BLOCK GRAPHS WITH LARGE PAIRED DOMINATION MULTISUBDIVISION NUMBER

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Abstract

The paired domination multisubdivision number of a nonempty graph $G$, denoted by $\text{msd}_{pr}(G)$, is the smallest positive integer $k$ such that there exists an edge which must be subdivided $k$ times to increase the paired domination number of $G$. It is known that $\text{msd}_{pr}(G) \leq 4$ for all graphs $G$. We characterize block graphs with $\text{msd}_{pr}(G) = 4$.

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1. Introduction

The study of changes that occur in domination-related parameters of a graph when its edges are subdivided$^1$ was initiated in [11]. If $\pi$ is a domination-type parameter of $G$, the smallest number of edges that must be subdivided, where each edge of $G$ can be subdivided at most once, in order to increase $\pi$ is called

$^1$See Section 2 for definitions of terms used in this section.
the \( \pi \)-subdivision number, denoted by \( \text{sd}_\pi(G) \). Subdivision numbers have been studied for the domination number \([6, 11]\), as well as for connected \([4]\), double \([1]\), Roman \([10]\), total \([7, 9]\) and paired domination numbers \([5]\).

Instead of subdividing multiple edges once each, one may wish to subdivide a single edge multiple times. The smallest number of times that a single edge of \( G \) must be subdivided to increase \( \pi \) is called the \( \pi \)-multisubdivision number, denoted by \( \text{msd}_\pi(G) \). Domination and paired domination multisubdivision numbers were studied in \([3]\) and \([2]\), respectively. In particular, it was shown in \([2]\) that the paired domination multisubdivision number \( \text{msd}_{pr}(G) \) of any graph \( G \) is at most four. For brevity we refer to a graph \( G \) with \( \text{msd}_{pr}(G) = 4 \) as an msd-4 graph. Msd-4 trees were characterized in \([2]\).

We discuss methods of combining msd-4 graphs to yield new msd-4 graphs and use our results, combined with results from \([2]\), to characterize msd-4 block graphs. Definitions and previous results are given in Section 2. We state the characterization of msd-4 block graphs in Section 3, but defer its proof to Section 6 to allow us to prove a number of results used in the proof; results that apply to general msd-4 graphs are given in Section 4, while results specific to block graphs can be found in Section 5.

2. Definitions and Previous Results

We refer the reader to \([8]\) for domination parameters not defined here. A set \( S \) of vertices of a graph \( G = (V, E) \) without isolated vertices is a paired dominating set of \( G \) if every vertex of \( G \) is adjacent to a vertex in \( S \), and the subgraph \( G[S] \) of \( G \) induced by \( S \) has a perfect matching. If \( u, v \in S \) and there exists a perfect matching \( M \) of \( G[S] \) such that \( uv \in M \), we say that \( u \) and \( v \) are paired in \( S \). The smallest cardinality of a paired dominating set of \( G \) is the paired domination number of \( G \), denoted by \( \gamma_{pr}(G) \). If \( S \) is a paired dominating set of \( G \) such that \( |S| = \gamma_{pr}(G) \), we call \( S \) a \( \gamma_{pr}(G) \)-set, or simply a \( \gamma_{pr} \)-set if the graph is clear from the context. If \( u \) is a vertex of \( G \) such that \( G - u \) has no isolated vertices and \( \gamma_{pr}(G - u) < \gamma_{pr}(G) \) (in which case \( \gamma_{pr}(G - u) = \gamma_{pr}(G) - 2 \)), we say that \( u \) is a \( \gamma_{pr}(G) \)-critical vertex, or simply a \( \gamma_{pr} \)-critical vertex, and define \( \text{Cr}(G) = \{ u \in V(G) : u \text{ is a } \gamma_{pr} \text{-critical vertex} \} \).

A neighbour of a vertex \( u \in V(G) \) is a vertex adjacent to \( u \). The (open) neighbourhood \( N(u) \) of a vertex \( u \) is the set of all vertices adjacent to \( u \), and its closed neighbourhood is \( N[u] = N(u) \cup \{ u \} \). For a set \( S \subseteq V(G) \), the (open) neighbourhood of \( S \) is \( N(S) = \bigcup_{u \in S} N(u) \), and its closed neighbourhood is \( N[S] = N(S) \cup S \). For a vertex \( u \in S \), the private neighbourhood of \( u \) with respect to \( S \) is the set \( \text{PN}(u, S) = N[u] \setminus N[S \setminus \{ u \}] \). It is possible that \( u \in \text{PN}(u, S) \), but if \( S \) is a paired dominating set, then \( u \) is adjacent to the vertex it is paired with,
so \( u \notin PN(u, S) \) in this case.

An edge \( uv \) of a graph \( G \) is **subdivided** if it is replaced by a path \((u, x, v)\), where \( x \) is a new vertex, and **multisubdivided** if it is replaced by a path \((u, x_1, \ldots, x_k, v)\), \( k \geq 2 \), where \( x_1, \ldots, x_k \) are new vertices; we also say that \( uv \) is subdivided \( k \) times.

Let \( G_{uv,k} \) denote the graph obtained from \( G \) by subdividing the edge \( uv \) \( k \) times. The **paired domination multisubdivision number** \( msd_{pr}(G) \) of a graph \( G \) without isolated vertices is the smallest positive integer \( k \) such that there exists an edge \( uv \) which must be subdivided \( k \) times for \( \gamma_{pr}(G_{uv,k}) \) to exceed \( \gamma_{pr}(G) \).

As mentioned above, \( msd_{pr}(G) \leq 4 \) for all graphs. The three graphs in Figure 1 are all \( msd-4 \) graphs; the red vertices form \( \gamma_{pr} \)-sets.

![Figure 1](image-url)

**Figure 1.** (a) The spider \( S(2, 2, 6) \) (b) the corona \( K_3 \circ K_1 \) (c) a flared corona \( K_4 \circ^2 K_1 \).

A **leaf** of a graph is a vertex of degree one, and its neighbour is called a **stem**.

The following properties of \( msd-4 \) graphs were proved in [2].

**Theorem 1** [2]. Let \( G \) be an \( msd-4 \) graph. Then

(i) each edge of \( G \) belongs to a matching of a minimum paired dominating set of \( G \);

(ii) any leaf of \( G \) is a \( \gamma_{pr} \)-critical vertex;

(iii) each stem is adjacent to exactly one leaf.

The complete bipartite graph \( K_{1,k} \), \( k \geq 2 \), is called a **star**. Let \( K_{1,k} \) have partite sets \( \{u\} \) and \( \{v_1, \ldots, v_k\} \). The **spider** \( S(\ell_1, \ldots, \ell_k) \), \( \ell_i \geq 1 \), \( k \geq 2 \), is a tree obtained from \( K_{1,k} \) by subdividing the edge \( uv_i \) \( \ell_i - 1 \) times, \( i = 1, \ldots, k \). Note that \( S(2, 2) \cong P_5 \). See Figure 1(a) for \( S(2, 2, 6) \). The characterization of \( msd-4 \) trees in [2] immediately gives the following result.
Proposition 2 [2]. The spider $T = S(2, \ldots, 2)$ satisfies $\text{msd}_{pr}(T) = 4$, and $\text{Cr}(T)$ consists of the leaves of $T$.

The corona $G \circ K_1$ of a graph $G$ is the graph obtained by joining each vertex of $G$ to a new leaf; $K_3 \circ K_1$ is illustrated in Figure 1(b). A flared corona $G \circ^{st} K_1$ of $G$ is a graph obtained by joining each vertex of $G$, except one vertex $w$, to a new leaf, while $w$ is joined to a single vertex of each of $t \geq 1$ copies of $K_2$. The flared corona $K_3 \circ^{st} K_1$ is depicted in Figure 1(c). The following facts can be verified easily and are stated without proof.

Remark 3. 
(i) A corona $K_n \circ K_1$, $n \geq 2$, is an msd-4 graph if and only if $n$ is odd.
(ii) A flared corona $K_n \circ^{st} K_1$, $n \geq 2$, is an msd-4 graph if and only if $n$ is even.
(iii) A vertex of $K_{2n+1} \circ K_1$ or $K_{2n} \circ^{st} K_1$ is $\gamma_{pr}$-critical if and only if it is a leaf (see Theorem 1).

A block of a graph is a maximal connected subgraph with no cut-vertex, and a block graph is a graph, each of whose blocks is a complete graph. Thus, trees are block graphs since each block of a nontrivial tree is a $K_2$. Evidently, coronas and flared coronas are also block graphs. To characterize msd-4 block graphs, we use spiders $S(2, \ldots, 2)$, coronas $K_{2n+1} \circ K_1$ and flared coronas $K_{2n} \circ^{st} K_1$, combining them by identifying vertices and edges in a prescribed way.

