Discussiones Mathematicae
Graph Theory xx (xxxx) 1–16
doi:10.7151/dmgt.2235

OPEN LOCATING-DOMINATING SETS IN CIRCULANT GRAPHS

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Abstract

Location detection problems have been studied for a variety of applications including finding faults in multiprocessors, contaminants in public utilities, intruders in buildings and facilities, and for environmental monitoring using wireless sensor networks. In each of these applications, the system or structure can be modeled as a graph, and sensors placed strategically at a subset of vertices can locate and detect anomalies in the system.

An open locating-dominating set (OLD-set) is a subset of vertices in a graph in which every vertex in the graph has a non-empty and unique set of neighbors in the subset. Sensors placed at OLD-set vertices can uniquely detect and locate disturbances in a system. These sensors can be expensive and, as a result, minimizing the size of the OLD-set is critical. Circulant graphs, a group of regular cyclic graphs, are often used to model parallel networks. We prove the optimal OLD-set size for a particular circulant graph using Hall's Theorem.

We also consider the mixed-weight OLD-set introduced in [R.M. Givens, R.K. Kincaid, W. Mao and G. Yu, *Mixed-weight open locating-dominating sets*, in: 2017 Annual Conference on Information Science and Systems, (IEEE, Baltimor, 2017) 1–6] which models a system with sensors of varying strengths. To model these systems, we place weights on the vertices in the graph, representing the strength of a sensor placed at the corresponding location in the system. We study particular mixed-weight OLD-sets in cycles, which behave similarly to OLD-sets in circulant graphs, and show the optimal mixed-weight OLD-set size using the discharging method.

Keywords: open locating-dominating sets, circulant graphs, Hall's Matching Theorem, mixed-weight open locating-dominating sets.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

Location detection problems have been studied for a variety of problems that can be modeled as a graph [11, 15, 20]. For these problems, a system is modeled as a graph $G = (V, E)$, with the goal of finding a minimally-sized subset of vertices such that if sensors are placed at locations represented by this subset, then the sensors are able to uniquely detect and locate anomalies in the system. Open locating-dominating sets (OLD-sets), defined in Section 2, work under the assumption that if an anomaly occurs at the location of a sensor node, then by the nature of the anomaly, that sensor is unable to detect it. For example, a nefarious individual may release a contaminant into a public waterworks system and destroy the sensor at the same location.

The OLD-set problem, defined in [21], attempts to find the smallest subset of vertices in a graph in which every vertex in the graph has a non-empty and unique set of neighbors in the subset. Minimum OLD-set sizes have been studied for a variety of graphs including grid-like graphs [22] and infinite triangular grids [13]. A dynamic bibliography of results in open locating-dominating sets and the related fields of identifying codes and locating-dominating sets can be found in [16].

The mixed-weight open locating-dominating set (mixed-weight OLD-set) is an extension of the open locating-dominating set that allows multiple integer weights to be given to vertices in the graph [9]. The mixed-weight OLD-set problem models a situation in which sensors of different strengths, and potentially different costs, are strategically placed throughout the system. An increase in the weight expands the reach of the vertex by an equivalent amount of edges in the graph. Wireless sensor networks often use multiple types of sensors, such as in systems that monitor natural habitats [17]. Mixed-weight OLD-sets can aid in the development and cost management of these networks. We use the discharging method [13] to provide a lower bound for mixed-weight OLD-sets in cycles and provide a construction of a mixed-weight OLD-set at that size. The mixed-weight OLD-set in a cycle behaves as a directed subgraph of the generalized circulant graph, as defined in Section 4.2.

Environmental monitoring using wireless sensor networks (WSNs) covers a range of important research areas including the study of glaciers [19], marine pollution [1], animal behavior and welfare [17], and the effect of climate change on farming [7]. Environments can be modeled as a graph by dividing the physical space into regions and adding vertices to the graph representing each of these regions. An edge is added to the graph to represent two regions that would be within communication range if a sensor was placed in one of the regions. The benefits of using location detection problems in the design of WSNs, including efficiency and ease of monitoring, are discussed in [15]. Location detection problems have been studied in other environments such as intruder detection in facilities [21], survivor location in emergency situations [20], and contaminant detection in public utilities [3].

A system of connected microprocessors can be modeled by an undirected graph where a processor is represented by a vertex and a network connection between two processors is represented by an edge. Fault detection in such networks has been studied using location detection problems [11]. Using OLD-sets we are able to consider the additional problem of a sensor failing to detect a fault in the processor where it is located. This type of sensor failure can be the result of the fault causing the sensor to fail, or, by design, when faults are only detected via routing messages directly between two connected processors.

Several topologies have been studied for fault location in multiprocessor system including trees, hypercubes, and meshes [11]. Circulant graphs, defined in Section 2, are another topology considered for multiprocessors and other massively parallel systems [23]. Connected circulant graphs are regular and cyclic with symmetric adjacencies, making them attractive for both design and study. Other related results for circulant graphs were presented in [18].

