BLOCK GRAPHS WITH LARGE PAIRED DOMINATION MULTISUBDIVISION NUMBER

CHRISTINA M. MYNHWARTD

Department of Mathematics and Statistics
University of Victoria
Victoria BC, Canada

e-mail: kieka@uvic.ca

AND

JOANNA RACZEK

Faculty of Applied Physics and Mathematics
Gdańsk University of Technology
ul. Narutowicza 11/12, 80-233 Gdańsk, Poland

e-mail: joanna.raczek@pg.edu.pl

Abstract

The paired domination multisubdivision number of a nonempty graph $G$, denoted by $msd_{pr}(G)$, is the smallest positive integer $k$ such that there exists an edge which must be subdivided $k$ times to increase the paired domination number of $G$. It is known that $msd_{pr}(G) \leq 4$ for all graphs $G$. We characterize block graphs with $msd_{pr}(G) = 4$.

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1. Introduction

The study of changes that occur in domination-related parameters of a graph when its edges are subdivided\footnote{See Section 2 for definitions of terms used in this section.} was initiated in [11]. If $\pi$ is a domination-type parameter of $G$, the smallest number of edges that must be subdivided, where each edge of $G$ can be subdivided at most once, in order to increase $\pi$ is called
the $\pi$-subdivision number, denoted by $\text{sd}_\pi(G)$. Subdivision numbers have been studied for the domination number $[6, 11]$, as well as for connected $[4]$, double $[1]$, Roman $[10]$, total $[7, 9]$ and paired domination numbers $[5]$.

Instead of subdividing multiple edges once each, one may wish to subdivide a single edge multiple times. The smallest number of times that a single edge of $G$ must be subdivided to increase $\pi$ is called the $\pi$-multisubdivision number, denoted by $\text{msd}_\pi(G)$. Domination and paired domination multisubdivision numbers were studied in $[3]$ and $[2]$, respectively. In particular, it was shown in $[2]$ that the paired domination multisubdivision number $\text{msd}_{\text{pr}}(G)$ of any graph $G$ is at most four. For brevity we refer to a graph $G$ with $\text{msd}_{\text{pr}}(G) = 4$ as an msd-4 graph. Msd-4 trees were characterized in $[2]$.

We discuss methods of combining msd-4 graphs to yield new msd-4 graphs and use our results, combined with results from $[2]$, to characterize msd-4 block graphs. Definitions and previous results are given in Section 2. We state the characterization of msd-4 block graphs in Section 3, but defer its proof to Section 6 to allow us to prove a number of results used in the proof; results that apply to general msd-4 graphs are given in Section 4, while results specific to block graphs can be found in Section 5.

## 2. Definitions and Previous Results

We refer the reader to $[8]$ for domination parameters not defined here. A set $S$ of vertices of a graph $G = (V, E)$ without isolated vertices is a paired dominating set of $G$ if every vertex of $G$ is adjacent to a vertex in $S$, and the subgraph $G[S]$ of $G$ induced by $S$ has a perfect matching. If $u, v \in S$ and there exists a perfect matching $M$ of $G[S]$ such that $uv \in M$, we say that $u$ and $v$ are paired in $S$. The smallest cardinality of a paired dominating set of $G$ is the paired domination number of $G$, denoted by $\gamma_{\text{pr}}(G)$. If $S$ is a paired dominating set of $G$ such that $|S| = \gamma_{\text{pr}}(G)$, we call $S$ a $\gamma_{\text{pr}}(G)$-set, or simply a $\gamma_{\text{pr}}$-set if the graph is clear from the context. If $u$ is a vertex of $G$ such that $G - u$ has no isolated vertices and $\gamma_{\text{pr}}(G - u) < \gamma_{\text{pr}}(G)$ (in which case $\gamma_{\text{pr}}(G - u) = \gamma_{\text{pr}}(G) - 2$), we say that $u$ is a $\gamma_{\text{pr}}(G)$-critical vertex, or simply a $\gamma_{\text{pr}}$-critical vertex, and define $\text{Cr}(G) = \{u \in V(G) : u$ is a $\gamma_{\text{pr}}$-critical vertex$\}$.

A neighbour of a vertex $u \in V(G)$ is a vertex adjacent to $u$. The (open) neighbourhood $N(u)$ of a vertex $u$ is the set of all vertices adjacent to $u$, and its closed neighbourhood is $N[u] = N(u) \cup \{u\}$. For a set $S \subseteq V(G)$, the (open) neighbourhood of $S$ is $N(S) = \bigcup_{u \in S} N(u)$, and its closed neighbourhood is $N[S] = N(S) \cup S$. For a vertex $u \in S$, the private neighbourhood of $u$ with respect to $S$ is the set $\text{PN}(u, S) = N[u] \setminus N[S \setminus \{u\}]$. It is possible that $u \in \text{PN}(u, S)$, but if $S$ is a paired dominating set, then $u$ is adjacent to the vertex it is paired with,
so \( u \notin \text{PN}(u, S) \) in this case.

An edge uv of a graph G is subdivided if it is replaced by a path \((u, x, v)\), where \( x \) is a new vertex, and multisubdivided if it is replaced by a path \((u, x_1, \ldots, x_k, v)\), \( k \geq 2 \), where \( x_1, \ldots, x_k \) are new vertices; we also say that uv is subdivided \( k \) times. Let \( G_{uv,k} \) denote the graph obtained from \( G \) by subdividing the edge uv \( k \) times. The paired domination multisubdivision number \( \text{msd}_{pr}(G) \) of a graph G without isolated vertices is the smallest positive integer \( k \) such that there exists an edge uv which must be subdivided \( k \) times for \( \gamma_{pr}(G_{uv,k}) \) to exceed \( \gamma_{pr}(G) \). As mentioned above, \( \text{msd}_{pr}(G) \leq 4 \) for all graphs. The three graphs in Figure 1 are all msd-4 graphs; the red vertices form \( \gamma_{pr} \)-sets.

![Figure 1](image_url)

Figure 1. (a) The spider \( S(2, 2, 6) \) (b) the corona \( K_3 \circ K_1 \) (c) a flared corona \( K_4 \circ^2 K_1 \).

A leaf of a graph is a vertex of degree one, and its neighbour is called a stem.

The following properties of msd-4 graphs were proved in [2].

**Theorem 1** [2]. Let G be an msd-4 graph. Then

(i) each edge of G belongs to a matching of a minimum paired dominating set of G;

(ii) any leaf of G is a \( \gamma_{pr} \)-critical vertex;

(iii) each stem is adjacent to exactly one leaf.

The complete bipartite graph \( K_{1,k} \), \( k \geq 2 \), is called a star. Let \( K_{1,k} \) have partite sets \( \{u\} \) and \( \{v_1, \ldots, v_k\} \). The spider \( S(\ell_1, \ldots, \ell_k) \), \( \ell_i \geq 1 \), \( k \geq 2 \), is a tree obtained from \( K_{1,k} \) by subdividing the edge \( uv_i \) \( \ell_i - 1 \) times, \( i = 1, \ldots, k \). Note that \( S(2, 2) \cong P_3 \). See Figure 1(a) for \( S(2, 2, 6) \). The characterization of msd-4 trees in [2] immediately gives the following result.
Proposition 2 [2]. The spider $T = S(2, \ldots, 2)$ satisfies $\text{msd}_{pr}(T) = 4$, and $\text{Cr}(T)$ consists of the leaves of $T$.

The corona $G \circ K_1$ of a graph $G$ is the graph obtained by joining each vertex of $G$ to a new leaf; $K_3 \circ K_1$ is illustrated in Figure 1(b). A flared corona $G \circ^{st} K_1$ of $G$ is a graph obtained by joining each vertex of $G$, except one vertex $w$, to a new leaf, while $w$ is joined to a single vertex of each of $t \geq 1$ copies of $K_2$. The flared corona $K_3 \circ^{st} K_1$ is depicted in Figure 1(c). The following facts can be verified easily and are stated without proof.

Remark 3.
(i) A corona $K_n \circ K_1$, $n \geq 2$, is an msd-4 graph if and only if $n$ is odd.
(ii) A flared corona $K_n \circ^{st} K_1$, $n \geq 2$, is an msd-4 graph if and only if $n$ is even.
(iii) A vertex of $K_{2n+1} \circ K_1$ or $K_{2n} \circ^{st} K_1$ is $\gamma_{pr}$-critical if and only if it is a leaf (see Theorem 1).

A block of a graph is a maximal connected subgraph with no cut-vertex, and a block graph is a graph, each of whose blocks is a complete graph. Thus, trees are block graphs since each block of a nontrivial tree is a $K_2$. Evidently, coronas and flared coronas are also block graphs. To characterize msd-4 block graphs, we use spiders $S(2, \ldots, 2)$, coronas $K_{2n+1} \circ K_1$ and flared coronas $K_{2n} \circ^{st} K_1$, combining them by identifying vertices and edges in a prescribed way.