We begin by describing two operations, collectively known as $\oplus$-operations, for joining disjoint graphs; since the operations can be performed on any graphs, we state them in their most general form. (The operations are well known but we need to define our notation.)

$G_1 \oplus^{u_i \nu_i} G_2$: Let $G_1$ and $G_2$ be vertex disjoint graphs and $u_i \in V(G_i)$ for $i \in \{1, 2\}$. We denote the graph obtained from $G_1$ and $G_2$ by identifying $u_1$ and $u_2$ into one vertex $u = u_1 = u_2$ by $G_1 \oplus^{u_1 \nu_2} G_2$ (or by $G_1 \oplus^{u_1 \nu_2} G_2$ if the label $u$ is unimportant).

$G_1 \oplus^{e_{i \nu}} G_2$: Let $G_1$ and $G_2$ be vertex disjoint graphs and $e_i = u_i v_i \in E(G_i)$. We denote the graph obtained from $G_1$ and $G_2$ by identifying $u_1$ and $u_2$ into one vertex $u = u_1 = u_2$, $v_1$ and $v_2$ into one vertex $v = v_1 = v_2$, and $e_1$ and $e_2$ into one edge $e = uv$ by $G_1 \oplus^{e_{1 \nu}} G_2$ (or by $G_1 \oplus^{e_{1 \nu}} G_2$ if the label $e$ is unimportant).

The graph $G_1 \oplus^{e_{1 \nu}} G_2$, where $G_1 = S(2, 2, 6), G_2 = K_3 \circ K_1$, and $e_i = u_i v_i$ for $i = 1, 2$, is illustrated in Figure 2. Note that $u_i$ is $\gamma_{pr}(G_i)$-critical for $i = 1, 2$, and $u_1 = u_2$ is $\gamma_{pr}$-critical in $G_1 \oplus^{e_{1 \nu}} G_2$. The spider $S(2, 2, 6)$, in turn, is obtained as $H_1 \oplus^{u_1 \nu_2} H_2$, where $H_1 = S(2, 2, 2), H_2 = P_5 = S(2, 2)$, and $u_i$ is a leaf of $H_i$, $i = 1, 2$. 

3. Characterization of msd-4 Block Graphs

We now state our main result — the characterization of msd-4 block graphs. The proof is deferred to Section 6.

Let $\mathcal{U}$ be the collection of all spiders $S(2, \ldots, 2)$, coronas $K_{2n+1} \circ K_1$ and flared coronas $K_{2n} \circ^* K_1$, $n \geq 1$. Define $\mathcal{B}$ to be the family of all block graphs $G$ that can be obtained as a graph $G_j$, $j \geq 1$, from a sequence $G_1, \ldots, G_j$ of graphs, where $H_1 = G_1 \in \mathcal{U}$, and, if $j > 1$, $G_{i+1}$ can be constructed recursively from $G_i$ by

- adding a graph $H_{i+1} \in \mathcal{U}$,
- choosing vertices $u_1 \in \text{Cr}(G_i)$, $u_2 \in \text{Cr}(H_{i+1})$, and if necessary, $v_1 \in N(u_1)$, $v_2 \in N(u_2)$,
- performing the operation $G_i \oplus_{u_1u_2} H_{i+1}$ or $G_i \oplus_{u_1v_1} u_2v_2 H_{i+1}$.

**Theorem 4.** Let $G$ be a connected block graph. Then $G$ is an msd-4 graph if and only if $G \in \mathcal{B}$. Moreover, if $G$ is an msd-4 graph constructed from the graphs $H_1, \ldots, H_j \in \mathcal{U}$, then $\text{Cr}(G) = \bigcup_{i=1}^j \text{Cr}(H_i)$.

The second statement of Theorem 4 implies that any $\gamma_{pr}$-critical vertex $v$ of an msd-4 block graph remains $\gamma_{pr}$-critical after the $\oplus$-operations have been performed any number of times, whether $v$ was identified with another vertex or not. The following corollary of Theorem 4 was proved in [2].

**Corollary 5.** A tree $T$ is an msd-4 graph if and only if $T \in \mathcal{B}$, that is, if and only if $T$ can be constructed as described, using only spiders $S(2, \ldots, 2)$.
4. General Results

In this section we discuss ways of constructing larger msd-4 graphs from smaller ones. We first prove a useful lemma.

Lemma 6. Let \( G \) be a graph with \( \text{msd}_{pr}(G) = 4 \). For any edge \( uv \) of \( G \), subdivide \( uv \) by replacing it with the path \( (u, x_1, x_2, x_3, v) \). If \( D \) is any \( \gamma_{pr}(G_{uv,3}) \)-set, then \( D \cap \{u, x_1, x_2, x_3, v\} = \)

(i) \( \{x_1, x_2\} \) or \( \{x_2, x_3\} \), or

(ii) \( \{u, x_1, v\} \) or \( \{u, x_3, v\} \).

If the first part of (i) holds, then \( u \) is \( \gamma_{pr} \)-critical, and if the second part of (i) holds, then \( v \) is \( \gamma_{pr} \)-critical.

Proof. Let \( X = \{x_1, x_2, x_3\} \). To dominate \( x_2 \), \( X \cap D \neq \emptyset \). We consider three cases.

Case 1. \( X \cap D = X \). Without loss of generality assume that \( x_1 \) is paired with \( u \in D \), and \( x_2 \) and \( x_3 \) are paired. Then \( v \notin D \), otherwise \( D \setminus \{x_2, x_3\} \) is also a paired dominating set of \( G_{uv,3} \), contradicting the minimality of \( D \). But now \( D' = (D \setminus X) \cup \{v\} \) is a paired dominating set of \( G \), which is impossible because \( \text{msd}_{pr}(G) = 4 \).

Case 2. \( |X \cap D| = 2 \). If \( X \cap D = \{x_1, x_3\} \), then \( \{u, v\} \subseteq D \) with \( u \) paired with \( x_1 \), and \( v \) with \( x_3 \). However, then \( D \setminus \{x_1, x_3\} \) is a paired dominating set of \( G \), contradicting \( \text{msd}_{pr}(G) = 4 \). Suppose \( X \cap D = \{x_1, x_2\} \). Then \( x_1 \) and \( x_2 \) are paired in \( D \). If \( \{u, v\} \cap D \neq \emptyset \), then \( D \setminus \{x_1, x_2\} \) is a paired dominating set of \( G \), which is a contradiction. Hence \( D \setminus \{u, x_1, x_2, x_3, v\} = \{x_1, x_2\} \). Now \( D' \setminus \{x_1, x_2\} \) is a paired dominating set of \( G - u \), so \( \gamma_{pr}(G - u) < \gamma_{pr}(G_{uv,3}) = \gamma_{pr}(G) \). We conclude that \( u \) is \( \gamma_{pr} \)-critical. Arguing similarly if \( X \cap D = \{x_2, x_3\} \), we conclude that (i) and the last part of the statement of the lemma hold.

Case 3. \( |X \cap D| = 1 \). Then \( x_2 \notin D \). If \( x_1 \in D \), then \( x_1 \) is paired with \( u \in D \), while \( v \) to dominate \( x_3 \). Consequently, \( D \cap \{u, x_1, x_2, x_3, v\} = \{u, x_1, v\} \). Similarly, if \( x_3 \in D \), then \( D \cap \{u, x_1, x_2, x_3, v\} = \{u, x_3, v\} \).

Our first result regarding the construction of msd-4 graphs from smaller graphs shows that subdividing any edge of an msd-4 graph four times produces another msd-4 graph. Repeatedly subdividing edges of an msd-4 graph thus yields, for example, msd-4 graphs of arbitrary large girth. In fact, we prove a stronger result: subdividing any edge of any graph \( G \) without isolated vertices four times produces a graph that has the same multisubdivision number as \( G \).

Proposition 7. For any graph \( G \) and any edge \( e \) of \( G \), \( \text{msd}_{pr}(G_{e,4}) = \text{msd}_{pr}(G) \).
Proof. Say msd \( pr(G) = t \leq 4 \) and \( e = uv \) has been subdivided by replacing it with the path \((u, x_1, x_2, x_3, x_4, v)\). Then \( \gamma_{pr}(G_{e,4}) = \gamma_{pr}(G) + 2 \) and there exists an edge \( e' \) of \( G \) such that \( \gamma_{pr}(G_{e',t}) = \gamma_{pr}(G) + 2 \). If \( e \neq e' \), then subdividing \( e \in E(G_{e',t}) \) four times yields the graph \((G_{e',t})_{e,4}\). Since \( msd_{pr}(G_{e',t}) \leq 4 \), \( \gamma_{pr}((G_{e',t})_{e,4}) = \gamma_{pr}(G_{e',t}) + 2 \). But \((G_{e',t})_{e,4} = (G_{e,4})_{e,t}'\), hence \( \gamma_{pr}((G_{e,4})_{e,t}') = \gamma_{pr}(G) + 4 = \gamma_{pr}(G_{e,4}) + 2 \). If \( e = e' \), say \( uv \) has been subdivided, in \( G \), by replacing it with \((u, x_1, \ldots, x_t, v)\). Subdividing (without loss of generality) the edge \( x_t v \) four times by replacing it with \((x_t, x_{t+1}, \ldots, x_{t+4}, v)\), we obtain the graph \((G_{e,t})_{x_t v,4} = (G_{e,4})_{x_{t}v,t}\) with \( \gamma_{pr}((G_{e,4})_{x_{t}v,t}) = \gamma_{pr}(G_{e,4}) + 2 \). It follows that \( msd_{pr}(G_{e,4}) \leq t \).

