{C}$  is an isometric path partition of the  $(r \times r)$ -dimensional torus when  $r$  is odd. Therefore,  $\text{ip}_p(G) \leq |\mathcal{C}| = r + 1 = \left\lceil \frac{|V(G)|}{\text{diam}(G)+1} \right\rceil + 1$  when  $r$  is odd. Thus, by Theorem 6,  $\left\lceil \frac{|V(G)|}{\text{diam}(G)+1} \right\rceil \leq \text{ip}_c(G) \leq \text{ip}_p(G) \leq \left\lceil \frac{|V(G)|}{\text{diam}(G)+1} \right\rceil + 1$ . In other words,  $r \leq \text{ip}_c(G) \leq \text{ip}_p(G) \leq r + 1$ . Therefore, when  $r$  is odd,  $\text{ip}_c(G) = r$  or  $r + 1$  and  $\text{ip}_p(G) = r$  or  $r + 1$ . ■

The above technique does not provide a solution to solve the isometric path (partition) problem for the  $(r \times s)$ -dimensional torus when  $r \neq s$ . Thus, the status of isometric path cover (partition) problems on rectangular torus remains unknown.

## 6. THE ISOMETRIC PATH PARTITION PROBLEM ON BENES NETWORKS

The network architectures Butterfly and Benes are known for their versatile and parallel implementations in the field of digital communication systems such as

wireless communication, fiber-optic communication, and on-chip communications [18, 25]. Let  $\mathbb{Z}_k = \{0, 1, \dots, k-1\}$  and  $\mathbb{Z}_2^k = \{x_0x_1 \cdots x_{k-1} \mid x_i = 0 \text{ or } 1\}$ . When we say  $i \in \mathbb{Z}_k$ , it means  $i \bmod k$ . The vertex set of an  $r$ -dimensional Butterfly  $BF(r)$  is  $\{\langle w, i \rangle \mid w \in \mathbb{Z}_2^r \text{ and } i \in \mathbb{Z}_{2r+1}\}$ . Two vertices  $\langle w, i \rangle$  and  $\langle w', i' \rangle$  of  $BF(r)$  are linked by an edge if  $i' = i + 1$  and either  $w = w'$  or  $w$  and  $w'$  differ only in the bit in position  $i$ .

An  $r$ -dimensional Benes network  $BN(r)$  is called back-to-back butterflies and is obtained by merging the  $r$ -level vertices of two Butterfly networks  $BF(r)$ . The vertex set of  $BN(r)$  is  $\{\langle w, i \rangle \mid w \in \mathbb{Z}_2^r \text{ and } i \in \mathbb{Z}_{2r+1}\}$ . Thus, the vertices  $\langle w, 0 \rangle$ ,  $w \in \mathbb{Z}_2^r$  have degree 2. Figure 4 displays  $BN(3)$ . We apply the lower bound of Theorem 6 to find the exact value for  $\text{ip}_p(BN(r))$  and  $\text{ip}_c(BN(r))$  of Benes networks.

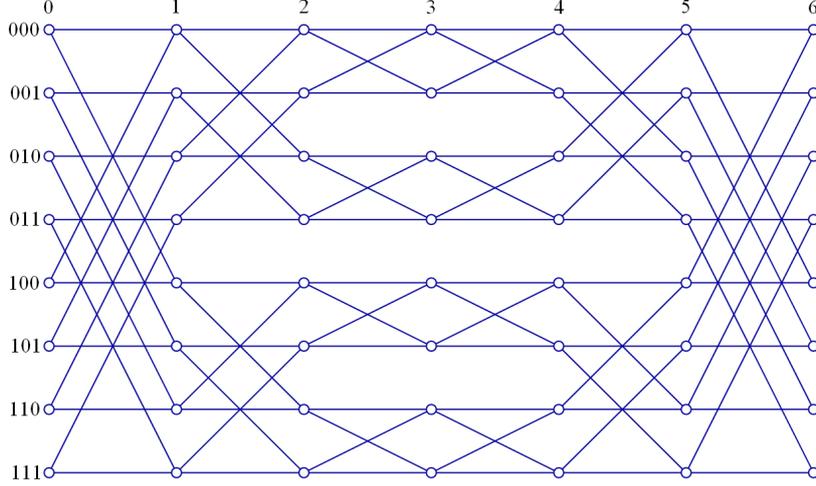


Figure 4. A 3-dimensional Benes network.

**Theorem 16.** *The isometric path cover (partition) number of an  $r$ -dimensional Benes network is  $2^r$ .*

**Proof.** An  $r$ -dimensional Benes network  $BN(r)$  consists of  $(2r+1)2^r$  vertices. The diameter of  $BN(r)$  is  $2r$  [18, 25]. Thus, it is enough to identify  $2^r$  number of vertex-disjoint isometric paths of length equal to  $\text{diam}(BN(r))$  in  $BN(r)$ . Given a binary string  $w \in \mathbb{Z}_2^r$ , let us define a path  $P_w$  as  $\langle w, 0 \rangle \langle w, 1 \rangle \cdots \langle w, 2r \rangle$ . Consider the collection  $S = \{P_w \mid w \in \mathbb{Z}_2^r\}$ . It is known that each member of the collection  $S$  is an isometric path of length equal to  $\text{diam}(BN(r))$  in  $BN(r)$  [18, 25]. Thus,  $S$  is an isometric path partition of  $BN(r)$ . Since each member of  $S$  is an isometric path of length equal to  $\text{diam}(BN(r))$  in  $BN(r)$ ,  $\text{ip}_p(BN(r)) = 2^r = \left\lceil \frac{|V(BN(r))|}{\text{diam}(BN(r))+1} \right\rceil$ . ■

## 7. CONCLUSION

We have proved that the isometric path partition problem is NP-complete. We have shown that the isometric  $k$ -path partition problem is also NP-complete on general graphs for  $k \geq 3$ . However, the complexity status of the isometric path cover problem is still unknown.

The aim of this manuscript is to determine or to derive properties that allow to calculate the considered parameters, and these parameters have been studied for graphs that can be expressed as Cartesian product.

The isometric path cover number of the  $(r \times r)$ -dimensional grid is  $\lceil 2r/3 \rceil$  [7]. We have established that the isometric path cover (partition) number of the  $(r \times s)$ -dimensional grid is  $s$  when  $r \geq s(s-1)$ . Thus the isometric path cover (partition) number of the  $(r \times s)$ -dimensional grid is still unknown for  $r < s(s-1)$ . Moreover, the isometric path partition number of the  $(r \times r)$ -dimensional grid is unknown. We have also studied the isometric path cover (partition) problem for cylinders. We have proved in Section 5 that the isometric path cover (partition) number of the  $(r \times r)$ -dimensional torus is  $r$  when  $r$  is even and is either  $r$  or  $r+1$  when  $r$  is odd. However, the isometric path cover number of the  $(r \times s)$ -dimensional torus remains unknown.

We have demonstrated that the isometric path cover (partition) number of an  $r$ -dimensional Benes network is  $2^r$ . These results are unknown for other fixed interconnection networks such as hypercubes, butterfly, and circulant networks.

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