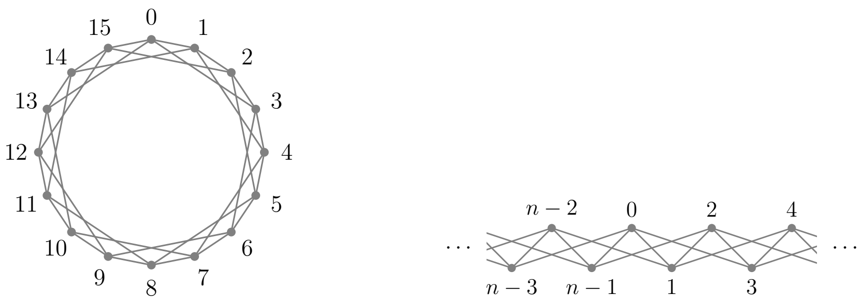
Hall's Matching Theorem [10] gives the conditions necessary and sufficient to find a matching or pairing in a bipartite graph. We use Hall's Matching Theorem to prove the following theorem: the optimal OLD-set density in $C_n(1,3)$ is $1/2$. To the best of our knowledge this is the first time Hall's Theorem has been used to prove a result for open locating-dominating sets. Recently, a result using matching theory in a different context was published in [8]. This further illustrates that matching is a valuable proof technique in addressing location detection.

Proof techniques in location detection problems have been dominated by the discharging method [13] and other similar methods. The discharging method was first used and made famous in the proof of the Four-Color Theorem [2]. It has recently been used to provide lower bounds on the size of an identifying code in infinite grids [5, 6]. Similarly, we use the discharging method to find the lower bound on the mixed-weight OLD-set size in cycles with weights 1 and 2, and provide an OLD-set construction at that same size, thus providing the optimal OLD-set density.

This paper is organized as follows. Section 2 provides background, definitions, and construction. In Section 3 we use Hall's Matching Theorem to prove the optimal OLD-set density in $C_n(1,3)$. We consider mixed-weight OLD-sets in cycles in Section 4, and we conclude in Section 5.

2. Background

A *circulant graph* $C_n(1,t)$ is a degree four, undirected graph containing n vertices labeled $\{0,1,\ldots,n-1\}$ where each vertex x is adjacent to vertices $x \pm 1 \mod n$ and $x \pm t \mod n$. In Figure 1(a) we depict circulant graph $C_{16}(1,3)$ with vertices drawn in the typical circular way. For graphs with $n \gg t$, edges are contained locally, so we can draw segments of the graph linearly to get a better view of this locality as seen in Figure 1(b).



(a) $C_{16}(1,3)$ circulant graph. (b) $C_n(1,3)$ drawn linearly.

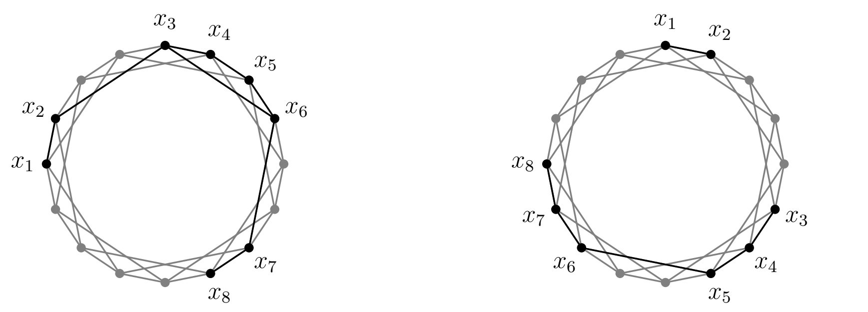
Figure 1.

The *neighborhood* of a vertex x, $N(x)$, is the set of vertices that are adjacent to x in the graph, not including x, i.e., $x \notin N(x)$ (the open property). For a set of vertices A, the neighborhood of the set, $N(A)$, contains all the vertices adjacent to the set and not in the set, i.e., $N(A) = \bigcup_{x \in A} N(x) \backslash A$. If $y \in N(x)$ or $N(A)$, we say that y is a *neighbor* of x or A.

A set S of vertices in a graph is an *open locating-dominating set* or *OLD-set*, if for every vertex x in the graph $N(x) \cap S \neq \emptyset$, and for any two vertices x and y in the graph such that $x \neq y$, $N(x) \cap S \neq N(y) \cap S$. Thus S is an OLD-set of the graph if every vertex in the graph has at least one neighbor in S (the dominating property), and if for every pair of vertices in the graph, x and y, there is at least one vertex in S that is adjacent to either x or y but not both (the locating property). We call the neighbors of a vertex x that are in the OLD-set S, $N(x) \cap S$, the *OLD-set neighborhood*. In determining if S is an OLD-set, we say two vertices *share* OLD-set neighborhoods if $N(x) \cap S = N(y) \cap S$, thus if S is an OLD-set any two vertices in the graph must not share OLD-set neighborhoods. We say an OLD-set *covers* a set of vertices if it has the locating and the dominating properties in the graph.

The *OLD-set density* is the proportion of vertices in an OLD-set S to the total number of vertices in the graph, $|S|/n$ for a circulant graph $C_n(1,t)$. The *optimal OLD-set density* is the achievable minimum OLD-set density in a graph, i.e., for optimal OLD-set density d, every OLD-set has density at least d, and there exists an OLD-set with density d.

By construction the upper bound on the minimum OLD-set density for $C_n(1,3)$ is $1/2$, as seen in Figures 2(a) and 2(b) for two OLD-sets in $C_{16}(1,3)$. If we consider these OLD-sets as $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, then the set $\{x_j + 16i | 1 \le j \le 8, 0 \le i < k\}$ is an OLD-set in $C_{16k}(1,3)$ of density $1/2$.



