DISTRIBUTION OF CONTRACTIBLE EDGES AND
THE STRUCTURE OF NONCONTRACTIBLE EDGES
HAVING ENDVERTEXES WITH LARGE DEGREE
IN A 4-CONNECTED GRAPH

SHUNSUKE NAKAMURA
Department of Mathematics
Tokyo University of Science
Kagurazaka 1-3
Shinjuku-Ku, Tokyo 162-8601, Japan

Abstract

Let \( G \) be a 4-connected graph, and let \( E_c(G) \) denote the set of 4-contractible edges of \( G \). We prove results concerning the distribution of edges in \( E_c(G) \). Roughly speaking, we show that there exists a set \( K_0 \) and a mapping \( \varphi : K_0 \to E_c(G) \) such that \( |\varphi^{-1}(e)| \leq 4 \) for each \( e \in E_c(G) \).

Keywords: 4-connected graph, contractible edge, cross free.

2010 Mathematics Subject Classification: 05C40.

1. Introduction

In this paper, we consider only finite undirected simple graphs with no loops and no multiple edges.

Let \( G = (V(G), E(G)) \) be a graph. For \( e \in E(G) \), we let \( V(e) \) denote the set of endvertices of \( e \). For \( x \in V(G) \), \( N_G(x) \) denotes the neighborhood of \( x \) and \( \deg_G(x) \) denotes the degree of \( x \); thus \( \deg_G(x) = |N_G(x)| \). For \( X \subseteq V(G) \), we let \( N_G(X) = \bigcup_{x \in X} N_G(x) \), and the subgraph induced by \( X \) in \( G \) is denoted by \( G[X] \). For an integer \( i \geq 0 \), we let \( V_i(G) \) denote the set of vertices \( x \) of \( G \) with \( \deg_G(x) = i \) and we let \( V_{\geq i}(G) = \bigcup_{j \geq i} V_j(G) \). A subset \( S \) of \( V(G) \) is called a cutset if \( G - S \) is disconnected. A cutset with cardinality \( i \) is simply referred to as an \( i \)-cutset. For an integer \( k \geq 1 \), we say that \( G \) is \( k \)-connected if \( |V(G)| \geq k + 1 \) and \( G \) has no \( (k - 1) \)-cutset.
Let $G$ be a 4-connected graph. For two distinct 4-cutsets $S, T$, we say that $S$ crosses $T$ if $S$ intersects with every component of $G − T$. It is easy to see that $S$ crosses $T$ if and only if $T$ crosses $S$, which is in turn equivalent to saying that $S$ intersects at least two components of $G − T$. Furthermore, we call a family of 4-cutsets $S$ cross free if no two members of $S$ cross. A 4-cutset $S$ of $G$ is said to be trivial if there exists a vertex $z$ of degree 4 such that $N_G(z) = S$; otherwise it is said to be nontrivial. For $e \in E(G)$, we let $G/e$ denote the graph obtained from $G$ by contracting $e$ into one vertex (and replacing each resulting pair of double edges by a simple edge). We say that $e$ is 4-contractible or 4-noncontractible according as $G/e$ is 4-connected or not. A 4-noncontractible edge $e = ab$ is said to be trivially 4-noncontractible if there exists a vertex $z$ of degree 4 such that $za, zb \in E(G)$. We let $E_c(G)$, $E_n(G)$ and $E_{ln}(G)$ denote the set of 4-contractible edges, the set of 4-noncontractible edges and the set of trivially 4-noncontractible edges, respectively. Note that if $|V(G)| \geq 6$, then $e \in E_n(G)$ if and only if there exists a 4-cutset $S$ such that $V(e) \subseteq S$, and $e \in E_{ln}(G)$ if and only if there exists a trivial 4-cutset $S$ such that $V(e) \subseteq S$.

The following theorem concerning the number of 4-contractible edges in a 4-connected graph was proved in [2].

**Theorem A.** If $G$ is a 4-connected graph, then $|E_c(G)| \geq (1/68) \sum_{u \in V(G)} (\deg_G(u) - 4)$.

The coefficient 1/68 in Theorem A seems far from best possible. The purpose of this paper is to prove two results which will be useful in refining Theorem A. Our results can be also seen as a “large-degree version” of the two structure theorems proved in [1] concerning edges not contained in triangles (see Theorems C and D below).