We show that \( msd_{pr}(G_{e,4}) \geq t \). If \( t = 1 \), this is obvious, hence assume \( t > 2 \). Consider any \( e' \in E(G) \). Suppose first that \( e' \neq e \). Since \( msd_{pr}(G) = t \), \( \gamma_{pr}(G_{e',t-1}) = \gamma_{pr}(G) \). If \( D' \) is any \( \gamma_{pr}(G_{e',t-1}) \)-set, then \( D = D' \cup \{x_1, x_2\} \) (if \( u \) and \( v \) are paired in \( D' \)) or \( D = D' \cup \{x_2, x_3\} \) (otherwise) is a paired dominating set of \( (G_{e,4})_{e,t-1}' \) of cardinality \( |D| = \gamma_{pr}(G_{e',t-1}) + 2 = \gamma_{pr}(G) + 2 = \gamma_{pr}(G_{e,4}) \).

Assume \( e' = e \). Without loss of generality subdivide the edge \( x_t v \) of \( G_{e,4} \) \( t-1 \) times by replacing it with the path \((x_{t}, x_{t+1}, \ldots, x_{t+4}, v)\) and denote the resulting graph \((G_{e,4})_{x_t v, t-1}\) by \( G_{e,3+t} \). Also consider the graph \( G_{e, t-1} \) obtained from \( G \) by subdividing \( e = uv \) by replacing it with \((u, x_1, \ldots, x_{t-1}, v)\). Since \( msd_{pr}(G) = t \), \( \gamma_{pr}(G_{e,t-1}) = \gamma_{pr}(G) \). Let \( S' \) be any \( \gamma_{pr}(G_{e,t-1}) \)-set. We consider three cases. In each case we construct a paired dominating set \( S \) of \( G_{e,3+t} \) such that \( |S| = |S'| + 2 = \gamma_{pr}(G_{e,4}) \); this shows that \( msd_{pr}(G_{e,4}) \geq t \).

Case 1. \( t = 2 \). If \( x_1 \notin S' \), then without loss of generality \( u \in S' \) to dominate \( x_1 \), and \( S' \setminus \{u\} \) dominates \( v \). Let \( S = S' \cup \{x_3, x_4\} \). If \( x_1 \in S' \), then again without loss of generality \( x_1 \) is paired with \( u \). Let \( S = S' \cup \{x_4, x_5\} \).

Case 2. \( t = 3 \). If \( S' \cap \{x_1, x_2\} = \emptyset \), then \( u \) dominates \( x_1 \) while \( v \) dominates \( x_2 \); let \( S = S' \cup \{x_3, x_4\} \) (so \( v \) dominates \( x_6 \)). If \( \) (without loss of generality) \( S' \cap \{x_1, x_2\} = \{x_1, x_2\} \), then \( u \) and \( x_1 \) are paired, and \( S' \setminus \{u, x_1\} \) dominates \( v \). Let \( S = S' \cup \{x_4, x_5\} \). If \( \{x_1, x_2\} \subseteq S' \), then \( x_1 \) and \( x_2 \) are paired (otherwise \( S' \cap \{x_1, x_2\} \) is a paired dominating set of \( G \), which is not the case). Let \( S = S' \cup \{x_5, x_6\} \).

Case 3. \( t = 4 \). By Lemma 6, without loss of generality \( S' \cap \{u, x_1, x_2, x_3, v\} = \{x_1, x_2\} \) or \( \{u, x_1, v\} \). In the former case, let \( S = S' \cup \{x_5, x_6\} \), and in the latter case, let \( S = S' \cup \{x_4, x_5\} \).

In all cases, \( S \) is a paired dominating set of \( G_{e,3+t} \) of cardinality \( \gamma_{pr}(G) + 2 = \gamma_{pr}(G_{e,4}) \), and \( msd_{pr}(G_{e,4}) \geq t \). It follows that \( msd_{pr}(G_{e,4}) = t \), as required.

We next prove results pertaining to the \( \oplus \)-operations defined above that hold for general \( msd-4 \) graphs, not only block graphs. We show that the \( \oplus \)-operations can be used to construct new connected \( msd-4 \) graphs from smaller ones.
Our next result shows that performing the operation \( G_1 \oplus_{u_1u_2} G_2 \) on msd-4 graphs \( G_1 \) and \( G_2 \) with \( \gamma_{pr}(G_i) \)-critical vertices \( u_1 \) and \( u_2 \), respectively, results in an msd-4 graph in which each \( \gamma_{pr}(G_i) \)-critical vertex is \( \gamma_{pr}(G) \)-critical.

**Proposition 8.** Let \( G_1 \) and \( G_2 \) be disjoint msd-4 graphs with \( \gamma_{pr}(G_i) \)-critical vertices \( u_i, i = 1, 2 \). Then for the graph \( G = G_1 \oplus_{u_1u_2} G_2 \), \( \gamma_{pr}(G) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2 \), any \( \gamma_{pr}(G_i) \)-critical vertex (including \( u \)) is \( \gamma_{pr}(G) \)-critical and

\[
\text{msd}_{pr}(G) = 4.
\]

**Proof.** Since \( u_i \in V(G_i) \) is \( \gamma_{pr}(G_i) \)-critical, \( \gamma_{pr}(G_1 - u_1) + \gamma_{pr}(G_2 - u_2) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 4 \), and at most two more vertices are needed to pairwise dominate \( G \). Therefore \( \gamma_{pr}(G) \leq \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2 \).

Suppose there exists a paired dominating set \( S \) of \( G \) such that \( |S| < \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2 \) and let \( S = S \cap V(G_i) \). First suppose that \( u \notin S \). Assume without loss of generality that \( S_1 \) dominates \( u \). Then \( S_1 \) is a paired dominating set of \( G_1 \) and \( S_2 \) is a paired dominating set of \( G_2 - u_2 \). Hence \( |S_1| \geq \gamma_{pr}(G_1) \) and \( |S_2| \geq \gamma_{pr}(G_2) - 2 \). But then \( |S| = |S_1| + |S_2| \geq \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2 \), which is not the case. Therefore we may assume that \( u \in S \) (in this case \( u_i \in S_i, i = 1, 2 \)) and \( |S_1| + |S_2| = |S| + 1 \). Without loss of generality, \( u \) is paired with \( v \in V(G_1) \), hence \( S_1 \) is a paired dominating set of \( G_1 \). Therefore \( |S_1| \geq \gamma_{pr}(G_1) \) so that \( |S_2| \leq \gamma_{pr}(G_2) - 3 \). If \( N_{G_2}(u_2) \subseteq S_2 \), then \( S_2 \setminus \{u_2\} \) is a paired dominating set of \( G_2 \), and if there exists \( w \in N_{G_2}(u_2) \setminus S_2 \), then \( S_2 \cup \{w\} \) is a paired dominating set of \( G_2 \). This is impossible because \( |S_2 \cup \{w\}| \leq \gamma_{pr}(G_2) - 2 \). Hence

\[
\gamma_{pr}(G) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2.
\]

If \( w_i \) is \( \gamma_{pr}(G_i) \)-critical, then, for \( j \neq i \), the union of any \( \gamma_{pr}(G_i - w_i) \)-set and any \( \gamma_{pr}(G_j - u_j) \)-set is a paired dominating set of \( G - w_i \) (this holds for \( w_i = u_i = u \) also), so

\[
\gamma_{pr}(G - w_i) \leq \gamma_{pr}(G_i - w_i) + \gamma_{pr}(G_j - u_j) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 4 < \gamma_{pr}(G).
\]

Therefore \( u_i \) is \( \gamma_{pr}(G) \)-critical.