(a) Connected, size 8 OLD-set in $C_{16}(1,3)$.

(b) Size 8 OLD-set in $C_{16}(1,3)$ with 2 connected components.

Figure 2. OLD-set vertices are shown in black with connections also shown in black.

3. Hall's Matching Theorem

We prove the optimal OLD-set density for $C_n(1,3)$ is $1/2$. Hall's Theorem states that for bipartite graphs with vertices partitioned into sets R and S there is a matching of size $|R|$ if and only if for every subset $A \subseteq R$, $|A| \le |N(A)|$. We use Hall's Theorem to show there is a matching from vertices not in an OLD-set to vertices in the OLD-set for graph $C_n(1,3)$. If such a matching exists, then the number of vertices in the OLD-set is at least $n/2$.

For the rest of this section we will consider x and y to be vertices such that $x = 2i$ and $y = 2i + 1$ for $0 \le i \le n/2$, so that x and y may be neighbors of each other, but x and y do not share neighbors. We say that two vertices have different parity if their label is a different parity, so all vertices x and y have different parity. We note that for any vertex x in $C_n(1,3)$, x does not have the same parity as its neighbors, $x \pm 1$, $x \pm 3$. In particular, two vertices of different parity do not share neighbors.

3.1. Preliminary results

Lemma 1. *If $|x_i - x_j| \geq 8$, i.e., x_i and x_j have three or more vertices of the same parity between them, then they do not share neighbors.*

Proof. If $|x_i - x_j| \geq 8$ and $x_i < x_j$, then the largest neighbor of x_i is $x_i + 3$ and the smallest neighbor of x_j is $x_j - 3$. However, $x_i + 3 \leq x_j - 5$, therefore x_i and x_j do not share neighbors. ∎

Lemma 2. *If $A = \{x_1, x_2, \ldots, x_k\}$ such that $x_i = x_{i-1} + 2$, then $|N(A)| = |A| + 3$. Subsequently, if $|A| = k \geq 4$, then there is a set $B = \{y_1, y_2, \ldots, y_{k-3}\}$ of size $k - 3$ such that $y_i = y_{i-1} + 2$ and $N(B) = A$.*

Proof. If $A = \{x_1, x_2, \ldots, x_k\}$ such that $x_i = x_{i-1} + 2$, then $N(A) = \{x_1 - 3, x_1 - 1, x_1 + 1, \ldots, x_1 + 2k - 1, x_1 + 2k + 1\} = \{y_{-2}, y_{-1}, y_0, y_1, \ldots, y_k\}$ and $|N(A)| = |A| + 3$. We note that $y_i = y_{i-1} + 2$, the set $B = \{y_1, y_2, \ldots, y_{k-3}\}$ has neighbors $N(B) = \{x_1, x_2, \ldots, x_k\} = A$, and $|B| = k - 3$. ∎

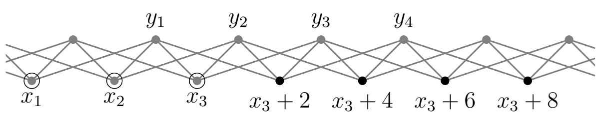


Figure 3. Three consecutive vertices of the same parity not in the OLD-set are followed by four consecutive vertices in the OLD-set.

Lemma 3. *If vertices x_1, x_2, and x_3, such that $x_3 = x_2 + 2$ and $x_2 = x_1 + 2$, are not in an OLD-set, then $x_3 + i$ for $i = 2, 4, 6, 8$ must be in the OLD-set. It follows that the four preceding vertices of the same parity must also be in the OLD-set.*

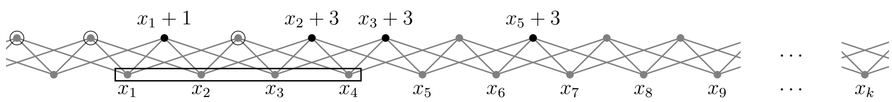
Proof. As seen in Figure 3 neighbor y_1 only has one possible vertex in the OLD-set, so $x_3 + 2$ must be in the OLD-set. Vertices y_1 and y_2 cannot share OLD-set neighborhood $\{x_3 + 2\}$, so $x_3 + 4$ must be in the OLD-set. Vertices y_2 and y_3 cannot share OLD-set neighborhood $\{x_3 + 2, x_3 + 4\}$, so $x_3 + 6$ must be in the OLD-set. Finally, vertices y_3 and y_4 cannot share OLD-set neighborhood $\{x_3 + 2, x_3 + 4, x_3 + 6\}$, so $x_3 + 8$ must be in the OLD-set. The proof follows similarly for the four preceding vertices of the same parity. ∎

Lemma 4. *If $A = \{x_1, x_2, \ldots, x_k\}$ such that $x_i = x_{i-1} + 2$ for $i < k$, and S is an OLD-set on $C_n(1, 3)$, then $|N(A) \cap S| \geq \left\lfloor \frac{k}{2} \right\rfloor + 1$.*

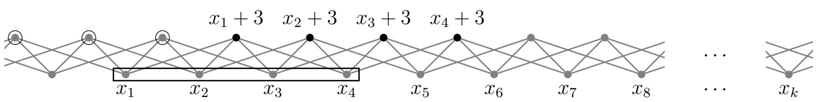
Proof. For $k = 1$, the vertex must be dominated by S, so one neighbor must be in S. For $k = 2$, one vertex in S cannot locate two vertices, so 2 neighbors must be in S. It is easy to show that for $k \in \{2, 3\}$ neighbors must be in S, $k = 4$ requires 3 neighbors, $k \in \{5, 6, 7\}$ requires 4 neighbors, and $k = 8$ requires 5 neighbors.