We begin by describing two operations, collectively known as $\oplus$-operations, for joining disjoint graphs; since the operations can be performed on any graphs, we state them in their most general form. (The operations are well known but we need to define our notation.)

$G_1 \oplus^{u_1v_2} G_2$: Let $G_1$ and $G_2$ be vertex disjoint graphs and $u_i \in V(G_i)$ for $i \in \{1, 2\}$. We denote the graph obtained from $G_1$ and $G_2$ by identifying $u_1$ and $u_2$ into one vertex $u = u_1 = u_2$ by $G_1 \oplus^{u_1v_2} G_2$ (or by $G_1 \oplus^{u_1v_2} G_2$ if the label $u$ is unimportant).

$G_1 \oplus^{e_1e_2} G_2$: Let $G_1$ and $G_2$ be vertex disjoint graphs and $e_i = u_iv_i \in E(G_i)$. We denote the graph obtained from $G_1$ and $G_2$ by identifying $u_1$ and $u_2$ into one vertex $u = u_1 = u_2$, $v_1$ and $v_2$ into one vertex $v = v_1 = v_2$, and $e_1$ and $e_2$ into one edge $e = uv$ by $G_1 \oplus^{e_1e_2} G_2$ (or by $G_1 \oplus^{e_1e_2} G_2$ if the label $e$ is unimportant).

The graph $G_1 \oplus^{e_1e_2} G_2$, where $G_1 = S(2, 2, 6)$, $G_2 = K_3 \circ K_1$, and $e_i = u_iv_i$ for $i = 1, 2$, is illustrated in Figure 2. Note that $u_i$ is $\gamma_{pr}(G_i)$-critical for $i = 1, 2$, and $u_1 = u_2$ is $\gamma_{pr}$-critical in $G_1 \oplus^{e_1e_2} G_2$. The spider $S(2, 2, 6)$, in turn, is obtained as $H_1 \oplus^{u_1v_2} H_2$, where $H_1 = S(2, 2, 2)$, $H_2 = P_5 = S(2, 2)$, and $u_i$ is a leaf of $H_i$, $i = 1, 2$. 

3. Characterization of msd-4 Block Graphs

We now state our main result — the characterization of msd-4 block graphs. The proof is deferred to Section 6.

Let \( \mathcal{U} \) be the collection of all spiders \( S(2, \ldots, 2) \), coronas \( K_{2n+1} \circ K_1 \), and flared coronas \( K_{2n} \circ^* K_1 \), \( n \geq 1 \). Define \( \mathcal{B} \) to be the family of all block graphs \( G \) that can be obtained as a graph \( G_j, j \geq 1 \), from a sequence \( G_1, \ldots, G_j \) of graphs, where \( H_1 = G_1 \in \mathcal{U} \), and, if \( j > 1 \), \( G_{i+1} \) can be constructed recursively from \( G_i \) by

- adding a graph \( H_{i+1} \in \mathcal{U} \),
- choosing vertices \( u_1 \in \text{Cr}(G_i), u_2 \in \text{Cr}(H_{i+1}) \), and if necessary, \( v_1 \in N(u_1), v_2 \in N(u_2) \),
- performing the operation \( G_i \oplus^{u_1 u_2} H_{i+1} \) or \( G_i \oplus^{u_1 v_1} u_2 v_2 H_{i+1} \).

**Theorem 4.** Let \( G \) be a connected block graph. Then \( G \) is an msd-4 graph if and only if \( G \in \mathcal{B} \). Moreover, if \( G \) is an msd-4 graph constructed from the graphs \( H_1, \ldots, H_j \in \mathcal{U} \), then \( \text{Cr}(G) = \bigcup_{i=1}^j \text{Cr}(H_i) \).

The second statement of Theorem 4 implies that any \( \gamma_{pr} \)-critical vertex \( v \) of an msd-4 block graph remains \( \gamma_{pr} \)-critical after the \( \oplus \)-operations have been performed any number of times, whether \( v \) was identified with another vertex or not. The following corollary of Theorem 4 was proved in [2].

**Corollary 5.** A tree \( T \) is an msd-4 graph if and only if \( T \in \mathcal{B} \), that is, if and only if \( T \) can be constructed as described, using only spiders \( S(2, \ldots, 2) \).
4. General Results

In this section we discuss ways of constructing larger msd-4 graphs from smaller ones. We first prove a useful lemma.

**Lemma 6.** Let $G$ be a graph with $msd_{pr}(G) = 4$. For any edge $uv$ of $G$, subdivide $uv$ by replacing it with the path $(u, x_1, x_2, x_3, v)$. If $D$ is any $\gamma_{pr}(G_{uv,3})$-set, then $D \cap \{u, x_1, x_2, x_3, v\} =$

(i) $\{x_1, x_2\}$ or $\{x_2, x_3\}$, or

(ii) $\{u, x_1, v\}$ or $\{u, x_3, v\}$.

If the first part of (i) holds, then $u$ is $\gamma_{pr}$-critical, and if the second part of (i) holds, then $v$ is $\gamma_{pr}$-critical.

**Proof.** Let $X = \{x_1, x_2, x_3\}$. To dominate $x_2$, $X \cap D \neq \emptyset$. We consider three cases.

**Case 1.** $X \cap D = X$. Without loss of generality assume that $x_1$ is paired with $u \in D$, and $x_2$ and $x_3$ are paired. Then $v \notin D$, otherwise $D\setminus\{x_2, x_3\}$ is also a paired dominating set of $G_{uv,3}$, contradicting the minimality of $D$. But now $D' = (D\setminus X) \cup \{v\}$ is a paired dominating set of $G$, which is impossible because $msd_{pr}(G) = 4$.

**Case 2.** $|X \cap D| = 2$. If $X \cap D = \{x_1, x_3\}$, then $\{u, v\} \subseteq D$ with $u$ paired with $x_1$, and $v$ with $x_3$. However, then $D\setminus\{x_1, x_3\}$ is a paired dominating set of $G$, contradicting $msd_{pr}(G) = 4$. Suppose $X \cap D = \{x_1, x_2\}$. Then $x_1$ and $x_2$ are paired in $D$. If $\{u, v\} \cap D \neq \emptyset$, then $D\setminus\{x_1, x_2\}$ is a paired dominating set of $G$, which is a contradiction. Hence $D\cap\{u, x_1, x_2, x_3, v\} = \{x_1, x_2\}$. Now $D\setminus\{x_1, x_2\}$ is a paired dominating set of $G - u$, so $\gamma_{pr}(G - u) < \gamma_{pr}(G_{uv,3}) = \gamma_{pr}(G)$. We conclude that $u$ is $\gamma_{pr}$-critical. Arguing similarly if $X \cap D = \{x_2, x_3\}$, we conclude that (i) and the last part of the statement of the lemma hold.

**Case 3.** $|X \cap D| = 1$. Then $x_2 \notin D$. If $x_1 \in D$, then $x_1$ is paired with $u \in D$, while $v \in D$ to dominate $x_3$. Consequently, $D \cap \{u, x_1, x_2, x_3, v\} = \{u, x_1, v\}$. Similarly, if $x_3 \in D$, then $D \cap \{u, x_1, x_2, x_3, v\} = \{u, x_3, v\}$. $\blacksquare$

Our first result regarding the construction of msd-4 graphs from smaller graphs shows that subdividing any edge of an msd-4 graph four times produces another msd-4 graph. Repeatedly subdividing edges of an msd-4 graph thus yields, for example, msd-4 graphs of arbitrary large girth. In fact, we prove a stronger result: subdividing any edge of any graph $G$ without isolated vertices four times produces a graph that has the same multisubdivision number as $G$.

**Proposition 7.** For any graph $G$ and any edge $e$ of $G$, $msd_{pr}(G_{e,4}) = msd_{pr}(G)$.
Proof. Say msd\(_{\text{pr}}(G) = t \leq 4\) and \(e = uv\) has been subdivided by replacing it with the path \((u, x_1, x_2, x_3, x_4, v)\). Then \(\gamma_{\text{pr}}(G_{e,4}) = \gamma_{\text{pr}}(G) + 2\) and there exists an edge \(e'\) of \(G\) such that \(\gamma_{\text{pr}}(G_{e',4}) = \gamma_{\text{pr}}(G) + 2\). If \(e \neq e'\), then subdividing \(e \in E(G_{e',4})\) four times yields the graph \((G_{e',4})_{e,4}\). Since msd\(_{\text{pr}}(G_{e',4}) \leq 4\), \(\gamma_{\text{pr}}((G_{e',4})_{e,4}) = \gamma_{\text{pr}}(G_{e',4}) + 2 = \gamma_{\text{pr}}(G) + 4\). But \((G_{e',4})_{e,4} = (G_{e,4})_{e',t}\), hence \(\gamma_{\text{pr}}((G_{e,4})_{e',t}) = \gamma_{\text{pr}}(G) + 4 = \gamma_{\text{pr}}(G_{e,4}) + 2\). If \(e = e'\), say \(uv\) has been subdivided, in \(G\), by replacing it with \((u, x_1, \ldots, x_t, v)\). Subdividing (without loss of generality) the edge \(x_tz\) four times by replacing it with \((x_t, x_{t+1}, \ldots, x_{t+4}, v)\), we obtain the graph \((G_{e,t})_{x_tz,t} = (G_{e,4})_{x_tz,t}\) with \(\gamma_{\text{pr}}((G_{e,4})_{x_tz,t}) = \gamma_{\text{pr}}(G_{e,4}) + 2\). It follows that msd\(_{\text{pr}}(G_{e,4}) \leq t\).