Without loss of generality consider \( e \in E(G_1) \) and subdivide \( e \) three times. Then, since \( \text{msd}_{pr}(G_1) = 4 \) and \( u_2 \) is \( \gamma_{pr}(G_2) \)-critical, we obtain

\[
\gamma_{pr}(G_{e,3}) \leq \gamma_{pr}(G_{1,e,3}) + \gamma_{pr}(G_2 - u_2) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2 = \gamma_{pr}(G).
\]

Therefore \( \text{msd}_{pr}(G) = 4. \)

We show next that performing the operation \( G_1 \oplus^{e_1e_2} G_2 \) on msd-4 graphs \( G_i, i = 1, 2 \), with edges \( e_i = x_iy_i, \) where \( x_i \) is a \( \gamma_{pr}(G_i) \)-critical vertex, results in an msd-4 graph in which each \( \gamma_{pr}(G_i) \)-critical vertex is \( \gamma_{pr}(G) \)-critical.
Proposition 9. Let $G_i$, $i = 1, 2$, be disjoint msd-4 graphs with $e_i = x_i y_i \in E(G_i)$, where $x_i \in \text{Cr}(G_i)$. Then for the graph $G = G_1 \oplus e_1 \oplus e_2 G_2$, $\gamma_{pr}(G) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2$, any $\gamma_{pr}(G_i)$-critical vertex (including $x = x_1 = x_2$) is $\gamma_{pr}(G)$-critical and $\text{msd}_{pr}(G) = 4$.

Proof. By Theorem 1, there exists a $\gamma_{pr}(G_i)$-set in which $x_i$ and $y_i$ are matched. Therefore

$$\gamma_{pr}(G) \leq \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2.$$  

On the other hand, it suffices to add two vertices to a $\gamma_{pr}(G)$-set when splitting it into paired dominating sets of $G_1$ and $G_2$. Hence we have equality in (1). As in the proof of Proposition 8, any $\gamma_{pr}(G_i)$-critical vertex is $\gamma_{pr}(G)$-critical.

Let $e \in E(G)$ be any edge. If $e \in E(G_1) \setminus \{e_1\}$, then

$$\gamma_{pr}(G_{e,3}) \leq \gamma_{pr}(G_{1e,3}) + \gamma_{pr}(G_2 - x_2) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2 = \gamma_{pr}(G).$$

The case when $e \in E(G_2) \setminus \{e_2\}$ is analogous. Thus assume $e = xy$ and subdivide $e$ by replacing it with the path $(x, u, v, w, y)$. Let $S$ be any $\gamma_{pr}(G - x)$-set. As shown above, $|S| = \gamma_{pr}(G) - 2$. Now $S \cup \{u, v\}$ is a paired dominating set of $G_{e,3}$ of cardinality $\gamma_{pr}(G)$. It follows that $G$ is an msd-4 graph. 

We now describe a type of “reverse” operation, called a split operation, for each of the $\oplus$-operations.

$G \ominus u$. Let $G$ be a connected graph with a cut-vertex $u$. Denote the components of $G - u$ by $F_1, F_2, \ldots, F_k$. For each $i$, let $G_i$ be the graph obtained from $F_i$ by adding a new vertex $u_i$, joining $u_i$ to $v_i \in V(F_i)$ if and only if $uv_i \in E(G)$. Denote the disjoint union $G_1 + \cdots + G_k$ by $G \ominus u$.

$G \ominus xy$. Let $G$ be a connected graph containing a vertex-cut $\{x, y\}$, where $xy \in E(G)$. Denote the components of $G - \{x, y\}$ by $F_1, F_2, \ldots, F_k$. For each $i$, let $G_i$ be the graph obtained from $F_i$ by adding the edge $x_i y_i$, joining $x_i$ ($y_i$, respectively) to $v_i \in V(F_i)$ if and only if $xv_i \in E(G)$ ($yv_i \in E(G)$, respectively). Denote the disjoint union $G_1 + \cdots + G_k$ by $G \ominus xy$.

The next proposition shows that if an msd-4 graph $G$ is split at a $\gamma_{pr}$-critical cut-vertex $u$, the components of $G \ominus u$ are msd-4 graphs having the copies of $u$ as $\gamma_{pr}$-critical vertices.

Proposition 10. Let $G$ be an msd-4 graph with a $\gamma_{pr}$-critical cut-vertex $u$. Denote the components of $G \ominus u$ by $G_1, \ldots, G_k$. Then for each $i = 1, \ldots, k$, $u_i$ is a $\gamma_{pr}(G_i)$-critical vertex and $\text{msd}_{pr}(G_i) = 4$. 

Proof. Since $u$ is $\gamma_{pr}(G)$-critical and $G - u$ is the disjoint union of $G_i - u_i$, $i = 1, \ldots, k$,

$$
\gamma_{pr}(G) - 2 = \gamma_{pr}(G - u) = \sum_{i=1}^{k} \gamma_{pr}(G_i - u_i).
$$

Suppose $\gamma_{pr}(G_1 - u_1) \geq \gamma_{pr}(G_1)$. Let $R_1$ be a $\gamma_{pr}(G_1)$-set and, for $i \geq 2$, let $R_i$ be a $\gamma_{pr}(G_i - u_i)$-set. Since $R_1$ dominates $u_1$, $R = \bigcup_{i=1}^{k} R_i$ is a paired dominating set of $G$. But then

$$
\gamma_{pr}(G) \leq |R| \leq \gamma_{pr}(G_1) + \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i) \leq \sum_{i=1}^{k} \gamma_{pr}(G_i - u_i) = \gamma_{pr}(G) - 2,
$$

which is impossible. Thus $u_1$ is $\gamma_{pr}(G_1)$-critical. The same argument works for each $i \in \{2, \ldots, k\}$.

Consider an arbitrary edge $e \in E(G_1)$ and subdivide $e$ three times. Then

$$
\gamma_{pr}(G_1e, 3) + \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i) \leq |S| = \gamma_{pr}(G_1e, 3).
$$

We show that equality holds in (2). Let $S$ be any $\gamma_{pr}(G_1e, 3)$-set and define $S_1 = S \cap V(G_1e, 3)$ and $S_i = S \cap V(G_i)$ for $i = 2, \ldots, k$ (if $u \in S$, then $u_i \in S_i$ for each $i$). First suppose that $u \notin S$. If $S_1$ dominates $u$, then $S_1$ is a paired dominating set of $G_1e, 3$ and $S_i$, $i \geq 2$, is a paired dominating set of $G_i - u_i$.

Hence $|S_1| \geq \gamma_{pr}(G_1e, 3)$ and $|S_i| \geq \gamma_{pr}(G_i - u_i)$, so that $\gamma_{pr}(G_1e, 3) = |S| = \sum_{i=1}^{k} |S_i| \geq \gamma_{pr}(G_1e, 3) + \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i)$ as required. On the other hand, if $S_1$ does not dominate $u$, then $S_j$ is a paired dominating set of $G_j$ for some $j \geq 2$, so that $|S_j| \geq \gamma_{pr}(G_j) = \gamma_{pr}(G_j - u_j) + 2$ (since $u_j$ is $\gamma_{pr}(G_j)$-critical). Let $S_j'$ be a $\gamma_{pr}(G_j - u_j)$-set, $S_j' = S_1 \cup \{u, u'\}$ for some $u' \in N_{G_j}(u)$, and $S' = (S \setminus S_j \cup S_j') \cup S_j' \cup S_j'$.

Then $|S'| = |S|$, $S_j'$ is a paired dominating set of $G_1e, 3$, and the result follows as before.

Now suppose that $u \in S$. Then $|S_1| + \sum_{i=2}^{k} |S_i| = |S| + k - 1$ and $u$ is paired with a vertex in exactly one of the graphs $G_1e, 3$ or $G_i$, $i \geq 2$. For each of the $k - 1$ other graphs, either $S_i \cup \{u_i\}$, for some neighbour $u_i \notin S_i$ of $u_i$, or $S_i \setminus \{u_i\}$ (if all neighbours of $u_i$ in $G_i$ belong to $S_i$) is a paired dominating set. Hence

$$
\gamma_{pr}(G_1e, 3) + \sum_{i=2}^{k} \gamma_{pr}(G_i) \leq |S| + 2(k - 1).
$$

Since $u_i$ is $\gamma_{pr}(G_i)$-critical for each $i = 2, 3, \ldots, k$, $\gamma_{pr}(G_i - u_i) = \gamma_{pr}(G_i) - 2$. This gives

$$
\gamma_{pr}(G_1e, 3) + \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i) \leq |S| = \gamma_{pr}(G_1e, 3).
$$

Therefore we have equality (2). Now
\[
\gamma_{pr}(G_{1e,3}) = \gamma_{pr}(G_{e,3}) - \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i) = \gamma_{pr}(G) - \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i) = \gamma_{pr}(G) + \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i) - \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i) = \gamma_{pr}(G_1).
\]

Hence, for any edge \( e \in E(G_1) \), \( \gamma_{pr}(G_{1e,3}) = \gamma_{pr}(G) \). Thus \( \text{msd}_{pr}(G_1) = 4 \).

Similar reasoning may be applied to \( G_i \) for \( i \in \{2, 3, \ldots, k\} \).