Assume for all $8 \leq \ell < k$ that ℓ vertices $x_i = x_{i-1} + 2$ for $i < \ell$ have $\left\lfloor \frac{\ell}{2} \right\rfloor + 1$ neighbors in the OLD-set. Consider k vertices $A = \{x_1, x_2, \ldots, x_k\}$ and by induction $\{x_3, \ldots, x_k\}$ contains $\left\lfloor \frac{k-2}{2} \right\rfloor + 1 = \left\lfloor \frac{k}{2} \right\rfloor$ neighbors in S. Vertices x_1 and x_2 introduce two new neighbors to $N(A)$, namely $\{x_1 - 3, x_1 - 1\}$. We may assume that neither of the neighbors is in S.

Case 1. Neighbor $x_1 + 1$ is in the OLD-set. Suppose $x_1 + 1$ is in the OLD-set as seen in Figure 4(a). Vertices x_1 and x_2 cannot share OLD-set neighborhood $\{x_1+1\}$, so x_2+3 must be in the OLD-set. Vertices x_2 and x_3 cannot share OLD-set neighborhood $\{x_1+1, x_2+3\}$, so x_3+3 must be in the OLD-set. And vertices x_4 and x_5 cannot share OLD-set neighborhood $\{x_2+3, x_3+3\}$, so x_5+3 must be in the OLD-set. The OLD-set vertices $\{x_1+1, x_2+3, x_3+3, x_5+3\}$ cover vertices $\{x_1, x_2, \ldots, x_7\}$. Consider the set $\{x_9, x_{10}, \ldots, x_k\}$ which is size $k - 8$ and does not have neighbors in the set of OLD-set vertices $\{x_1 + 1, x_2 + 3, x_3 + 3, x_5 + 3\}$. This set of $k-8$ needs at least $\left\lfloor \frac{k-8}{2} \right\rfloor + 1$ of its neighbors in the OLD-set. Thus the set of k vertices need at least $4 + \left\lfloor \frac{k-8}{2} \right\rfloor + 1 = \left\lfloor \frac{k}{2} \right\rfloor + 1$ neighbors in the OLD-set.



(a) $x_1 + 1$ is in the OLD-set.



(b) $x_1 + 3$ is in the OLD-set.

Figure 4.

Case 2. Neighbor x_1+3 is in the OLD-set. If x_1+3 is in the OLD-set, then the 3 neighbors of x_1 are not in the OLD-set, and by Lemma 3 the next four vertices of that parity, including x_1+3, must be in the OLD-set as shown in Figure 4(b). The first 7 vertices of the k total vertices are covered by these 4 OLD-set neighbors. The remaining $k - 7$ vertices do not currently share any OLD-set neighbors with the first 7 vertices, but will require $\left\lfloor \frac{k-7}{2} \right\rfloor + 1$ neighbors in the OLD-set to cover them. Thus the set of k vertices need at least $4 + \left\lfloor \frac{k-7}{2} \right\rfloor + 1 \geq \left\lfloor \frac{k}{2} \right\rfloor + 1$ neighbors in the OLD-set. ■

3.2. Main results

Theorem 5. *The optimal OLD-set density in $C_n(1,3)$ is $1/2$.*

Proof. We note that by construction the upper bound for the minimum OLD-set size is 1/2. To show the lower bound is 1/2, we show there is a matching from vertices not in the OLD-set to vertices in the OLD-set for $C_n(1,3)$. We build an auxiliary bipartite graph from $C_n(1,3)$, $B = (R, S)$, where $R = V \backslash S$, with edges $E(B) = \{(u,v) | u \in R, v \in S, (u,v) \in E(C_n(1,3))\} \cup \{(x, x+5) | x-4, x-2, x \notin S, x+5 \in S\}$. We only need to consider $A \subseteq R$ such that the vertices in A share neighbors. If A has sets of vertices that do not share neighbors, A can be separated into sets of vertices with disjoint neighbor sets, and the size of the neighbors of A in the OLD-set will be the sum of the size of the neighbors in the OLD-set of the individual sets.

Case 1. $|A| = 1$. The bipartite graph contains A in one partition, $N(A) \cap S$ in the other, and all edges between those sets that occur in the graph. For $|A| = 1$, the vertex in A must be covered by at least 1 vertex in the OLD-set, thus $|N(A) \cap S| \geq |A|$.

Case 2. $|A| = 2$. The bipartite graph contains A in one partition, $N(A) \cap S$ in the other, and all edges between those sets that occur in the graph. For $|A| = 2$, each vertex must be covered by 1 vertex, but they cannot be covered by the same vertex or they would not be distinguishable by the OLD-set. Thus they must be covered by at least 2 vertices and $|N(A) \cap S| \geq |A|$.

Case 3. $|A| \geq 3$.