We show that msd\(_{\text{pr}}(G_{e,4}) \geq t\). If \(t = 1\), this is obvious, hence assume \(t \geq 2\). Consider any \(e' \in E(G)\). Suppose first that \(e' \neq e\). Since msd\(_{\text{pr}}(G) = t\), \(\gamma_{\text{pr}}(G_{e',t-1}) = \gamma_{\text{pr}}(G)\). If \(D'\) is any \(\gamma_{\text{pr}}(G_{e',t-1})\)-set, then \(D = D' \cup \{x_1, x_2\}\) (if \(u\) and \(v\) are paired in \(D'\)) or \(D = D' \cup \{x_2, x_3\}\) (otherwise) is a paired dominating set of \((G_{e,4})_{e',t-1}\) of cardinality \(|D| = \gamma_{\text{pr}}(G_{e',t-1}) + 2 = \gamma_{\text{pr}}(G) + 2 = \gamma_{\text{pr}}(G_{e,4})\).

Assume \(e' = e\). Without loss of generality subdivide the edge \(x_4z\) of \(G_{e,4}\) \(t-1\) times by replacing it with the path \((x_4, \ldots, x_{4t+1}, v)\) and denote the resulting graph \((G_{e,4})_{x_4z,t-1}\) by \(G_{e,3+t}\) for simplicity. Also consider the graph \(G_{e,t-1}\) obtained from \(G\) by subdividing \(e = uv\) by replacing it with \((u, x_1, \ldots, x_{t-1}, v)\). Since msd\(_{\text{pr}}(G) = t\), \(\gamma_{\text{pr}}(G_{e,t-1}) = \gamma_{\text{pr}}(G)\). Let \(S'\) be any \(\gamma_{\text{pr}}(G_{e,t-1})\)-set. We consider three cases. In each case we construct a paired dominating set \(S\) of \(G_{e,3+t}\) such that \(|S| = |S'| + 2 = \gamma_{\text{pr}}(G_{e,4})\); this shows that msd\(_{\text{pr}}(G_{e,4}) \geq t\).

Case 1. \(t = 2\). If \(x_1 \not\in S'\), then without loss of generality \(u \in S'\) to dominate \(x_1\), and \(S'\{u\}\) dominates \(v\). Let \(S = S' \cup \{x_3, x_4\}\). If \(x_1 \in S'\), then again without loss of generality \(x_1\) is paired with \(u\). Let \(S = S' \cup \{x_4, x_5\}\).

Case 2. \(t = 3\). If \(S' \cap \{x_1, x_2\} = \emptyset\), then \(u\) dominates \(x_1\) while \(v\) dominates \(x_2\); let \(S = S' \cup \{x_3, x_4\}\) (so \(v\) dominates \(x_6\)). If (without loss of generality) \(S' \cap \{x_1, x_2\} = \{x_1\}\), then \(u\) and \(x_1\) are paired, and \(S' \setminus \{u, x_1\}\) dominates \(v\). Let \(S = S' \cup \{x_4, x_5\}\). If \(\{x_1, x_2\} \subseteq S'\), then \(x_1\) and \(x_2\) are paired (otherwise \(S' \cap \{x_1, x_2\}\) is a paired dominating set of \(G\), which is not the case). Let \(S = S' \cup \{x_5, x_6\}\).

Case 3. \(t = 4\). By Lemma 6, without loss of generality \(S' \cap \{u, x_1, x_2, x_3, v\} = \{x_1, x_2\}\) or \(\{u, x_1, v\}\). In the former case, let \(S = S' \cup \{x_5, x_6\}\), and in the latter case, let \(S = S' \cup \{x_4, x_5\}\).

In all cases, \(S\) is a paired dominating set of \(G_{e,3+t}\) of cardinality \(\gamma_{\text{pr}}(G) + 2 = \gamma_{\text{pr}}(G_{e,4})\), and msd\(_{\text{pr}}(G_{e,4}) \geq t\). It follows that msd\(_{\text{pr}}(G_{e,4}) = t\), as required.

We next prove results pertaining to the \(\oplus\)-operations defined above that hold for general msd-4 graphs, not only block graphs. We show that the \(\oplus\)-operations can be used to construct new connected msd-4 graphs from smaller ones.
Our next result shows that performing the operation $G_1 \oplus_{u_1u_2} G_2$ on msd-4 graphs $G_1$ and $G_2$ with $\gamma_{pr}(G_i)$-critical vertices $u_1$ and $u_2$, respectively, results in an msd-4 graph in which each $\gamma_{pr}(G_i)$-critical vertex is $\gamma_{pr}(G)$-critical.

**Proposition 8.** Let $G_1$ and $G_2$ be disjoint msd-4 graphs with $\gamma_{pr}(G_i)$-critical vertices $u_i$, $i = 1, 2$. Then for the graph $G = G_1 \oplus_{u_1u_2} G_2$, $\gamma_{pr}(G) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2$, any $\gamma_{pr}(G_i)$-critical vertex (including $u$) is $\gamma_{pr}(G)$-critical and

$$\text{msd}_{pr}(G) = 4.$$ 

**Proof.** Since $u_i \in V(G_i)$ is $\gamma_{pr}(G_i)$-critical, $\gamma_{pr}(G_1 - u_1) + \gamma_{pr}(G_2 - u_2) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 4$, and at most two more vertices are needed to pairwise dominate $G$. Therefore $\gamma_{pr}(G) \leq \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2$.

Suppose there exists a paired dominating set $S$ of $G$ such that $|S| < \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2$ and let $S = S \cap V(G_i)$. First suppose that $u \notin S$. Assume without loss of generality that $S_1$ dominates $u$. Then $S_1$ is a paired dominating set of $G_1$ and $S_2$ is a paired dominating set of $G_2 - u_2$. Hence $|S_1| \geq \gamma_{pr}(G_1)$ and $|S_2| \geq \gamma_{pr}(G_2) - 2$. But then $|S| = |S_1| + |S_2| \geq \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2$, which is not the case. Therefore we may assume that $u \in S$ (in this case $u_i \in S_i$, $i = 1, 2$) and $|S_1| + |S_2| = |S| + 1$. Without loss of generality, $u$ is paired with $v \in V(G_1)$, hence $S_1$ is a paired dominating set of $G_1$. Therefore $|S_1| \geq \gamma_{pr}(G_1)$ so that $|S_2| \leq \gamma_{pr}(G_2) - 3$. If $N_{G_2}(u_2) \subseteq S_2$, then $S_2 \setminus \{u_2\}$ is a paired dominating set of $G_2$, and if there exists $w \in N_{G_2}(u_2) \setminus S_2$, then $S_2 \cup \{w\}$ is a paired dominating set of $G_2$. This is impossible because $|S_2 \cup \{w\}| \leq \gamma_{pr}(G_2) - 2$. Hence

$$\gamma_{pr}(G) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2.$$ 

If $w_i$ is $\gamma_{pr}(G_i)$-critical, then, for $j \neq i$, the union of any $\gamma_{pr}(G_i - w_i)$-set and any $\gamma_{pr}(G_j - u_j)$-set is a paired dominating set of $G - w_i$ (this holds for $w_i = u_i = u$ also), so

$$\gamma_{pr}(G - w_i) \leq \gamma_{pr}(G_i - w_i) + \gamma_{pr}(G_j - u_j) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 4 < \gamma_{pr}(G).$$ 

Therefore $u_i$ is $\gamma_{pr}(G)$-critical.

Without loss of generality consider $e \in E(G_1)$ and subdivide $e$ three times. Then, since $\text{msd}_{pr}(G_1) = 4$ and $u_2$ is $\gamma_{pr}(G_2)$-critical, we obtain

$$\gamma_{pr}(G_{e,3}) \leq \gamma_{pr}(G_{1e,3}) + \gamma_{pr}(G_2 - u_2) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2 = \gamma_{pr}(G).$$ 

Therefore $\text{msd}_{pr}(G) = 4$. 