Throughout the rest of this paper, we let $G$ be a 4-connected graph. Set

$$\mathcal{L} = \{(S, A) \mid S \text{ is a 4-cutset, } A \text{ is the union of the vertex set of some components of } G - S, \emptyset \neq A \neq V(G) - S\},$$

$$\mathcal{L}_0 = \{(S, A) \in \mathcal{L} \mid S \text{ is a nontrivial 4-cutset}\}.$$

For $(S, A) \in \mathcal{L}$, we let $\overline{A} = V(G) - S - A$. Thus if $(S, A) \in \mathcal{L}$, then $(S, \overline{A}) \in \mathcal{L}$ and $N_G(A) - A = N_G(\overline{A}) - \overline{A} = S$.

Let $F$ be a subset of $E_n(G) - E_{ln}(G)$. Let $\tilde{V}(G)$ denote the set of those vertices of $G$ which are incident with an edge in $F$, and let $\tilde{G}$ denote the spanning subgraph of $G$ with edge set $F$; that is to say, $\tilde{V}(G) = \bigcup_{e \in F} V(e)$ and $\tilde{G} = (V(G), F)$. Now take $(S_1, A_1), \ldots, (S_k, A_k) \in \mathcal{L}$ so that for each $e \in F$, there exists $S_i$ such that $V(e) \subseteq S_i$. We choose $(S_1, A_1), \ldots, (S_k, A_k)$ so that $k$ is minimum and so that $(|A_1|, \ldots, |A_k|)$ is lexicographically minimum, subject to the condition that $k$ is minimum (thus if $F = \emptyset$, then $k = 0$). Note that the
minimality of $k$ implies that for each $1 \leq i \leq k$, we have $E(G[S_i]) \cap F \neq \emptyset$ and hence $(S_i, A_i) \in \mathcal{L}_0$. Set $\mathcal{S} = \{S_1, \ldots, S_k\}$. Further set
\[
\mathcal{K} = \{(u, S, A) \mid u \in V(G), S \in \mathcal{S}, (S, A) \in \mathcal{L}_0, \text{ there exists } e \in F \text{ such that } u \in V(e) \subseteq S\},
\]
\[
\mathcal{K}^* = \{(u, S, A) \in \mathcal{K} \mid \text{ there is no } (v, T, B) \in \mathcal{K} \text{ with } v = u \text{ and } (T, B) \neq (S, A) \text{ such that } B \subseteq A\}.
\]
Moreover let $\mathcal{K}_0$ be the set of those members $(u, S, A) \in \mathcal{K}^*$ which satisfy one of the following two conditions:
(1) $\deg_G(u) \geq 5$; or
(2) $\deg_G(u) = 4$, $|N_G(u) \cap A| = 1$ and, if we write $N_G(u) \cap A = \{a\}$, then $ua \in E_c(G)$.

We say that $F$ is admissible if the following statement is true (note that this definition implies that if $F = \emptyset$, then $F$ is admissible).

**Statement B.** Let $uv \in F$, and let $S$ be a 4-cutset with $u, v \in S$, and let $A$ be the vertex set of a component of $G - S$. Then there exists $e \in E_c(G)$ such that either $e$ is incident with $u$ or there exists $a \in N_G(u) \cap (S \cup A) \cap V_4(G)$ such that $e$ is incident with $a$.

Now we let $\tilde{E}(G)$ denote the set of those edges of a 4-connected graph $G$ which are not contained in a triangle. The following result appears as Theorem 1 in [1].

**Theorem C.** The set $\tilde{E}(G) \cap E_n(G)$ is admissible.

Let $L$ be the set of edges $e$ such that both endvertices of $e$ have degree 4. In this paper, we prove the following theorem.

**Theorem 1.** Let $F = E_n(G) - E_{tn}(G) - L$. Let $\mathcal{S}$ be as above, and suppose that $\mathcal{S}$ is cross free. Then $F$ is admissible.

Note that in the case where $F = \tilde{E}(G) \cap E_n(G)$, we can show that $\mathcal{S}$ is cross free (see Claim 4.1 in [1]), and this is why we do not need the assumption that $\mathcal{S}$ is cross free in Theorem C.

The following theorem appears as Theorem 2 in [1].

**Theorem D.** Let $\mathcal{K}_0$ be as above with $F = \tilde{E}(G) \cap E_n(G)$. Then we can assign to each $(u, S, A) \in \mathcal{K}_0$ a 4-contractible edge $\varphi(u, S, A)$ having the property stated in Statement B, so that for each $e \in E_c(G)$ there are at most two members $(u, S, A)$ of $\mathcal{K}_0$ such that $\varphi(u, S, A) = e$. 
The following theorem is our main result.