Subcase 3a. A is $\{x_1, x_2, x_3\}$ with $x_i = x_{i-1} + 2$. The bipartite graph contains A in one partition, $(N(A) \cap S) \cup \{x_3 + 5\}$ in the other, all edges between those sets that occur in the graph, and edge $(x_3, x_3 + 5)$ if $x_3 + 5 \in S$. By Lemma 4 we know that the three vertices can be covered by two OLD-set vertices. However, because A only contains vertices not in the OLD-set, by Lemma 3 the next four vertices to be in the OLD-set, as shown in Figure 5. Thus $x_3 + 5$ will not be in $N(A) \cap S$ for any A. If A only has 2 neighbors in the OLD-set in $C_n(1,3)$, it is also the case that $x_3 + 5$ must be in the OLD-set in order to cover $x_4 = x_3 + 2$, otherwise x_3 and x_4 share OLD-set neighborhoods. For any subset $A = \{x_1, x_2, x_3\}$ with $x_i = x_{i-1} + 2$, if $x_3 + 5 \in S$, the bipartite graph includes an edge from x_3 to $x_3 + 5$, also shown in Figure 5. If $x_3 + 5 \notin S$, a third neighbor of A must be in S. Thus $|N(A) \cap S| \geq |A|$.

Subcase 3b. $A = \{x_1, x_2, \ldots, x_{|A|}\}$ such that $x_i + 2j \mod n = x_{i+1}$ for $j > 0$ with at least one $j > 1$, and for any vertex y such that $y \notin A$, $x_1 + 2 \leq y \leq x_{|A|} - 2$,

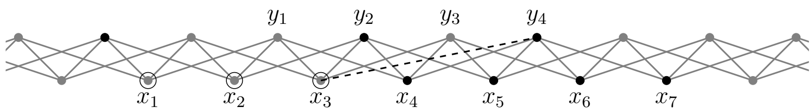


Figure 5. $A = \{x_1, x_2, x_3\}$ is three consecutive vertices of the same parity.

y either does not share neighbors with vertices in A or y is in the OLD-set. For example, $A = \{x_1, x_1 + 2, x_1 + 6, x_1 + 8\}$, where $x_1 + 4$ is in the OLD-set, is one set covered by this case.

We note that if a subset of A has two vertices x_i and x_{i+1} that do not share neighbors, A can be split into at least two different sets with disjoint neighbors. Then the size of the OLD-set neighborhood A would be the sum of the size of the OLD-set neighbors of the individual sets. Because of this consideration, $j < 4$ by Lemma 1. And by Lemma 3, A will not contain three consecutive vertices of the same parity, as in Case 3a, as these vertices would not share neighbors with any other vertex not in the OLD-set.

Let S' be the set of OLD-set vertices $\left\{x'_1, x'_2, \ldots, x'_j\right\}$ such that $A \cup S'$ is the set of all vertices $\left\{x_1 + 2i \,|\, 0 \le i \le \frac{x_{|A|} - x_1}{2}\right\}$. We note that $|A \cup S'| = k \ge 4$ and $A \cup S'$ is a neighborhood to $k - 3$ consecutive vertices of the same parity by Lemma 2. Thus at least $\left\lfloor \frac{k-3}{2} \right\rfloor + 1$ vertices of $A \cup S'$ must be in the OLD-set, and $|A| \le k - \left(\left\lfloor \frac{k-3}{2} \right\rfloor + 1\right)$. We note that because each x_i in A shares at least one neighbor with x_{i+1}, $N(S') \subset N(A)$ and thus $N(A) = N(A \cup S')$. At least $\left\lfloor \frac{k}{2} \right\rfloor + 1$ neighbors of $A \cup S'$ must be in the OLD-set, and thus at least $\left\lfloor \frac{k}{2} \right\rfloor + 1$ neighbors of A must be in the OLD-set. For k even or odd we find that $|A| \le k - \left(\left\lfloor \frac{k-3}{2} \right\rfloor + 1\right) = \left\lfloor \frac{k}{2} \right\rfloor + 1 = |N(A \cup S') \cap S| = |N(A) \cap S|$.

Thus we have created a bipartite graph from R to S such that for all $A \subseteq R$, $|N(A) \cap S| \ge |A|$. By Hall's Theorem there is a matching from R to S of size $|R|$, and therefore an OLD-set on $C_n(1, 3)$ needs at least $n/2$ vertices. By construction an OLD-set needs at most $n/2$ vertices. ■

4. Mixed-Weight OLD-sets

The mixed-weight open locating-dominating set (mixed-weight OLD-set) models a system in which sensors have varying strengths, represented by placing weights on vertices in the graph. Mixed-weight OLD-sets are related to the weighted or d-identifying code, where all vertices receive the same weight d, which have been studied for paths and cycles [4], hypercubes [12], and other graphs. The mixed-weight OLD-set is also similar to an OLD-set on a directed graph: increased weight can be represented by adding arcs to other vertices. Identifying codes have also been studied for directed graphs in [14]. We note that the weighted OLD-set problem is a special case of the mixed-weight OLD-set problem, and, to our knowledge, there is no literature for weighted OLD-sets or OLD-sets in directed graphs.

4.1. Definitions and properties

Definition 6. The *weight function*, w, such that $w(x) \geq 1$ for every $x \in V$, provides an integer value, or *weight*, for each vertex in the graph.