We show next that performing the operation $G_1 \oplus_{e_1e_2} G_2$ on msd-4 graphs $G_i$, $i = 1, 2$, with edges $e_i = x_iy_i$, where $x_i$ is a $\gamma_{pr}(G_i)$-critical vertex, results in an msd-4 graph in which each $\gamma_{pr}(G_i)$-critical vertex is $\gamma_{pr}(G)$-critical.
Proposition 9. Let $G_i, i = 1, 2$, be disjoint msd-4 graphs with $e_i = x_i y_i \in E(G_i)$, where $x_i \in \text{Cr}(G_i)$. Then for the graph $G = G_1 \oplus e_1 e_2 G_2$, $\gamma_{pr}(G) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2$, any $\gamma_{pr}(G_i)$-critical vertex (including $x = x_1 = x_2$) is $\gamma_{pr}(G)$-critical and $\text{msd}_{pr}(G) = 4$.

Proof. By Theorem 1, there exists a $\gamma_{pr}(G_i)$-set in which $x_i$ and $y_i$ are matched. Therefore

$$\gamma_{pr}(G) \leq \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2.$$ 

On the other hand, it suffices to add two vertices to a $\gamma_{pr}(G)$-set when splitting it into paired dominating sets of $G_1$ and $G_2$. Hence we have equality in (1). As in the proof of Proposition 8, any $\gamma_{pr}(G_i)$-critical vertex is $\gamma_{pr}(G)$-critical.

Let $e \in E(G)$ be any edge. If $e \in E(G_1) \backslash \{e_1\}$, then

$$\gamma_{pr}(G_{e,3}) \leq \gamma_{pr}(G_{e,3}) + \gamma_{pr}(G_2 - x_2) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2 = \gamma_{pr}(G).$$

The case when $e \in E(G_2) \backslash \{e_2\}$ is analogous. Thus assume $e = xy$ and subdivide $e$ by replacing it with the path $(x, u, v, w, y)$. Let $S$ be any $\gamma_{pr}(G - x)$-set. As shown above, $|S| = \gamma_{pr}(G) - 2$. Now $S \cup \{u, v\}$ is a paired dominating set of $G_{e,3}$ of cardinality $\gamma_{pr}(G)$. It follows that $G$ is an msd-4 graph.

We now describe a type of “reverse” operation, called a split operation, for each of the $\oplus$-operations.

$G \ominus u$. Let $G$ be a connected graph with a cut-vertex $u$. Denote the components of $G - u$ by $F_1, F_2, \ldots, F_k$. For each $i$, let $G_i$ be the graph obtained from $F_i$ by adding a new vertex $u_i$ joining $u_i$ to $v_i \in V(F_i)$ if and only if $uv_i \in E(G)$. Denote the disjoint union $G_1 + \cdots + G_k$ by $G \ominus u$.

$G \ominus xy$. Let $G$ be a connected graph containing a vertex-cut $\{x, y\}$, where $xy \in E(G)$. Denote the components of $G - \{x, y\}$ by $F_1, F_2, \ldots, F_k$. For each $i$, let $G_i$ be the graph obtained from $F_i$ by adding the edge $x_i y_i$, joining $x_i$ ($y_i$, respectively) to $v_i \in V(F_i)$ if and only if $xv_i \in E(G)$ ($yv_i \in E(G)$, respectively). Denote the disjoint union $G_1 + \cdots + G_k$ by $G \ominus xy$.

The next proposition shows that if an msd-4 graph $G$ is split at a $\gamma_{pr}$-critical cut-vertex $u$, the components of $G \ominus u$ are msd-4 graphs having the copies of $u$ as $\gamma_{pr}$-critical vertices.

Proposition 10. Let $G$ be an msd-4 graph with a $\gamma_{pr}$-critical cut-vertex $u$. Denote the components of $G \ominus u$ by $G_1, \ldots, G_k$. Then for each $i = 1, \ldots, k$, $u_i$ is a $\gamma_{pr}(G_i)$-critical vertex and $\text{msd}_{pr}(G_i) = 4$. 

Proof. Since $u$ is $\gamma_{pr}(G)$-critical and $G - u$ is the disjoint union of $G_i - u_i$, $i = 1, \ldots, k$,

$$\gamma_{pr}(G) - 2 = \gamma_{pr}(G - u) = \sum_{i=1}^{k} \gamma_{pr}(G_i - u_i).$$

Suppose $\gamma_{pr}(G_1 - u_1) \geq \gamma_{pr}(G_1)$. Let $R_1$ be a $\gamma_{pr}(G_1)$-set and, for $i \geq 2$, let $R_i$ be a $\gamma_{pr}(G_i - u_i)$-set. Since $R_1$ dominates $u_1$, $R = \bigcup_{i=1}^{k} R_i$ is a paired dominating set of $G$. But then

$$\gamma_{pr}(G) \leq |R| \leq \sum_{i=1}^{k} \gamma_{pr}(G_i - u_i) \leq \sum_{i=1}^{k} \gamma_{pr}(G_i - u_i) = \gamma_{pr}(G) - 2,$$

which is impossible. Thus $u_1$ is $\gamma_{pr}(G_1)$-critical. The same argument works for each $i \in \{2, \ldots, k\}$.

Consider an arbitrary edge $e \in E(G_1)$ and subdivide $e$ three times. Then

$$\gamma_{pr}(G_{e,3}) \leq \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i).$$

We show that equality holds in (2). Let $S$ be any $\gamma_{pr}(G_{e,3})$-set and define $S_1 = S \cap V(G_{1,e,3})$ and $S_i = S \cap V(G_i)$ for $i = 2, \ldots, k$ (if $u \in S$, then $u_i \in S_i$ for each $i$). First suppose that $u \notin S$. If $S_1$ dominates $u$, then $S_1$ is a paired dominating set of $G_{1,e,3}$ and $S_i$, $i \geq 2$, is a paired dominating set of $G_i - u_i$. Hence $|S_1| \geq \gamma_{pr}(G_{1,e,3})$ and $|S_i| \geq \gamma_{pr}(G_i - u_i)$, so that $\gamma_{pr}(G_{e,3}) = |S| = \sum_{i=1}^{k} |S_i| \geq \gamma_{pr}(G_{1,e,3}) + \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i)$ as required. On the other hand, if $S_1$ does not dominate $u$, then $S_1$ is a paired dominating set of $G_j$ for some $j \geq 2$, so that $|S_j| \geq \gamma_{pr}(G_j) = \gamma_{pr}(G_j - u_j) + 2$ (since $u_j$ is $\gamma_{pr}(G_j)$-critical). Let $S_j'$ be a $\gamma_{pr}(G_j - u_j)$-set, $S_j' = S_1 \cup \{u, u'\}$ for some $u' \in N_{G_j}(u)$, and $S' = (S \setminus S_j) \cup S_j' \cup S'$. Then $|S'| = |S|$, $S'$ is a paired dominating set of $G_{1,e,3}$ and the result follows as before.

Now suppose that $u \in S$. Then $|S_1| + \sum_{i=2}^{k} |S_i| = |S| + k - 1$ and $u$ is paired with a vertex in exactly one of the graphs $G_{1,e,3}$ or $G_i$, $i \geq 2$. For each of the $k - 1$ other graphs, either $u \notin S_i \setminus \{u_i\}$, for some neighbour $u_i \notin S_i$ of $u_i$, or $S_i \setminus \{u_i\}$ (if all neighbours of $u_i$ in $G_i$ belong to $S_i$) is a paired dominating set. Hence

$$\gamma_{pr}(G_{1,e,3}) + \sum_{i=2}^{k} \gamma_{pr}(G_i) \leq |S| + 2(k - 1).$$

Since $u_i$ is $\gamma_{pr}(G_i)$-critical for each $i = 2, 3, \ldots, k$, $\gamma_{pr}(G_i - u_i) = \gamma_{pr}(G_i) - 2$. This gives

$$\gamma_{pr}(G_{1,e,3}) + \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i) \leq |S| = \gamma_{pr}(G_{e,3}).$$

Therefore we have equality (2). Now
\[
\gamma_{pr}(G_{1e,3}) = \gamma_{pr}(G_{e,3}) - \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i) = \gamma_{pr}(G) - \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i)
\]
\[
= \gamma_{pr}(G_1) + \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i) - \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i) = \gamma_{pr}(G_1).
\]

Hence, for any edge \( e \in E(G_1) \), \( \gamma_{pr}(G_{1e,3}) = \gamma_{pr}(G) \). Thus \( msd_{pr}(G_1) = 4 \). Similar reasoning may be applied to \( G_i \) for \( i \in \{2, 3, \ldots, k\} \).

5. MSD-4 Block Graphs

The last three results we need for the proof of Theorem 4 concern block graphs. In the first result we prove that every non-leaf vertex of an msd-4 block graph is a cut-vertex.