5. MSD-4 Block Graphs

The last three results we need for the proof of Theorem 4 concern block graphs. In the first result we prove that every non-leaf vertex of an msd-4 block graph is a cut-vertex.

**Theorem 11.** Let \( G \) be a graph containing a block \( B \cong K_n \), where \( n \geq 3 \), such that some vertex of \( B \) is not adjacent to any vertex of \( G - B \). Then

\[ \text{msd}_{pr}(G) < 4. \]

**Proof.** Suppose the hypothesis of the theorem holds but \( \text{msd}_{pr}(G) = 4 \). Let \( V(B) = \{v_0, \ldots, v_{n-1}\} \) and say \( u = v_0 \) is not adjacent to any vertex of \( G - B \). Subdivide the edge \( uv_2 \) by replacing it with the path \( (u, x_3, x_2, x_1, v_2) \) (see Figure 3). Denote \( X = \{x_1, x_2, x_3\} \) and let \( D \) be a \( \gamma_{pr} \)-set of \( G_{uv,3} \). By Lemma 6 we only have to consider the cases

\[ D \cap \{u, x_1, x_2, x_3, v_2\} \in \{\{u, x_1, v_2\}, \{u, x_3, v_2\}, \{x_1, x_2\}, \{x_2, x_3\}\}. \]

Figure 3. The block \( B \) with the edge \( uv_2 \) subdivided with vertices \( x_1, x_2, x_3 \).
Case 1. $|X \cap D| = 1$. If $D \cap \{u, x_1, x_2, x_3, v_2\} = \{u, x_1, v_2\}$, then $x_1$ and $v_2$ are paired in $D$, while $u$ is paired with $v_i$ for some $i \neq 0, 2$. However, then $D \setminus \{x_1, u\}$, with $v_2$ and $v_1$ paired, is a smaller paired dominating set of $G$. If $D \cap \{u, x_1, x_2, x_3, v_2\} = \{u, x_3, v_2\}$, then $D \setminus \{x_3, u\}$ is a smaller paired dominating set of $G$. In either case $\text{msd}_{pr}(G) < 4$, contrary to our assumption.

Case 2. $|X \cap D| = 2$. If $D \cap \{u, x_1, x_2, x_3, v_2\} = \{x_1, x_2\}$, then $x_1$ and $x_2$ are paired in $D$. To pairwise dominate $u$, $v_i \in D$ for some $i \neq 0, 2$. But then $D \setminus \{x_1, x_2\}$ is a paired dominating set of $G$ (with $v_1$ paired as in $D$) and $\text{msd}_{pr}(G) < 4$, contrary to our assumption. Hence assume $D \cap \{u, x_1, x_2, x_3, v_2\} = \{x_2, x_3\}$. Then $x_2$ and $x_3$ are matched in $D$. If $v_i \in D$ for some $i$, then $D \setminus \{x_2, x_3\}$ is a paired dominating set of $G$ (again with $v_1$ paired as in $D$), a contradiction.

We therefore assume henceforth that

(i) $D$ contains $x_2$ and $x_3$, but neither $x_1$ nor any $v_0, \ldots, v_{n-1}$.

By Lemma 6, $u$ is $\gamma_{pr}$-critical, that is, 

(ii) $\gamma_{pr}(G - u) = \gamma_{pr}(G) - 2$.

For each $i = 1, \ldots, n - 1$, let $G_i$ be the component of $G - E(B)$ that contains $v_i$. Since $B$ is a block of $G$, the subgraphs $G_i$ are distinct and pairwise vertex-disjoint. Let $D_i = D \cap V(G_i)$. Then $|\bigcup_{i=1}^{n-1} D_i| = |D \setminus \{x_2, x_3\}| = \gamma_{pr}(G) - 2$. By (i), each $D_i$ is a $\gamma_{pr}(G_i)$-set that does not contain $v_i$.

We next show that

(iii) no $\gamma_{pr}(G)$-set contains $u = v_0$ and at least two $v_i$, $i \geq 1$.

Suppose there exists such a set $Z$: assume without loss of generality that $\{u, v_1, v_2, \ldots, v_k\} \subseteq Z$, $k \geq 2$. Necessarily, $u$ is paired with some $v_i$, $i = 1, \ldots, k$, in $Z$. Assume (again without loss of generality) $u$ is paired with $v_1$. Let $Z_1 = Z \cap V(G_1) \setminus \{v_1\}$ and, for $i \geq 2$, let $Z_i = Z \cap V(G_i)$. Then $\bigcup_{i=1}^{n-1} Z_i \subseteq V(G - u)$ and $|\bigcup_{i=1}^{n-1} Z_i| = |Z| - 2 = \gamma_{pr}(G - u) < \gamma_{pr}(G)$, by (ii). Since $v_1$ and $u$ are paired, $G_1[Z_1]$ contains a perfect matching, as does $G[\bigcup_{i=2}^{n-1} Z_i]$. Since $v_1$ is not adjacent to any vertex of $G - v_i$, $i \geq 2$, and $v_2$ dominates $B$ in $G$, $\bigcup_{i=2}^{n-1} Z_i$ is a paired dominating set of $G - G_1$.

Suppose $|Z_1| < |D_1|$. Since both $Z_1$ and $D_1$ have even cardinality, $|Z_1| \leq |D_1| - 2$. Then $Z_1$ does not dominate $G_1 - v_1$, otherwise $\bigcup_{i=1}^{n-1} Z_i$ is a paired dominating set of $G$ of cardinality less than $\gamma_{pr}(G)$, which is impossible. Since $Z_1 \cup \{v_1\}$ dominates $G_1$, there exists a vertex $w \in N_{G_1}(v_1)$ that is undominated by $Z_1$. Then $W_1 = Z_1 \cup \{w, v_1\}$ is a paired dominating set of $G_1$ of cardinality at most $|D_1|$ that contains $v_1$. But now $W_1 \cup D_2 \cup D_3 \cup \cdots \cup D_{n-1}$ is a paired dominating set of $G$ of cardinality at most $|D \setminus \{x_2, x_3\}| = \gamma_{pr}(G) - 2$, which is impossible. We conclude that $|Z_1| = |D_1|$.

Let $Z' = D_1 \cup \left( \bigcup_{i=2}^{n-1} Z_i \right)$. Since $\bigcup_{i=2}^{n-1} Z_i$ is a paired dominating set of $G - G_1$ and $D_1$ is a paired dominating set of $G$, $Z'$ is a paired dominating set of $G$. 

Moreover,

\[ |Z'| = \left| \bigcup_{i=2}^{n-1} Z_i \right| + |D_1| = \left| \bigcup_{i=1}^{n-1} Z_i \right| = |Z| - 2 = \gamma_{pr}(G - u) < \gamma_{pr}(G), \]

which is impossible. This concludes the proof of (iii).

Subdivide the edge \( v_1v_2 \) with vertices \( y_1, y_2, y_3 \), where \( y_1 \) is adjacent to \( v_1 \) and \( y_3 \) is adjacent to \( v_2 \) (see Figure 4). Denote \( Y = \{y_1, y_2, y_3\} \) and let \( Q \) be a \( \gamma_{pr} \)-set of \( G_{v_1v_2,3} \). Without loss of generality, by Lemma 6 we only have to consider the cases \( Q \cap \{v_1, v_2, y_1, y_2, y_3\} \in \{\{y_1, y_2\}, \{v_1, v_2, y_1\}\} \).

**Case 3a.** \( Q \cap \{v_1, v_2, y_1, y_2, y_3\} = \{y_1, y_2\} \). Then these two vertices are paired in \( Q \). To pairwise dominate \( u, v_i \in Q \) for some \( i \). It follows that \( Q \setminus \{y_1, y_2\} \) is a paired dominating set of \( G \), so \( msd_{pr}(G) < 4 \), contrary to our assumption.

**Case 3b.** \( Q \cap \{v_1, v_2, y_1, y_2, y_3\} = \{v_1, v_2, y_1\} \). Then \( y_1 \) is paired with \( v_1 \). If \( u \notin Q \), then \( Q' = (Q \setminus \{y_1\}) \cup \{u\} \) is a paired dominating set of \( G \) containing \( u, v_1, v_2 \). By (iii), \( Q' \) is not a \( \gamma_{pr} \)-set of \( G \), from which it follows that \( \gamma_{pr}(G) < |Q| \) and \( msd_{pr}(G) < 4 \). Assume therefore that \( u \in Q \). Then \( u \) is paired in \( Q \) with \( v_i \) for some \( i > 1 \). Now \( Q'' = Q \setminus \{y_1, u\} \) is a paired dominating set of \( G \) in which \( v_1 \) and \( v_i \) are paired. In both cases we again have a contradiction and the proof is complete.

The graph in Figure 5 shows that the statement of Theorem 11 is false if the complete subgraph \( B \) is not a block of \( G \).