For a particular vertex x, $w(x)$ is the weight of the vertex. For a set of vertices $A \subseteq V$, the weight of the set $w(A)$ is the sum of all the weights of vertices in that set, $\sum_{x \in A} w(x)$. The distance between two vertices, $d(x, y)$, is the length of the shortest path between vertex x and vertex y, where each edge is considered to be length 1.

Definition 7. The *open outgoing-ball*, $B^-(x)$ is the set of all vertices within a distance of $w(x)$ from x but not including x, i.e., $B^-(x) = \{y \in V | 0 < d(x, y) \leq w(x)\}$. The *open incoming-ball*, $B^+(x)$ is the set of all vertices that contain x in their open outgoing-ball, i.e., $B^+(x) = \{y \in V | 0 < d(x, y) \leq w(y)\}$.

Definition 8. A *mixed-weight open locating-dominating set*, *mixed-weight OLD-set*, or *MW-OLD-set* of a graph G is a set of vertices $S \subseteq V$ such that $B^+(x) \cap S$ is nonempty and unique for every $x \in V$. The *total weight* of the (mixed-weight) OLD-set S, $w(S)$, is the sum of all the weights of all vertices in S.

Definition 9. The *eccentricity* of a vertex x in a graph, $\epsilon(x)$, is the maximum distance of all the shortest paths between x and any other vertex in the graph.

The case in which $w(x) = 1$ for all $x \in V$ defines the non-weighted OLD-set. For the mixed-weight OLD-set, we say a vertex y is a *neighbor* of vertex x if $y \in B^+(x)$. In this case neighbors are not symmetric. We note that the weight of a vertex x not in mixed-weight OLD-set S does not affect S or the size of S, both theoretically and in the application of sensor monitoring.

In Figure 6, a vertex with weight greater than 1 becomes the neighbor of other vertices. In a wireless sensor network, a weighted vertex under this definition would represent a sensor that has a stronger antenna and can therefore receive data from transmitters that are further away.

Figure 7 shows an example of a mixed-weight OLD-set in a graph that does not contain a non-weighted OLD-set. Vertices x_1 and x_3 share open incoming-ball $\{x_2, x_4\}$ when $w(x) = 1$ for every $x \in V$. If vertex x_3 is given weight 2, then every open incoming-ball becomes unique, allowing for a mixed-weight OLD-set. One minimum sized mixed-weight OLD-set is $\{x_3, x_4, x_5, x_6\}$.

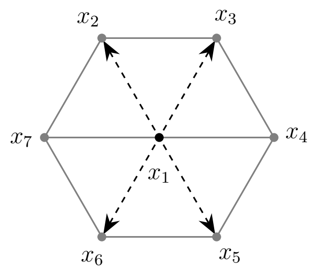
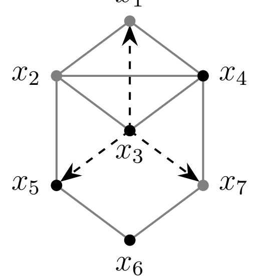


Figure 6. The effect of the weighted vertex, $w(x_1) = 2$, is indicated by dashed arrows.



(a)

x	$B^+(x)$, weighted
x_1	$\{x_2, x_3, x_4\}$
x_2	$\{x_1, x_3, x_4, x_5\}$
x_3	$\{x_2, x_4\}$
x_4	$\{x_1, x_2, x_3, x_7\}$
x_5	$\{x_2, x_3, x_6\}$
x_6	$\{x_5, x_7\}$
x_7	$\{x_3, x_4, x_6\}$

(b)

Figure 7. Mixed-Weight OLD-set Example. If there are no weighted vertices, x_1 and x_3 share open incoming-ball $\{x_2, x_4\}$, and the graph does not have an OLD-set. If $w(x_3) = 2$, as indicated by dashed arrows, then $\{x_3, x_4, x_5, x_6\}$ is one minimum mixed-weight OLD-set, as shown with black vertices.

As in the case of the OLD-set, we are interested in minimizing the size of the mixed-weight OLD-set. In [9] we showed that the decision problem for finding the minimum mixed-weight OLD-set, MW-OLD, is NP-Complete.

The following lemmas provide the necessary conditions for the existence of mixed-weight OLD-sets.

Lemma 10. *For graph $G = (V, E)$ and weight function w, a mixed-weight OLD-set $S \subseteq V$ exists if and only if for every $x, y \in V$, $x \neq y$, $B^+(x) \neq B^+(y)$, i.e., if the open incoming-ball is unique for all vertices in the graph.*

Lemma 11. *For graph $G = (V, E)$ and weight function w, if a mixed-weight OLD-set exists, then $S = V$ is a mixed-weight OLD-set.*

Lemma 12. *For every connected graph $G = (V, E)$ with $|V| > 1$, there exists a weight function such that the graph contains a mixed-weight OLD-set. If $w(x) = \epsilon(x)$, the eccentricity of x, then the graph has a mixed-weight OLD-set.*

4.2. Mixed-weight OLD-sets in cycles

The weight of a vertex can be represented by the addition of arcs from the weighted vertex to its new neighbors, as seen in Figure 6. A cycle is an undirected graph containing n vertices labeled $\{0, 1, \dots, n-1\}$ where each vertex x is adjacent to vertices $x \pm 1 \mod n$. Weighted vertices in a cycle C_n of size n, transform the graph into a directed subgraph of a generalized circulant graph $C_n(1, 2, \dots, \max(w))$ where $\max(w)$ is the maximum weight of a vertex in the cycle and $C_n(1, 2, \dots, \max(w))$ is a graph where every vertex is adjacent to $x \pm i \mod n$ for $1 \leq i \leq \max(w)$.