**Theorem 11.** Let \( G \) be a graph containing a block \( B \cong K_n \), where \( n \geq 3 \), such that some vertex of \( B \) is not adjacent to any vertex of \( G - B \). Then

\[
msd_{pr}(G) < 4.
\]

**Proof.** Suppose the hypothesis of the theorem holds but \( msd_{pr}(G) = 4 \). Let \( V(B) = \{v_0, \ldots, v_{n-1}\} \) and say \( u = v_0 \) is not adjacent to any vertex of \( G - B \). Subdivide the edge \( uv_2 \) by replacing it with the path \((u, x_3, x_2, x_1, v_2)\) (see Figure 3). Denote \( X = \{x_1, x_2, x_3\} \) and let \( D \) be a \( \gamma_{pr} \)-set of \( G_{uv_2,3} \). By Lemma 6 we only have to consider the cases \( D \cap \{u, x_1, x_2, x_3, v_2\} \in \{\{u, x_1, v_2\}, \{u, x_3, v_2\}, \{x_1, x_2\}, \{x_2, x_3\}\} \).

![Figure 3. The block B with the edge uv2 subdivided with vertices x1, x2, x3.](image-url)
Case 1. \(|X \cap D| = 1\). If \(D \cap \{u, x_1, x_2, x_3, v_2\} = \{u, x_1, v_2\}\), then \(x_1\) and \(v_2\) are paired in \(D\), while \(u\) is paired with \(v_i\) for some \(i \neq 0, 2\). However, then \(D \setminus \{x_1, u\}\), with \(v_2\) and \(v_i\) paired, is a smaller paired dominating set of \(G\). If \(D \cap \{u, x_1, x_2, x_3, v_2\} = \{u, x_3, v_2\}\), then \(D \setminus \{x_3, u\}\) is a smaller paired dominating set of \(G\). In either case \(\text{msd}_{pr}(G) < 4\), contrary to our assumption.

Case 2. \(|X \cap D| = 2\). If \(D \cap \{u, x_1, x_2, x_3, v_2\} = \{x_1, x_2\}\), then \(x_1\) and \(x_2\) are paired in \(D\). To pairwise dominate \(u\), \(v_i \in D\) for some \(i \neq 0, 2\). But then \(D \setminus \{x_1, x_2\}\) is a paired dominating set of \(G\) (with \(v_i\) paired as in \(D\)) and \(\text{msd}_{pr}(G) < 4\), contrary to our assumption. Hence assume \(D \cap \{u, x_1, x_2, x_3, v_2\} = \{x_2, x_3\}\). Then \(x_2\) and \(x_3\) are matched in \(D\). If \(v_i \in D\) for some \(i\), then \(D \setminus \{x_2, x_3\}\) is a paired dominating set of \(G\) (again with \(v_i\) paired as in \(D\)), a contradiction.

We therefore assume henceforth that

(i) \(D\) contains \(x_2\) and \(x_3\), but neither \(x_1\) nor any \(v_0, \ldots, v_{n-1}\).

By Lemma 6, \(u\) is \(\gamma_{pr}\)-critical, that is,

(ii) \(\gamma_{pr}(G - u) = \gamma_{pr}(G) - 2\).

For each \(i = 1, \ldots, n-1\), let \(G_i\) be the component of \(G - E(B)\) that contains \(v_i\). Since \(B\) is a block of \(G\), the subgraphs \(G_i\) are distinct and pairwise vertex-disjoint. Let \(D_i = D \cap V(G_i)\). Then \(|\bigcup_{i=1}^{n-1} D_i| = |D \setminus \{x_2, x_3\}| = \gamma_{pr}(G) - 2\). By (i), each \(D_i\) is a \(\gamma_{pr}(G_i)\)-set that does not contain \(v_i\).

We next show that

(iii) no \(\gamma_{pr}(G)\)-set contains \(u = v_0\) and at least two \(v_i, i \geq 1\).

Suppose there exists such a set \(Z\): assume without loss of generality that \(\{u, v_1, v_2, \ldots, v_k\} \subseteq Z\), \(k \geq 2\). Necessarily, \(u\) is paired with some \(v_i\), \(i = 1, \ldots, k\), in \(Z\). Assume (again without loss of generality) \(u\) is paired with \(v_1\). Let \(Z_1 = Z \cap V(G_1) \setminus \{v_1\}\) and, for \(i \geq 2\), let \(Z_i = Z \cap V(G_i)\). Then \(\bigcup_{i=1}^{n-1} Z_i \subseteq V(G - u)\) and \(|\bigcup_{i=1}^{n-1} Z_i| = |Z| - 2 = \gamma_{pr}(G - u) < \gamma_{pr}(G)\), by (ii). Since \(v_1\) and \(u\) are paired, \(G_1[Z_1]\) contains a perfect matching, as does \(G[\bigcup_{i=2}^{n-1} Z_i]\). Since \(v_1\) is not adjacent to any vertex of \(G - v_1\), \(i \geq 2\), and \(v_2\) dominates \(B\) in \(G\), \(\bigcup_{i=2}^{n-1} Z_i\) is a paired dominating set of \(G - G_1\).

Suppose \(|Z_1| < |D_1|\). Since both \(Z_1\) and \(D_1\) have even cardinality, \(|Z_1| \leq |D_1| - 2\). Then \(Z_1\) does not dominate \(G_1 - v_1\), otherwise \(\bigcup_{i=1}^{n-1} Z_i\) is a paired dominating set of \(G\) of cardinality less than \(\gamma_{pr}(G)\), which is impossible. Since \(Z_1 \cup \{v_1\}\) dominates \(G_1\), there exists a vertex \(w \in N_{G_1}(v_1)\) that is undominated by \(Z_1\). Then \(W_1 = Z_1 \cup \{w, v_1\}\) is a paired dominating set of \(G_1\) of cardinality at most \(|D_1|\) that contains \(v_1\). But now \(W_1 \cup D_2 \cup D_3 \cup \cdots \cup D_{n-1}\) is a paired dominating set of \(G\) of cardinality at most \(|D \setminus \{x_2, x_3\}| = \gamma_{pr}(G) - 2\), which is impossible. We conclude that \(|Z_1| = |D_1|\).

Let \(Z' = D_1 \cup \left(\bigcup_{i=2}^{n-1} Z_i\right)\). Since \(\bigcup_{i=2}^{n-1} Z_i\) is a paired dominating set of \(G - G_1\) and \(D_1\) is a paired dominating set of \(G_1\), \(Z'\) is a paired dominating set of \(G\).
Moreover,
\[ |Z'| = \bigcup_{i=2}^{n-1} Z_i + |D_1| = \bigcup_{i=1}^{n-1} Z_i = |Z| - 2 = \gamma_{pr}(G - u) < \gamma_{pr}(G), \]

which is impossible. This concludes the proof of (iii).

Subdivide the edge \( v_1v_2 \) with vertices \( y_1, y_2, y_3 \), where \( y_1 \) is adjacent to \( v_1 \) and \( y_3 \) is adjacent to \( v_2 \) (see Figure 4). Denote \( Y = \{y_1, y_2, y_3\} \) and let \( Q \) be a \( \gamma_{pr} \)-set of \( G_{v_1v_2,3} \). Without loss of generality, by Lemma 6 we only have to consider the cases \( Q \cap \{v_1, v_2, y_1, y_2, y_3\} \in \{\{y_1, y_2\}, \{v_1, v_2, y_1\}\} \).

Figure 4. The block \( B \) with the edge \( v_1v_2 \) subdivided with vertices \( y_1, y_2, y_3 \).

**Case 3a.** \( Q \cap \{v_1, v_2, y_1, y_2, y_3\} = \{y_1, y_2\} \). Then these two vertices are paired in \( Q \). To pairwise dominate \( u, v_i \in Q \) for some \( i \). It follows that \( Q \setminus \{y_1, y_2\} \) is a paired dominating set of \( G \), so \( msd_{pr}(G) < 4 \), contrary to our assumption.

**Case 3b.** \( Q \cap \{v_1, v_2, y_1, y_2, y_3\} = \{v_1, v_2, y_1\} \). Then \( y_1 \) is paired with \( v_1 \). If \( u \notin Q \), then \( Q' = (Q \setminus \{y_1\}) \cup \{u\} \) is a paired dominating set of \( G \) containing \( u, v_1, v_2 \). By (iii), \( Q' \) is not a \( \gamma_{pr} \)-set of \( G \), from which it follows that \( \gamma_{pr}(G) < |Q| \) and \( msd_{pr}(G) < 4 \). Assume therefore that \( u \in Q \). Then \( u \) is paired in \( Q \) with \( v_i \) for some \( i > 1 \). Now \( Q'' = Q \setminus \{y_1, u\} \) is a paired dominating set of \( G \) in which \( v_1 \) and \( v_i \) are paired. In both cases we again have a contradiction and the proof is complete.