The next result in this section shows that \( msd-4 \) block graphs have many \( \gamma_{pr} \)-critical vertices.
Figure 5. A graph $G$ with $\text{msd}_{pr}(G) = 4$ and a subgraph $K_3$ that is not a block of $G$.

**Theorem 12.** If $G$ is a block graph with $\text{msd}_{pr}(G) = 4$, then for any edge $uv \in E(G)$, 
\[(N_G[u] \cup N_G[v]) \cap \text{Cr}(G) \neq \emptyset.\]

**Proof.** Suppose there exists an edge $uv \in E(G)$ such that $(N_G[u] \cup N_G[v]) \cap \text{Cr}(G) = \emptyset$. By Theorem 1, no vertex in $N_G[u] \cup N_G[v]$ is a leaf. We subdivide the edge $uv$ by replacing it with the path $(u, x_1, x_2, x_3, v)$ to obtain the graph $G_{uv,3}$. By Lemma 6, for any $\gamma_{pr}$-set $S$ of $G_{uv,3}$, $S \cap \{u, v, x_1, x_2, x_3\} \in \{\{u, v, x_1\}, \{u, v, x_2\}, \{u, v, x_3\}\}$. Without loss of generality assume there exists such a set $S$ such that $S \cap \{u, v, x_1, x_2, x_3\} = \{u, v, x_1\}$, and among all such sets $S$, let $D$ be one for which $\text{PN}(u, D)$ is as small as possible. Then $x_1$ and $u$ are paired in $D$.

Say $v$ is paired with $v'$ and let $B$ be the block of $G$ that contains $uv$. If $v' \in V(G) \setminus V(B)$, let $G_v$ be the subgraph of $G - E(B)$ that contains $v$, and if $v' \in V(B)$, let $G_v$ be the subgraph of $G - (E(B) - \{vv\})$ that contains $v$. In either case, $v' \in V(G_v)$. Let $D_v = D \cap V(G_v)$ and $D' = D \setminus \{x_1, u\}$. Then $G[D']$ has a perfect matching and $D_v$ is a paired dominating set of $G_v$ containing $v$ and $v'$. In fact, $D_v$ is a $\gamma_{pr}(G_v)$-set, for if not, let $D''$ be a smaller paired dominating set of $G_v$. Consider $N_G(u) \setminus V(B)$. If $B \cong K_2$, then $N_G(u) \setminus V(B) = N_G(u) \setminus \{v\}$ is nonempty because $u$ is not a leaf, and if $B \cong K_n$ for $n \geq 3$, then $N_G(u) \setminus V(B)$ is nonempty by Theorem 11. If $N_G(u) \setminus V(B) \subseteq D$, then $D'$ is a paired dominating set of $G$, and if there exists $w \in N_G(u) \setminus V(B) \setminus D$, then $(D \setminus \{x_1\} \cup D_v) \cup D'' \cup \{w\}$ is a smaller paired dominating set of $G$ than $D$. In both cases we have a contradiction to $\text{msd}_{pr}(G) = 4$.

Since $\text{msd}_{pr}(G) = 4$, $|D'| = \gamma_{pr}(G_{uv,3}) - 2 = \gamma_{pr}(G) - 2$. Consequently, $D'$ does not dominate $G$. Since $v \in D'$ dominates $B$ in $G$, there exist vertices $w_1, \ldots, w_k \in N_G(u) \setminus N_G[v] \subseteq N_G(u) \setminus B$ that are undominated by $D'$, that is,
\{w_1, \ldots, w_k\} = PN(u, D). For i = 1, \ldots, k, let \(G_i\) be the component of \(G - u\) that contains \(w_i\). Possibly, \(G_i = G_j\) for \(i \neq j\); this happens exactly when \(w_i w_j \in E(G)\), and then \(w_i\) and \(w_j\) also belong to the same (complete) block of \(G_i\). Since no \(w_i\) is adjacent to \(v\) or \(v'\), \(V(G_i) \cap V(G_v) = \emptyset\) for each \(i\). Define \(D_i = D \cap V(G_i)\). Then \(G_i[D_i]\) has a perfect matching, but does not dominate \(w_i\). If it is nevertheless true that \(\gamma_{pr}(G_i) = |D_i|\) for some \(i\), let \(Q_i\) be a \(\gamma_{pr}(G_i)\) set. Then \(D^* = (D \setminus D_i) \cup Q_i\) is a \(\gamma_{pr}(G_{uv,3})\)-set such that \(PN(u, D^*) \subseteq PN(u, D) \setminus \{w_i\}\), contrary to the choice of \(D\). Therefore \(\gamma_{pr}(G_i) \geq |D_i| + 2\) for each \(i\).

Since each stem belongs to all paired dominating sets, no \(w_i\) is a stem, and by our initial assumption, no \(w_i\) is a leaf. Subdivide the edge \(uw_1\) by replacing it with the path \((u, y_1, y_2, y_3, w_1)\). Consider a \(\gamma_{pr}(G_{uw,1,3})\)-set \(S\). Since \(u, w_1 \notin Cr(G)\), Lemma 6 states that \(S \cap \{u, y_1, y_2, y_3, w_1\} \neq \emptyset\).

- In the former case, \(y_1\) is paired with \(u\) and \(S_1 = S \cap V(G_1)\) is a paired dominating set of \(G_1\); hence \(|S_1| \geq \gamma_{pr}(G) \geq |D_1| + 2\). Since \(w_1\) is adjacent to all \(w_i \in V(G_1)\), \(D_1 \cup \{w_i\}\) dominates \(G_1\) (but not pairwise). Now \(S' = (S \setminus S_1) \cup D_1 \cup \{w_1, y_3\}\) is a paired dominating set of \(G_{uw,3}\) such that \(|S'| \leq |S|\), hence \(S'\) is a \(\gamma_{pr}(G_{uw,1,3})\)-set. Moreover, \(S' \cap \{u, y_1, y_2, y_3, w_1\} = \{u, y_1, y_3, w_1\}\), contrary to Lemma 6.

- In the latter case, \(y_3\) is paired with \(w_1\). Then \(S_2 = (S \cap V(G_1)) \cup \{y_3\}\) is a paired dominating set of the graph obtained from \(G_1\) by joining \(y_3\) to \(w_1\). If all neighbours of \(w_1\) in \(G_1\) belong to \(S_2\), then \(S_2 \setminus \{w_1, y_3\}\) is a paired dominating set of \(G_1\). But then \(S'' = S \setminus \{w_1, y_3\}\) is a paired dominating set of \(G\) such that \(|S''| < |S|\), contradicting \(msd_{pr}(G) = 4\). Assume some neighbour \(z\) of \(w_1\) in \(G_1\) does not belong to \(S_2\). Then \(S_3 = (S_2 \setminus \{y_3\}) \cup \{z\}\) is a paired dominating set of \(G_1\), so that \(|S_2| = |S_3| \geq |D_1| + 2\). Since \(u \in S\) and \(\{w_1, \ldots, w_k\} \subseteq N(u)\), \(S^* = (S \setminus S_2) \cup D_1\) is a paired dominating set of \(G\) such that \(|S^*| < |S|\), again a contradiction.

This completes the proof of the theorem.

Although the graph \(G\) in Figure 5 satisfies \(msd_{pr}(G) = 4\) without being a block graph, Theorem 12 holds for \(G\) as well.

Our final result in this section concerns the reverse operation \(G \oplus xy\) for certain msd-4 block graphs.

**Proposition 13.** Let \(G\) be a connected msd-4 block graph such that the only \(\gamma_{pr}(G)\)-critical vertices are leaves. Let \(x\) be a leaf adjacent to the stem \(y\), where \(\{x, y\}\) is a vertex-cut, and denote the components of \(G \oplus xy\) by \(G_1, \ldots, G_k\). Then for each \(i = 1, \ldots, k\), \(G_i\) is an msd-4 graph and \(x_i \in Cr(G_i)\).

**Proof.** If \(G_i\) is an msd-4 graph, it will follow from Theorem 1(ii) that \(x_i \in Cr(G_i)\). However, we need the fact that \(x_i\) is \(\gamma_{pr}(G_i)\)-critical to show that \(msd_{pr}(G_i) = 4\), hence this is what we prove first.
Since $G$ is a block graph, $N_{G_i-x_i}(y_i)$ induces a clique for each $i = 1, \ldots, k$. Since $x$ is a leaf, $y$ belongs to every paired dominating set of $G$, and by Theorem 1(ii), $x \in Cr(G)$. Hence $y$ belongs to no $\gamma_{pr}(G-x)$-set (for such a set would dominate $x$ and thus $G$, contradicting $x \in Cr(G)$).