Suppose C_n has a weight function $w(x) \geq 1$. We consider the mixed-weight OLD-set problem for possible weights 1 and 2. Let $OLD(C_n, w)$ be the minimum size OLD-set in C_n with weight function $w(x) \leq 2$. We show that $OLD(C_n, w) \geq \frac{2n}{5}$ using the a discharging argument.

In the discharging method each vertex in a set of unknown size is given a charge of 1, and all other vertices in the graph are given a charge of 0. Thus the total charge in the graph is equivalent to the number of vertices in the set of unknown size. If the total charge in the graph can be redistributed from vertices in the set to vertices not in the set, so that each vertex has at least a fraction of charge f, then the size of the original set of vertices must be at least $f \cdot n$, where n is the total number of vertices in the graph.

We define an m-cluster as a weakly connected component of order m in the graph induced by a mixed-weight OLD-set S, where vertex x with $w(x) = 2$ is considered to have arcs from x to $x - 2$ and $x + 2$. We define a neighbor of a cluster as a vertex that has at least one vertex in its open incoming-ball that is in the cluster. All neighbors of a cluster cannot be in the mixed-weight OLD-set. For the following proof, we refer to the open incoming-ball intersected with the OLD-set, $B^+(x) \cap S$, as the mixed-weight neighborhood. We say two vertices share mixed-weight neighborhoods if $B^+(x) \cap S = B^+(y) \cap S$, thus if S is a mixed-weight OLD-set any two vertices in the graph must not share mixed-weight neighborhoods.

Theorem 13. *$OLD(C_n, w) \geq 2/5$ where $w(x) \leq 2$ for every x.*

Proof. Let S be a mixed-weight OLD-set on C_n for $C_n > 2$ and $w(x) \leq 2$. Assign charge 1 to vertices in the mixed-weight OLD-set S and 0 to all other vertices.

Redistribution Rule: If $x \notin S$ is adjacent to k clusters in S, then x gets $\frac{1}{k} \cdot \frac{2}{5}$ charge from each of the clusters.

By this rule, each vertex not in S will have a charge of at least $2/5$. For the proof we consider all the possible m-clusters in S with all possible mixed-weight functions with weights 1 and 2. The m-cluster must allow for its vertices and its neighbors to be open-located and dominated, and the weight function must allow it to be a cluster. If each possible m-cluster in S is left with $2m/5$ charge after redistribution, then all vertices in S will maintain $2/5$ charge. We note that an m-cluster can give at most $3m/5$ charge to its neighbors to maintain at least $2m/5$ charge, thus a cluster can provide enough charge for $\lfloor 3m/2 \rfloor$ neighbors. We also note that there are no 1-clusters in a mixed-weight OLD-set, as the vertex in that cluster would not be dominated by the mixed-weight OLD-set.

Case 1. 2-cluster. There are two 2-clusters with more than $\lfloor 3m/2 \rfloor = 3$ neighbors that can occur in a mixed-weight OLD-set. The first 2-cluster is $\{x_1, x_2\}$ with $x_2 = x_1+1$ and $w(x_i) = 2$ and has four neighbors $\{x_1-2, x_1-1, x_2+1, x_2+2\}$. In order for $x_1 - 1$ and $x_2 + 1$ to not share mixed-weight neighborhood $\{x_1, x_2\}$, without loss of generality, $x_1 - 3$ must be in the mixed-weight OLD-set with $w(x_1 - 3) = 2$. This also prevents $x_1 - 2$ and x_2 from sharing mixed-weight neighborhood $\{x_1\}$. In order for x_1 and $x_2 + 2$ to not share mixed-weight neighborhood $\{x_2\}$, x_2+2 must have x_2+3 or x_2+4 in its mixed-weight neighborhood. Thus $x_1 - 2$, $x_1 - 1$, and $x_2 + 2$ receive at most $\frac{1}{5}$ charge from the cluster, and x_2+1 receives at most $\frac{2}{5}$ charge from the cluster, leaving the cluster with at least $\frac{4}{5}$ charge.

The second 2-cluster is $\{x_1, x_2\}$ with $x_2 = x_1 + 2$ and $w(x_i) = 2$ and has five neighbors $\{x_1-2, x_1-1, x_1+1, x_2+1, x_2+2\}$. In order for x_1-2, x_1-1, and x_2 to not share mixed-weight neighborhood $\{x_1\}$, $x_1 - 3$ and $x_1 - 4$ must be in the mixed-weight OLD-set with $w(x_1 - 3) = w(x_1 - 4) = 2$. Similarly, in order for x_1, $x_2 + 1$, and $x_2 + 2$ to not share mixed-weight neighborhood $\{x_2\}$, $x_2 + 3$ and $x_2 + 4$ must be in the mixed-weight OLD-set with $w(x_2 + 3) = w(x_2 + 4) = 2$. Thus x_1-2, x_1-1, x_2+1, and x_2+2 receive at most $\frac{1}{5}$ charge from the cluster, and $x_1 + 1$ receives at most $\frac{2}{5}$ charge from the cluster, leaving the cluster with at least $\frac{4}{5}$ charge.