The graph in Figure 5 shows that the statement of Theorem 11 is false if the complete subgraph \( B \) is not a block of \( G \).

The next result in this section shows that \( msd-4 \) block graphs have many \( \gamma_{pr} \)-critical vertices.
Figure 5. A graph $G$ with $\text{msd}_{pr}(G) = 4$ and a subgraph $K_3$ that is not a block of $G$.

**Theorem 12.** If $G$ is a block graph with $\text{msd}_{pr}(G) = 4$, then for any edge $uv \in E(G)$,

$$(N_G[u] \cup N_G[v]) \cap \text{Cr}(G) \neq \emptyset.$$  

**Proof.** Suppose there exists an edge $uv \in E(G)$ such that $(N_G[u] \cup N_G[v]) \cap \text{Cr}(G) = \emptyset$. By Theorem 1, no vertex in $N_G[u] \cup N_G[v]$ is a leaf. We subdivide the edge $uv$ by replacing it with the path $(u, x_1, x_2, x_3, v)$ to obtain the graph $G_{uv,3}$. By Lemma 6, for any $\gamma_{pr}$-set $\mathcal{S}$ of $G_{uv,3}$, $\mathcal{S} \cap \{u, v, x_1, x_2, x_3\} \in \{\{u, v, x_1\}, \{u, v, x_3\}\}$. Without loss of generality assume there exists such a set $\mathcal{S}$ such that $\mathcal{S} \cap \{u, v, x_1, x_2, x_3\} = \{u, v, x_1\}$, and among all such sets $\mathcal{S}$, let $D$ be one for which $\text{PN}(u, D)$ is as small as possible. Then $x_1$ and $u$ are paired in $D$.

Say $v$ is paired with $v'$ and let $B$ be the block of $G$ that contains $uv$. If $v' \in V(G) \setminus V(B)$, let $G_v$ be the subgraph of $G - E(B)$ that contains $v$, and if $v' \in V(B)$, let $G_v$ be the subgraph of $G - (E(B) - \{vv'\})$ that contains $v$. In either case, $v' \in V(G_v)$. Let $D_v = D \cap V(G_v)$ and $D' = D \setminus \{x_1, u\}$. Then $G[D']$ has a perfect matching and $D_v$ is a paired dominating set of $G_v$ containing $v$ and $v'$. In fact, $D_v$ is a $\gamma_{pr}(G_v)$-set, for if not, let $D''$ be a smaller paired dominating set of $G_v$.

Consider $N_G(u) \setminus V(B)$. If $B \cong K_2$, then $N_G(u) \setminus V(B) = N_G(u) \setminus \{v\}$ is nonempty because $u$ is not a leaf, and if $B \cong K_n$ for $n \geq 3$, then $N_G(u) \setminus V(B)$ is nonempty by Theorem 11. If $N_G(u) \setminus V(B) \subseteq D$, then $D'$ is a paired dominating set of $G$, and if there exists $w \in N_G(u) \setminus V(B) \setminus D$, then $(D \setminus \{x_1\} \setminus D_v) \cup D'' \cup \{w\}$ is a smaller paired dominating set of $G$ than $D$. In both cases we have a contradiction to $\text{msd}_{pr}(G) = 4$.

Since $\text{msd}_{pr}(G) = 4$, $|D'| = \gamma_{pr}(G_{uv,3}) - 2 = \gamma_{pr}(G) - 2$. Consequently, $D'$ does not dominate $G$. Since $v \in D'$ dominates $B$ in $G$, there exist vertices $w_1, \ldots, w_k \in N_G(u) \setminus N_G[v] \subseteq N_G(u) \setminus B$ that are undominated by $D'$, that is,
\{w_1, \ldots, w_k\} = \text{PN}(u, D). For i = 1, \ldots, k, let G_i be the component of G - u that contains w_i. Possibly, G_i = G_j for i \neq j; this happens exactly when w_i w_j \in E(G), and then w_i and w_j also belong to the same (complete) block of G_i. Since no w_i is adjacent to v or \nu', V(G_i) \cap V(G_v) = \emptyset for each i. Define D_i = D \cap V(G_i). Then \text{G}_i[D_i] has a perfect matching, but does not dominate w_i. If it is nevertheless true that \gamma_{pr}(G_i) = |D_i| for some i, let Q_i be a \gamma_{pr}(G_i) set. Then \text{D}^* = (D \setminus D_i) \cup Q_i is a \gamma_{pr}(G_{uw, 3})-set such that \text{PN}(u, D^*) \subseteq \text{PN}(u, D) \setminus \{w_i\}, contrary to the choice of D. Therefore \gamma_{pr}(G_i) \geq |D_i| + 2 for each i.

Since each stem belongs to all paired dominating sets, no w_i is a stem, and by our initial assumption, no w_i is a leaf. Subdivide the edge \text{uw}_1 by replacing it with the path \{u, y_1, y_2, y_3, w_1\}. Consider a \gamma_{pr}(G_{uw, 3})-set S. Since u, w_1 \notin \text{Cr}(G), Lemma 6 states that S \cap \{u, y_1, y_2, y_3, w_1\} \subseteq \{u, w_1, y_1, y_3, w_1\}.

- In the former case, y_1 is paired with u and S_1 = S \cap V(G_1) is a paired dominating set of G_1; hence \big|S_1\big| \geq \gamma_{pr}(G_1) \geq |D_1| + 2. Since w_3 is adjacent to all w_i \in V(G_1), D_i \cup \{w_i\} dominates G_1 (but not pairwise). Now S' = (S \setminus S_1) \cup D_1 \cup \{w_1, y_3\} is a paired dominating set of G_{uw, 3} such that \big|S'\big| \leq \big|S\big|, hence S' is a \gamma_{pr}(G_{uw, 3})-set.

Moreover, S' \cap \{u, y_1, y_2, y_3, w_1\} = \{u, y_1, y_3, w_1\}, contrary to Lemma 6.

- In the latter case, y_3 is paired with w_1. Then S_2 = (S \cap V(G_1)) \cup \{y_3\} is a paired dominating set of the graph obtained from G_1 by joining y_3 to w_1. If all neighbours of w_1 in G_1 belong to S_2, then S_2 \setminus \{y_3\} is a paired dominating set of G_1. But then \big|S''\big| = \big|S \setminus \{y_3\}\big| is a paired dominating set of G such that \big|S''\big| < \big|S\big|, contradicting \text{msd}_{pr}(G) = 4. Assume some neighbour z of w_1 in G_1 does not belong to S_2. Then S_3 = (S_2 \setminus \{y_3\}) \cup \{z\} is a paired dominating set of G_1, so that \big|S_2\big| = \big|S_3\big| \geq |D_1| + 2. Since u \in S and \{w_1, \ldots, w_k\} \subseteq N(u), S' = (S \setminus S_2) \cup D_1 is a paired dominating set of G such that \big|S'\big| < \big|S\big|, again a contradiction.

This completes the proof of the theorem.

Although the graph G in Figure 5 satisfies \text{msd}_{pr}(G) = 4 without being a block graph, Theorem 12 holds for G as well.

Our final result in this section concerns the reverse operation \text{G} \ominus xy for certain \text{msd}-4 block graphs.

**Proposition 13.** Let G be a connected \text{msd}-4 block graph such that the only \gamma_{pr}(G)-critical vertices are leaves. Let x be a leaf adjacent to the stem y, where \{x, y\} is a vertex-cut, and denote the components of \text{G} \ominus xy by G_1, \ldots, G_k. Then for each i = 1, \ldots, k, G_i is an \text{msd}-4 graph and x_i \in \text{Cr}(G_i).

**Proof.** If G_i is an \text{msd}-4 graph, it will follow from Theorem 1(ii) that x_i \in \text{Cr}(G_i). However, we need the fact that x_i is \gamma_{pr}(G_i)-critical to show that \text{msd}_{pr}(G_i) = 4, hence this is what we prove first.
Since $G$ is a block graph, $N_{G_i-x_i}(y_i)$ induces a clique for each $i = 1, \ldots, k$. Since $x$ is a leaf, $y$ belongs to every paired dominating set of $G$, and by Theorem 1(ii), $x \in \text{Cr}(G)$. Hence $y$ belongs to no $\gamma_{pr}(G-x)$-set (for such a set would dominate $x$ and thus $G$, contradicting $x \in \text{Cr}(G)$).

Let $D$ be a $\gamma_{pr}(G-x)$ set such that $|D \cap N(y)|$ is maximum and let $D_i = D \cap V(G_i)$, $i = 1, \ldots, k$. Since $x \in \text{Cr}(G)$ and $y \not\in D$, $|D| = \sum_{i=1}^{k} |D_i| = \gamma_{pr}(G) - 2$.