Let $D$ be a $\gamma_{pr}(G-x)$ set such that $|D \cap N(y)|$ is maximum and let $D_i = D \cap V(G_i)$, $i = 1, \ldots, k$. Since $x \in Cr(G)$ and $y \notin D$, $|D| = \sum_{i=1}^{k} |D_i| = \gamma_{pr}(G) - 2$. Also, $D_i$ is a paired dominating set of $G_i - \{x_i, y_i\}$ for each $i$, and a paired dominating set of $G_i - x_i$ for at least one $i$. We show that, in fact,

(A) $D_i$ is a paired dominating set of $G_i - x_i$ for each $i$.

First suppose $|N_{G_i-x_i}(y_i)| \geq 2$; say $z_1, z_2 \in N_{G_i-x_i}(y_i)$. Since $N_{G_i-x_i}(y_i)$ induces a clique, $z_1 z_2 \in E(G)$. By Theorem 12, $(N_G[z_1] \cup N_G[z_2]) \cap Cr(G) \neq \emptyset$. Since $N_G[z_i] = N_{G_i-x_i}[z_i]$ and $z_i$ is not a leaf (and thus, by the hypothesis, not $\gamma_{pr}(G)$-critical), $z_1$ or $z_2$ is adjacent to a $\gamma_{pr}(G)$-critical vertex, i.e., a leaf. Say $z_1$ is adjacent to a leaf $z'$. Then $z_1$ belongs to any paired dominating set of any subgraph of $G$ containing both $z_1$ and $z'$, so $z_1 \in D$. Therefore $D_i$ dominates $y_i$ and (A) holds.

Assume therefore that $|N_{G_i-x_i}(y_i)| = 1$, say $N_{G_i-x_i}(y_i) = \{z\}$. If $z \in D$, we are done, hence assume $z \notin D$. By Theorem 1(iii), $z$ is not a leaf, hence there exists a vertex $z' \in N_{G_i-x_i}(z) \{y_i\}$. By Theorem 1(i), $G$ has a $\gamma_{pr}$-set $X$ such that $zz'$ belongs to a matching of $G[X]$. Now $y \in X$, but $y$ is not paired with any vertex of $G_i-x_i$, since $N_{G_i-x_i}(y_i) = \{z\}$. Therefore $X_i = (X \setminus \{x, y\}) \cap V(G_i)$ is a paired dominating set of $G_i - x_i$. Moreover, $|X_i| \leq |D_i|$, otherwise $(X - X_i) \cup D_i$ is a smaller paired dominating set of $G$, which is impossible. However, now $D' = (D \setminus D_i) \cup X_i$ is a paired dominating set of $G - x$, hence a $\gamma_{pr}(G-x)$-set, containing more neighbours of $y$ than $D$, contrary to the choice of $D$. Hence (A) holds in this case as well.

Therefore $\gamma_{pr}(G_i - x_i) \leq |D_i|$ for each $i$, so that

$$\sum_{i=1}^{k} \gamma_{pr}(G_i - x_i) \leq \sum_{i=1}^{k} |D_i| = |D| = \gamma_{pr}(G-x).$$

Suppose there exists a $\gamma_{pr}(G_i - x_i)$-set $Y_i$ containing $y_i$. Since no $D_j$ contains $y_j$, $D' = (D \setminus D_i) \cup Y_i$ is a paired dominating set of $G - x$ such that $|D'| \leq |D| = \gamma_{pr}(G) - 2$ and $D'$ dominates $x$. Then $D''$ is a paired dominating set of $G$, which is impossible. Therefore no $\gamma_{pr}(G_i - x_i)$-set contains $y_i$. Similarly, if $\gamma_{pr}(G_i - x_i) < |D_i|$ for some $i$ and $Z_i$ is a $\gamma_{pr}(G_i - x_i)$-set, then $D'' = (D \setminus D_i) \cup Z_i$ is a paired dominating set of $G - x$ such that $|D''| < |D|$, which is also impossible. From these two facts we deduce that $D_i$ is a $\gamma_{pr}(G_i - x_i)$-set, equality holds in (3) and $\gamma_{pr}(G_i) = \gamma_{pr}(G_i - x_i) + 2$, that is, $x_i$ is $\gamma_{pr}(G_i)$-critical for each $i$.

We show that $msd_{pr}(G_1) = 4$: it will follow similarly that $msd_{pr}(G_i) = 4$ for each $i$. Since $D_1$ is a $\gamma_{pr}(G_1 - x_1)$-set, it is easy to see that we can pairwise
dominate $G_{1,y,3}$ by $|D_1| + 2 = \gamma_{pr}(G_1)$ vertices. Hence consider any edge $e \in E(G_1 - x_1)$ and the graphs $G_{e,3}$ and $G_{1,e,3}$. Since combining any $\gamma_{pr}(G_{1,e,3})$-set with the sets $D_j, j = 2, \ldots, k$, produces a paired dominating set of $G_{e,3}$,

\begin{equation}
\gamma_{pr}(G_{e,3}) \leq \gamma_{pr}(G_{1,e,3}) + \sum_{i=2}^{k} \gamma_{pr}(G_i - x_i).
\end{equation}

We show that equality holds in (4). For convenience of notation, define $H_1 = G_{1,e,3}$ and $H_i = G_i, i \geq 2$. Let $S$ be a $\gamma_{pr}(G_{e,3})$-set and define $S_i = S \cap V(H_i)$ for $i = 1, \ldots, k$ (since $y \in S, y_i \in S_i$ for each $i$, and if $x \in S$, then $x_i \in S_i$ for each $i$). We consider two cases, depending on whether $x \in S$ or not.

**Case 1.** $x \notin S$. Then $\sum_{i=1}^{k} |S_i| = |S| + k - 1$. Note that $y$ is paired with $w \in V(H_1) \setminus \{x_i, y_i\}$ for exactly one $i$. Then $S_i$ is a paired dominating set of $H_i$. For $j \neq i, S_i \cup \{x_j\}$ is a paired dominating set of $H_j$. Therefore $\gamma_{pr}(H_i) \leq |S_i|$ and $\gamma_{pr}(H_j) \leq |S_j| + 1$ for $j \neq i$. For $\ell \geq 2, x_\ell$ is $\gamma_{pr}(H_\ell)$-critical, hence $\gamma_{pr}(H_\ell - x_\ell) \leq \gamma_{pr}(H_\ell) - 2$. Therefore

$$\gamma_{pr}(G_{1,e,3}) + \sum_{i=2}^{k} \gamma_{pr}(G_i - x_i) \leq \sum_{i=1}^{k} |S_i| - 2(k - 1) + (k - 1) = \sum_{i=1}^{k} |S_i| - (k - 1) = |S|$$

and equality holds in (4).

**Case 2.** $\{x, y\} \subseteq S$. Then $x$ and $y$ are paired in $S$, $\{x_i, y_i\} \subseteq S_i$ for each $i$, and $S_i$ is a paired dominating set of $H_i$. Also, $\sum_{i=2}^{k} |S_i| = |S| + 2(k - 1) - |S_1|$. Since $x_i$ is $\gamma_{pr}(G_i)$-critical,

$$\gamma_{pr}(G_{1,e,3}) + \sum_{i=2}^{k} \gamma_{pr}(G_i - x_i) \leq |S_1| + \sum_{i=2}^{k} |S_i| - 2(k - 1) = |S| = \gamma_{pr}(G_{e,3}),$$

giving equality in (4).

It now follows as in the proof of Proposition 10 that $\text{msd}(G_1) = 4$. Similarly, $\text{msd}(G_i) = 4$ for $i \geq 2$.

6. **Proof of Theorem 4**

We are now ready to prove our main theorem, the characterization of $\text{msd}$-4 block graphs. We restate the theorem here for convenience.

**Theorem 4** (again). Let $G$ be a connected block graph. Then $G$ is an $\text{msd}$-4 graph if and only if $G \in B$. Moreover, if $G$ is an $\text{msd}$-4 graph constructed from the graphs $H_1, \ldots, H_j \in U$, then $\text{Cr}(G) = \bigcup_{i=1}^{j} \text{Cr}(H_i)$. 


Proof. If \( G \in B \), it follows immediately from Propositions 8 and 9 that \( G \) is an msd-4 graph and \( \text{Cr}(G) = \bigcup_{i=1}^{3} \text{Cr}(H_i) \).

For the converse, let \( G \) be an msd-4 block graph. If \( G \) is a tree, the result follows from Corollary 5, hence we assume that \( B \cong K_n, n \geq 3 \), is a block of \( G \). By (the contrapositive of) Theorem 11, each vertex of \( B \) is a cut-vertex, so \( \text{deg}(v) \geq n \) for each \( v \in V(B) \). Since each non-leaf vertex of a \( K_2 \)-block is a cut-vertex, we deduce that each vertex of \( G \) is either a leaf or a cut-vertex.