Case 2. 3-cluster. There is one 3-cluster with more than $\lfloor 3m/2 \rfloor = 4$ neighbors, $\{x_1, x_2, x_3\}$ with $x_i = x_{i-1} + 2$ and at least one of x_1 or x_3 having a weight of 2. Without loss of generality, suppose $w(x_1) = 2$. If $w(x_2) = 1$, then x_1 will not have any vertices in its mixed-weight neighborhood. If $w(x_2) = w(x_3) = 2$, then x_1 and x_3 will share mixed-weight neighborhood $\{x_2\}$. If $w(x_2) = 2$ and $w(x_3) = 1$, then the neighbors of the cluster are x_1-2, x_1-1, x_1+1, x_2+1, and $x_3 + 1$. In order for vertices x_1 and x_3 to not share mixed-weight neighborhood $\{x_2\}$, $x_3 + 2$ must be in the OLD-set with $w(x_3 + 2) = 2$. In this case, $x_3 + 1$ will be the neighbor of two clusters. Thus $x_3 + 1$ will receive at most $\frac{1}{5}$ charge from the cluster and the remaining four neighbors will receive at most $\frac{2}{5}$ charge, leaving the cluster with at least $\frac{6}{5}$ charge.

Case 3. m-cluster with $m \in \{4, 5\}$. There is one 4-cluster and one 5-cluster with more than $\lfloor 3m/2 \rfloor = m + 2$, for $m \in \{4, 5\}$, neighbors that can occur in a mixed-weight OLD-set. The clusters are $\{x_1, x_2, \ldots, x_m\}$ with $x_i = x_{i-1} + 2$,

$w(x_1) = w(x_m) = 2$, $w(x_i) \leq 2$ for $1 < i < m$ and has $m+3$ neighbors $\{x_1 - 2, x_1 - 1, x_m + 2\} \cup \{x_1 + 1, \ldots, x_m + 1\}$. In order for $x_1 - 2$ and $x_1 - 1$ to not share mixed-weight neighborhood $\{x_1\}$, either $x_1 - 3$ with $w(x_1 - 3) = 1$ or $x_1 - 4$ with $w(x_1 - 4) = 2$ must be in the mixed-weight OLD-set. Similarly, in order for $x_m + 2$ and $x_m + 1$ to not share mixed-weight neighborhood $\{x_m\}$, either $x_m + 3$ with $w(x_m + 3) = 1$ or $x_1 + 4$ with $w(x_m + 4) = 2$ must be in the mixed-weight OLD-set. Thus $x_1 - 2$ and $x_m + 2$ receive at most $\frac{1}{5}$ charge from the cluster, and the remaining $m+1$ neighbors receive at most $\frac{2}{5}$ charge from the cluster, leaving the cluster with $m - \frac{2(m+1)}{5} - \frac{2}{5} = \frac{3m-4}{5} \geq \frac{2m}{5}$ charge.

Case 4. m-cluster with $m \geq 6$. For this case we note that for any m-cluster in C_n with weight function $w(x) \leq 2$, the cluster has at most $m+3$ neighbors. If the first vertex in the cluster is x, then the last vertex in the cluster, y, is at most $x + 2(m-1)$ given any possible weight function. This means there are at most $x + 2(m-1) - x - (m-1) = m-1$ vertices between x and y that are not in the mixed-weight OLD-set. Vertex x has at most 2 neighbors prior to it in the path, and vertex y has at most 2 neighbors after it in the path. Thus the cluster has at most $m+3$ neighbors, and is responsible for discharging at most $\frac{2(m+3)}{5}$ charge to its neighbors. This leaves the cluster with at least $m - \frac{2(m+3)}{5} = \frac{3m}{5} - \frac{6}{5} \geq \frac{2m}{5}$ charge. ∎

This lower bound can be achieved in the trivial case $w(x) = 2$ for $x \in S$. Consider the cycle C_{10n} for $n \geq 1$. The set $S = \{10i, 10i+2, 10i+4, 10i+6 | 0 \leq i < n\}$ where $w(x) = 2$ for $x \in S$ is a mixed-weight OLD-set of size $4n$. Vertices not in the OLD-set can have weights 1 or 2.

5. Conclusion

We showed the optimal OLD-set density in $C_n(1,3)$ is $1/2$. We provided the upper bound by construction and derived the lower bound using Hall's Theorem. To our knowledge this is the first time Hall's Theorem has been used to provide this type of a result in the open locating-dominating sets literature. This method has the potential to find OLD-set size bounds in other circulant graphs and topologies used in microprocessor systems. Finding these bounds will help determine best practices for location detection problems in a variety of applications.

We considered mixed-weight OLD-sets [9] in cycles which, with weighted vertices, behave like directed subgraphs of circulant graphs. We showed that the lower bound for the size of mixed-weight OLD-sets in cycles is $2/5$ for weight functions $w(x) \leq 2$ via the discharging method. We provided a construction to show this bound is optimal when $w(x) = 2$ for all vertices in the OLD-set.

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Received 11 November 2017
Revised 18 February 2019
Accepted 21 June 2019