Also, $D_i$ is a paired dominating set of $G_i - \{x_i, y_i\}$ for each $i$, and a paired dominating set of $G_i - x_i$ for at least one $i$. We show that, in fact,

(A) $D_i$ is a paired dominating set of $G_i - x_i$ for each $i$.

First suppose $|N_{G_i-x_i}(y_i)| \geq 2$; say $z_1, z_2 \in N_{G_i-x_i}(y_i)$. Since $N_{G_i-x_i}(y_i)$ induces a clique, $z_1z_2 \in E(G)$. By Theorem 12, $(N_G[z_1] \cup N_G[z_2]) \cap \text{Cr}(G) \neq \emptyset$.

Since $N_G[z_i] = N_{G_i-x_i}[z_i]$ and $z_i$ is not a leaf (and thus, by the hypothesis, not $\gamma_{pr}(G)$-critical), $z_1$ or $z_2$ is adjacent to a $\gamma_{pr}(G)$-critical vertex, i.e., a leaf. Say $z_1$ is adjacent to a leaf $z''$. Then $z_1$ belongs to any paired dominating set of any subgraph of $G$ containing both $z_1$ and $z''$, so $z_1 \in D$. Therefore $D_i$ dominates $y_i$, and (A) holds.

Assume therefore that $|N_{G_i-x_i}(y_i)| = 1$, say $N_{G_i-x_i}(y_i) = \{z\}$. If $z \in D$, we are done, hence assume $z \notin D$. By Theorem 1(iii), $z$ is not a leaf, hence there exists a vertex $z' \in N_{G_i-x_i}(z) \backslash \{y_i\}$. By Theorem 1(i), $G$ has a $\gamma_{pr}$-set $X$ such that $zz'$ belongs to a matching of $G[X]$. Now $y \in X$, but $y$ is not paired with any vertex of $G_i - x_i$, since $N_{G_i-x_i}(y_i) = \{z\}$. Therefore $X_i = (X \backslash \{x, y\}) \cap V(G_i)$ is a paired dominating set of $G_i - x_i$. Moreover, $|X_i| \leq |D_i|$, otherwise $(X - X_i) \cup D_i$ is a smaller paired dominating set of $G$, which is impossible. However, now $D' = (D \backslash D_i) \cup X_i$ is a paired dominating set of $G - x$, hence a $\gamma_{pr}(G - x)$-set, containing more neighbours of $y$ than $D$, contrary to the choice of $D$. Hence (A) holds in this case as well.

Therefore $\gamma_{pr}(G_i - x_i) \leq |D_i|$ for each $i$, so that

$$
\sum_{i=1}^{k} \gamma_{pr}(G_i - x_i) \leq \sum_{i=1}^{k} |D_i| = |D| = \gamma_{pr}(G - x).
$$

Suppose there exists a $\gamma_{pr}(G_i - x_i)$-set $Y_i$ containing $y_i$. Since no $D_j$ contains $y_j$, $D' = (D \backslash D_i) \cup Y_i$ is a paired dominating set of $G - x$ such that $|D'| \leq |D| = \gamma_{pr}(G) - 2$ and $D'$ dominates $x$. Then $D'$ is a paired dominating set of $G$, which is impossible. Therefore no $\gamma_{pr}(G_i - x_i)$-set contains $y_i$. Similarly, if $\gamma_{pr}(G_i - x_i) < |D_i|$ for some $i$ and $Z_i$ is a $\gamma_{pr}(G_i - x_i)$-set, then $D'' = (D \backslash D_i) \cup Z_i$ is a paired dominating set of $G - x$ such that $|D''| < |D|$, which is also impossible.

From these two facts we deduce that $D_i$ is a $\gamma_{pr}(G_i - x_i)$-set, equality holds in (3) and $\gamma_{pr}(G_i) = \gamma_{pr}(G_i - x_i) + 2$, that is, $x_i$ is $\gamma_{pr}(G_i)$-critical for each $i$.

We show that $\text{msd}_{pr}(G_1) = 4$: it will follow similarly that $\text{msd}_{pr}(G_i) = 4$ for each $i$. Since $D_1$ is a $\gamma_{pr}(G_1 - x_1)$-set, it is easy to see that we can pairwise
dominate \(G_{1,y,3}\) by \(|D_1| + 2 = \gamma_{pr}(G_1)\) vertices. Hence consider any edge \(e \in E(G_1 - x_1)\) and the graphs \(G_{e,3}\) and \(G_{1,e,3}\). Since combining any \(\gamma_{pr}(G_{1,e,3})\)-set with the sets \(D_j, j = 2, \ldots, k\), produces a paired dominating set of \(G_{e,3}\),

\[
(4) \quad \gamma_{pr}(G_{e,3}) \leq \gamma_{pr}(G_{1,e,3}) + \sum_{i=2}^{k} \gamma_{pr}(G_i - x_i).
\]

We show that equality holds in (4). For convenience of notation, define \(H_1 = G_{1,e,3}\) and \(H_i = G_i, i \geq 2\). Let \(S\) be a \(\gamma_{pr}(G_{e,3})\)-set and define \(S_i = S \cap V(H_i)\) for \(i = 1, \ldots, k\) (since \(y \in S, y_i \in S_i\) for each \(i\), and if \(x \in S\), then \(x_i \in S_i\) for each \(i\)). We consider two cases, depending on whether \(x \in S\) or not.

**Case 1.** \(x \notin S\). Then \(\sum_{i=1}^{k} |S_i| = |S| + k - 1\). Note that \(y\) is paired with \(w \in V(H_i)\) \(\{x_i, y_i\}\) for exactly one \(i\). Then \(S_i\) is a paired dominating set of \(H_i\). For \(j \neq i\), \(S_j \cup \{x_j\}\) is a paired dominating set of \(H_j\). Therefore \(\gamma_{pr}(H_j) \leq |S|\) and \(\gamma_{pr}(H_j) \leq |S_j| + 1\) for \(j \neq i\). For \(\ell \geq 2\), \(x_{\ell}\) is \(\gamma_{pr}(H_{\ell})\)-critical, hence \(\gamma_{pr}(H_{\ell} - x_{\ell}) \leq \gamma_{pr}(H_{\ell}) - 2\). Therefore

\[
\gamma_{pr}(G_{1,e,3}) + \sum_{i=2}^{k} \gamma_{pr}(G_i - x_i) \leq \sum_{i=1}^{k} |S_i| - 2(k-1) + (k-1) = \sum_{i=1}^{k} |S_i| - (k-1) = |S|
\]

and equality holds in (4).

**Case 2.** \(\{x, y\} \subseteq S\). Then \(x\) and \(y\) are paired in \(S\), \(\{x_i, y_i\} \subseteq S_i\) for each \(i\), and \(S_i\) is a paired dominating set of \(H_i\). Also, \(\sum_{i=2}^{k} |S_i| = |S| + 2(k-1) - |S_1|\). Since \(x_i\) is \(\gamma_{pr}(G_i)\)-critical,

\[
\gamma_{pr}(G_{1,e,3}) + \sum_{i=2}^{k} \gamma_{pr}(G_i - x_i) \leq |S_1| + \sum_{i=2}^{k} |S_i| - 2(k-1) = |S| = \gamma_{pr}(G_{e,3}),
\]

giving equality in (4).

It now follows as in the proof of Proposition 10 that \(\text{msd}(G_1) = 4\). Similarly, \(\text{msd}(G_i) = 4\) for \(i \geq 2\).

6. Proof of Theorem 4

We are now ready to prove our main theorem, the characterization of \(\text{msd}-4\) block graphs. We restate the theorem here for convenience.

**Theorem 4** (again). Let \(G\) be a connected block graph. Then \(G\) is an \(\text{msd}-4\) graph if and only if \(G \in B\). Moreover, if \(G\) is an \(\text{msd}-4\) graph constructed from the graphs \(H_1, \ldots, H_j \in U\), then \(\text{Cr}(G) = \bigcup_{i=1}^{j} \text{Cr}(H_i)\).
Proof. If \( G \in B \), it follows immediately from Propositions 8 and 9 that \( G \) is an msd-4 graph and \( \text{Cr}(G) = \bigcup_{i=1}^{j} \text{Cr}(H_i) \).

For the converse, let \( G \) be an msd-4 block graph. If \( G \) is a tree, the result follows from Corollary 5, hence we assume that \( B \cong K_n, n \geq 3 \), is a block of \( G \). By (the contrapositive of) Theorem 11, each vertex of \( B \) is a cut-vertex, so \( \text{deg}(v) \geq n \) for each \( v \in V(B) \). Since each non-leaf vertex of a \( K_2 \)-block is a cut-vertex, we deduce that each vertex of \( G \) is either a leaf or a cut-vertex.