Suppose \( v \in V(B) \) is \( \gamma_{pr} \)-critical. Applying Proposition 10 to \( v \) we obtain an msd-4 graph \( G_1 \) with \( v_1 = v \) and \( N_{G_1}[v_1] = B \), which contradicts Theorem 11. Thus every \( \gamma_{pr}(G) \)-critical vertex belongs only to \( K_2 \)-blocks.

We say that a vertex \( u \) is a type-A vertex if it is a \( \gamma_{pr}(G) \)-critical cut-vertex, and an edge \( uv \) is a type-A edge if \( u \) is a leaf (hence \( \gamma_{pr}(G) \)-critical) and \( G - \{u, v\} \) is disconnected. Denote the number of type-A elements (vertices and edges together) of \( G \) by \( a(G) \). First we show that

\[
\text{(B) if } a(G) = 0, \text{ then } G \in \mathcal{U}.
\]

Suppose \( a(G) = 0 \). Then every \( \gamma_{pr}(G) \)-critical vertex is a leaf. Say \( V(B) = \{v_1, \ldots, v_n\} \). Since no vertex of \( B \) is \( \gamma_{pr}(G) \)-critical, Theorem 12 implies that \( v_1 \) or \( v_n \) is adjacent to a \( \gamma_{pr}(G) \)-critical vertex. Without loss of generality we assume that \( v_1u_1 \in E(G), u_1 \notin V(B), \text{ and } u_1 \) is \( \gamma_{pr}(G) \)-critical. Similarly, without loss of generality, \( v_i \) is adjacent to a \( \gamma_{pr}(G) \)-critical vertex \( u_i \notin V(B) \) for \( i = 2, \ldots, n-1 \).

Since \( a(G) = 0 \) and each vertex of \( G \) is either a leaf or a cut-vertex, \( \text{deg}_G(u_i) = 1 \) for each \( i = 1, \ldots, n-1 \) and \( G - \{v_i, u_i\} \) is connected. Thus, \( v_i \) belongs to only the two blocks \( B \) and \( v_iu_i \), so \( \text{deg}_G(v_i) = n \) for each \( i = 1, \ldots, n-1 \).

Since \( v_n \) is a cut-vertex, \( N(v_n) \setminus V(B) \neq \emptyset \). If \( v_n \) is adjacent to a \( \gamma_{pr}(G) \)-critical vertex, say \( u_n \), then, arguing as above, \( \text{deg}(u_n) = 1 \), \( \text{deg}(v_n) = n \) and \( G = K_n \cup K_1 \). By Remark 3(i), \( n \) is odd, hence \( G \) belongs to the family \( \mathcal{U} \subseteq B \). If no vertex in \( N(v_n) \setminus V(B) \) is critical, let \( N(v_n) \setminus V(B) = \{w_1, \ldots, w_k\} \) for \( t \geq 1 \). By Theorem 12, each \( w_i \) is adjacent to a critical vertex \( w'_i \neq v_n \), and since \( a(G) = 0 \), \( w'_i \) is a leaf. We show that

\[
\text{(C) } \{w_1, \ldots, w_k\} \text{ is an independent set of } G.
\]

Suppose (without loss of generality) that \( w_1w_2 \in E(G) \) and consider \( G_{w_1w_2,3} \). Let \( u_1, x_1, x_2, x_3, w_2 \) be the \( w_1 \) - \( w_2 \) path in \( G_{w_1w_2,3} \) and let \( D \) be a \( \gamma_{pr}(G_{w_1w_2,3}) \)-set. Since \( w'_1 \) and \( w'_2 \) are leaves, \( w_1, w_2 \in D \). To dominate \( x_2, \{x_1, x_2, x_3\} \cap D \neq \emptyset \). If \( \{|x_1, x_2, x_3\} \cap D| = 2 \), then \( D \setminus \{x_1, x_2, x_3\} \) is a paired dominating set (with \( w_1 \) and \( w_2 \) paired) of \( G \) of smaller cardinality than \( D \), contrary to \( msd(G) = 4 \). Hence assume without loss of generality that \( \{|x_1, x_2, x_3\} \cap D = \{x_1\}, \) so \( w_1 \) and \( x_1 \) are paired (and \( w'_1 \notin D \)), while \( w_2 \) is paired with either \( w'_2 \) or \( v_n \). However, each vertex in \( N_G(v_n) \) is adjacent to a leaf and belongs to \( D \), thus \( D \setminus \{v_n\} \) dominates \( G \). Therefore, either \( D \setminus \{x_1, w'_2\} \) or \( D \setminus \{x_1, v_n\} \) is a paired dominating set of \( G \) in which \( w_1 \) and \( w_2 \) are paired, contrary to \( msd(G) = 4 \). It follows that (C) holds.
Since $G$ is a block graph, $w_i$ and $w_j$ belong to different components of $G - v_n$ for all $i \neq j$.

Consequently, if there exists a vertex $z \notin \{v_n, w'_i\}$ adjacent to $w_i$, then $z$ and $v_n$ belong to different components of $G - \{w_i, w'_i\}$. But now $w_i, w'_i$ is a type-A edge, which is not the case as $a(G) = 0$. Hence $\deg(w_i) = 2$ and $G \cong K_n \circ K_1$. Since $\msd(G) = 4$, $n$ is even, by Remark 3(ii). Therefore $G \in \mathcal{U} \subseteq \mathcal{B}$. Thus (B) holds.

Now suppose $a(G) > 1$. If $G$ has a type-A critical cut-vertex $u$, perform the operation $G \ominus u$; each resulting graph is an msd-4 graph by Proposition 10, and clearly a block graph. Moreover, the copies of $u$ in each graph are $\gamma_{pr}$-critical. Repeat this process until no resulting msd-4 block graph has a type-A critical cut-vertex. Let $G_1, \ldots, G_k$ be the resulting graphs. Then each critical vertex of each $G_i$ is a leaf. If any $G_i$ has a type-A critical edge $uv$, where $u$ is a leaf, perform the operation $G \ominus uv$. Each resulting graph is an msd-4 block graph by Proposition 13. Repeat this process until all resulting graphs $H_j$ satisfy $a(H_j) = 0$. If $H_j$ is a tree, then $H_j \cong S(2, \ldots, 2) \in \mathcal{U}$ by Corollary 5, otherwise $H_j \in \mathcal{U}$ by (B). Now $G$ can be reconstructed by performing the $\oplus$-operations on the $H_j$, hence $G \in \mathcal{B}$, as required. $
$

7. Open Problems

We conclude with a short list of open problems for future consideration.

**Question 1.** Does Theorem 12 hold for all msd-4 graphs?

Define another $\oplus$-operation as follows.

$\oplus_{u,Q}^{a_1Q_1, a_2Q_2}$: Let $G_1$ and $G_2$ be vertex disjoint graphs containing (not necessarily maximal) cliques $Q_1$ and $Q_2$ of equal size, and vertices $u_i \in V(Q_i)$ for $i \in \{1, 2\}$. We denote a graph obtained from $G_1$ and $G_2$ by identifying $Q_1$ and $Q_2$ into one clique $Q$, and $u_1$ and $u_2$ into one vertex $u = u_1 = u_2$, by $G_1 \oplus_{u,Q}^{a_1Q_1, a_2Q_2} G_2$ (or by $G_1 \oplus_{u,Q}^{a_1Q_1, a_2Q_2} G_2$ if $u$ and $Q$ are unimportant).

Note that if the cliques $Q_i$ have order at least three, then identifying the vertices of $Q_i - u_i$ in different ways may yield different graphs. Both operations $\oplus_{u}^{a_1u_1, a_2u_2}$ and $\oplus_{e}^{1e_1, 1e_2}$ are special cases of $\oplus_{u,Q}^{a_1Q_1, a_2Q_2}$.

**Question 2.** Let $G_1$ and $G_2$ be disjoint msd-4 graphs containing cliques $Q_1$ and $Q_2$ of equal size and $\gamma_{pr}(G_1)$-critical vertices $u_i \in V(Q_i)$, $i = 1, 2$. Is it true that for any graph $G = G_1 \oplus_{u,Q}^{a_1Q_1, a_2Q_2} G_2$, $u$ is $\gamma_{pr}(G)$-critical and $\msd_{pr}(G) = 4$?

If $G_1$ and $G_2$ are copies of the msd-4 graph in Figure 5, with $u_i = u$, which is $\gamma_{pr}$-critical, and $Q_i$ is the triangle containing $u$, then both graphs obtainable as $G_1 \oplus_{u,Q}^{a_1Q_1, a_2Q_2} G_2$ are msd-4 graphs having $u$ as critical vertex.
Question 3. Let $G$ be a graph with $\text{msd}_{\text{pr}}(G) = 4$. What is the largest number of edges of $G$ that can be subdivided three times before the paired domination number increases? If this number can be arbitrarily high, what is its ratio to the number of edges of $G$?

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