Suppose \( v \in V(B) \) is \( \gamma_{pr} \)-critical. Applying Proposition 10 to \( v \) we obtain an msd-4 graph \( G_1 \) with \( v_1 = v \) and \( N_{G_1}[v_1] = B \), which contradicts Theorem 11. Thus every \( \gamma_{pr}(G) \)-critical vertex belongs only to \( K_2 \)-blocks.

We say that a vertex \( u \) is a type-A vertex if it is a \( \gamma_{pr}(G) \)-critical cut-vertex, and an edge \( uv \) is a type-A edge if \( u \) is a leaf (hence \( \gamma_{pr}(G) \)-critical) and \( G - \{u, v\} \) is disconnected. Denote the number of type-A elements (vertices and edges together) of \( G \) by \( a(G) \). First we show that

(B) if \( a(G) = 0 \), then \( G \in \mathcal{U} \).

Suppose \( a(G) = 0 \). Then every \( \gamma_{pr}(G) \)-critical vertex is a leaf. Say \( V(B) = \{v_1, \ldots, v_n\} \). Since no vertex of \( B \) is \( \gamma_{pr}(G) \)-critical, Theorem 12 implies that \( v_1 \) or \( v_n \) is adjacent to a \( \gamma_{pr}(G) \)-critical vertex. Without loss of generality we assume that \( v_1u_1 \in E(G), u_1 \notin V(B) \), and \( u_1 \) is \( \gamma_{pr}(G) \)-critical. Similarly, without loss of generality, \( v_i \) is adjacent to a \( \gamma_{pr}(G) \)-critical vertex \( u_i \notin V(B) \) for \( i = 2, \ldots, n - 1 \). Since \( a(G) = 0 \) and each vertex of \( G \) is either a leaf or a cut-vertex, \( \text{deg}_G(u_i) = 1 \) for each \( i = 1, \ldots, n - 1 \) and \( G - \{v_i, u_i\} \) is connected. Thus, \( v_i \) belongs to only the two blocks \( B \) and \( v_1u_1 \), so \( \text{deg}_G(v_i) = n \) for each \( i = 1, \ldots, n - 1 \).

Since \( v_n \) is a cut-vertex, \( N(v_n) \setminus V(B) \neq \emptyset \). If \( v_n \) is adjacent to a \( \gamma_{pr}(G) \)-critical vertex, say \( u_n \), then, arguing as above, \( \text{deg}(u_n) = 1 \), \( \text{deg}(v_n) = n \) and \( G = K_n \circ K_1 \). By Remark 3(i), \( n \) is odd, hence \( G \) belongs to the family \( \mathcal{U} \subseteq B \). If no vertex in \( N(v_n) \setminus V(B) \) is critical, let \( N(v_n) \setminus V(B) = \{w_1, \ldots, w_t\} \) for \( t \geq 1 \). By Theorem 12, each \( w_i \) is adjacent to a critical vertex \( w'_i \neq v_n \), and since \( a(G) = 0 \), \( w'_i \) is a leaf. We show that

(C) \( \{w_1, \ldots, w_t\} \) is an independent set of \( G \).

Suppose (without loss of generality) that \( w_1w_2 \in E(G) \) and consider \( G_{w_1w_2,3} \). Let \( w_1, x_1, x_2, x_3, w_2 \) be the \( w_1 - w_2 \) path in \( G_{w_1w_2,3} \) and let \( D \) be a \( \gamma_{pr}(G_{w_1w_2,3}) \)-set. Since \( w'_1 \) and \( w'_2 \) are leaves, \( w_1, w_2 \in D \). To dominate \( x_2, \{x_1, x_2, x_3\} \cap D \neq \emptyset \). If \( |\{x_1, x_2, x_3\} \cap D| = 2 \), then \( D \setminus \{x_1, x_2, x_3\} \) is a paired dominating set (with \( w_1 \) and \( w_2 \) paired) of \( G \) of smaller cardinality than \( D \), contrary to \( \text{msd}(G) = 4 \). Hence assume without loss of generality that \( \{x_1, x_2, x_3\} \cap D = \{x_1\} \), so \( w_1 \) and \( x_1 \) are paired (and \( w'_1 \notin D \)), while \( w_2 \) is paired with either \( w'_2 \) or \( v_n \). However, each vertex in \( N_G(v_n) \) is adjacent to a leaf and belongs to \( D \), thus \( D \setminus \{v_n\} \) dominates \( G \). Therefore, either \( D \setminus \{x_1, w'_2\} \) or \( D \setminus \{x_1, v_n\} \) is a paired dominating set of \( G \) in which \( w_1 \) and \( w_2 \) are paired, contrary to \( \text{msd}(G) = 4 \). It follows that (C) holds.
Since $G$ is a block graph, $w_i$ and $w_j$ belong to different components of $G - v_n$ for all $i \neq j$.

Consequently, if there exists a vertex $z \notin \{v_n, w'_i\}$ adjacent to $w_i$, then $z$ and $v_n$ belong to different components of $G - \{w_i, w'_i\}$. But now $w_i, w'_i$ is a type-A edge, which is not the case as $a(G) = 0$. Hence $\deg(w_i) = 2$ and $G \cong K_n \circ^{t} K_1$. Since $\text{msd}(G) = 4$, $n$ is even, by Remark 3(ii). Therefore $G \in \mathcal{U} \subseteq \mathcal{B}$. Thus (B) holds.

Now suppose $a(G) \geq 1$. If $G$ has a type-A critical cut-vertex $u$, perform the operation $G \ominus u$; each resulting graph is an msd-4 graph by Proposition 10, and clearly a block graph. Moreover, the copies of $u$ in each graph are $\gamma_{pr}$-critical. Repeat this process until no resulting msd-4 block graph has a type-A critical cut-vertex. Let $G_1, \ldots, G_k$ be the resulting graphs. Then each critical vertex of each $G_i$ is a leaf. If any $G_i$ has a type-A critical edge $uv$, where $u$ is a leaf, perform the operation $G \ominus uv$. Each resulting graph is an msd-4 block graph by Proposition 13. Repeat this process until all resulting graphs $H_j$ satisfy $a(H_j) = 0$. If $H_j$ is a tree, then $H_j \cong S(2, \ldots, 2) \in \mathcal{U}$ by Corollary 5, otherwise $H_j \in \mathcal{U}$ by (B). Now $G$ can be reconstructed by performing the $\oplus$-operations on the $H_j$, hence $G \in \mathcal{B}$, as required.

\section{Open Problems}

We conclude with a short list of open problems for future consideration.

\textbf{Question 1.} Does Theorem 12 hold for all msd-4 graphs?

Define another $\oplus$-operation as follows.

$\oplus_{u_i}^{u_1Q_1, u_2Q_2}$. Let $G_1$ and $G_2$ be vertex disjoint graphs containing (not necessarily maximal) cliques $Q_1$ and $Q_2$ of equal size, and vertices $u_i \in V(Q_i)$ for $i \in \{1, 2\}$. We denote a graph obtained from $G_1$ and $G_2$ by identifying $Q_1$ and $Q_2$ into one clique $Q$, and $u_1$ and $u_2$ into one vertex $u = u_1 = u_2$, by $G_1 \oplus_{u_i}^{u_1Q_1, u_2Q_2} G_2$ (or by $G_1 \oplus_{u_i}^{u_1Q_1, u_2Q_2} G_2$ if $u$ and $Q$ are unimportant).

Note that if the cliques $Q_i$ have order at least three, then identifying the vertices of $Q_i - u_i$ in different ways may yield different graphs. Both operations $\oplus_{u_i}^{u_1u_2}$ and $\oplus_{t}^{u_1u_2}$ are special cases of $\oplus_{u_i}^{u_1Q_1, u_2Q_2}$.

\textbf{Question 2.} Let $G_1$ and $G_2$ be disjoint msd-4 graphs containing cliques $Q_1$ and $Q_2$ of equal size and $\gamma_{pr}(G_1)$-critical vertices $u_i \in V(Q_i)$, $i = 1, 2$. Is it true that for any graph $G = G_1 \oplus_{u_i}^{u_1Q_1, u_2Q_2} G_2$, $u$ is $\gamma_{pr}(G)$-critical and $\text{msd}_{pr}(G) = 4$?

If $G_1$ and $G_2$ are copies of the msd-4 graph in Figure 5, with $u_i = u$, which is $\gamma_{pr}$-critical, and $Q_i$ is the triangle containing $u$, then both graphs obtainable as $G_1 \oplus_{u_i}^{u_1Q_1, u_2Q_2} G_2$ are msd-4 graphs having $u$ as critical vertex.
Question 3. Let $G$ be a graph with $\text{msd}_{pr}(G) = 4$. What is the largest number of edges of $G$ that can be subdivided three times before the paired domination number increases? If this number can be arbitrarily high, what is its ratio to the number of edges of $G$?

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