**Theorem 2.** Let $S$ and $K_0$ be as above with $F = E_n(G) - E_{tn}(G) - L$, and suppose that $S$ is cross free. Then we can assign to each $(u, S, A) \in K_0$ a 4-contractible edge $\varphi(u, S, A)$ having the property stated in Statement B, so that for each $e \in E_c(G)$ there are at most four members $(u, S, A)$ of $K_0$ such that $\varphi(u, S, A) = e$.

We remark that in Theorem 2, situations in which there are three or four members $(u, S, A)$ of $K_0$ such that $\varphi(u, S, A) = e$ are rather limited (see Claim 4.17).

Recall that Theorems 1 and 2 will be useful in refining Theorem A. The reasons are as follows. Let $k$ be a maximum value with $|V_{\leq 5}(G)| \geq k \sum_{u \in V(G)}(\deg_G(u) - 4)$ for a 4-connected graph $G$. Note that we know that $1/68 \leq k \leq 1/13$, and hence assume $1/68 \leq k \leq 1/13$ throughout the rest of this argument. If $|V_{\leq 5}(G)| = 0$, then the above inequality holds immediately. Thus we now assume that $|V_{\leq 5}(G)| \geq 1$. Let $S$ be as above with $F = E_n(G) - E_{tn}(G) - L$. If $|E_c(G)| < k \sum_{u \in V(G)}(\deg_G(u) - 4)$, then we can show that $S$ is cross free by Theorem 1 in [4]. Suppose that $|E_c(G)| < (1/28) \sum_{u \in V(G)}(\deg_G(u) - 4)$. Then $S$ is cross free by the above argument. Hence we can use Theorem 2, and we can show that $|E_c(G)| \geq (1/28) \sum_{u \in V(G)}(\deg_G(u) - 4)$ by Theorem 2 (the verification of this statement involves lengthy calculations), which is a contradiction. Thus we have $|E_c(G)| \geq (1/28) \sum_{u \in V(G)}(\deg_G(u) - 4)$. However, it is likely that the coefficient $1/28$ can further be improved in view of the fact that situations in which there are three or four members $(u, S, A)$ of $K_0$ such that $\varphi(u, S, A) = e$ are limited. Thus matters concerning refinements of Theorem A will be discussed in a separate paper.

Our notation is standard, and is mostly taken from Diestel [3]. The organization of this paper is as follows. In Section 2, we introduce known results proved in [1], and prove some preliminary results. We prove Theorem 1 in Section 3, and Theorem 2 in Section 4.

2. Preliminaries

Throughout the rest of this paper, we let $G$ denote a 4-connected graph with $F = E_n(G) - E_{tn}(G) - L \neq \emptyset$ (note that in proving Theorems 1 and 2, we may clearly assume $F \neq \emptyset$). Thus $|V(G)| \geq 6$. Also let $L, L_0$ be as in the second paragraph following the statement of Theorem A.

In this section, we state several results which we use in the proof of Theorems 1 and 2.
2.1. Known results

In this subsection, we state results about the distribution of 4-contractible edges. The following lemmas follow from Lemmas 2.2 through 2.13, respectively, in [1].

Lemma 2.1. Let \((S, A), (T, B) \in \mathcal{L}_0\), and suppose that \(S \cap T \neq \emptyset\). Then either \(A \cap B \neq \emptyset\) and \(\overline{A} \cap \overline{B} \neq \emptyset\), or \(A \cap \overline{B} \neq \emptyset\) and \(\overline{A} \cap B \neq \emptyset\).

Lemma 2.2. Let \((S, A), (T, B) \in \mathcal{L}_0\), and suppose that \(A \cap B \neq \emptyset\) and \(\overline{A} \cap \overline{B} \neq \emptyset\). Then \(((S \cap T) \cup (S \cap B) \cup (A \cap T), A \cap B) \in \mathcal{L}_0\) and \(((S \cap T) \cup (S \cap \overline{B}) \cup (\overline{A} \cap T), \overline{A} \cap \overline{B}) \in \mathcal{L}\).

Lemma 2.3. Let \((S, A) \in \mathcal{L}\).

(i) If \(W \subseteq S\) and \(|W| \leq |A|\), then \(|N_G(W) \cap A| \geq |W|\). Further if \(|W| < |A|\) and \(|N_G(W) \cap A| = |W|\), then \(((S - W) \cup (N_G(W) \cap A), A - (N_G(W) \cap A)) \in \mathcal{L}\).

(ii) If \(x \in S\), then \(N_G(x) \cap A \neq \emptyset\). Further if \((S, A) \in \mathcal{L}_0\) and \(|N_G(x) \cap A| = 1\), then \(((S - \{x\}) \cup (N_G(x) \cap A), A - (N_G(x) \cap A)) \in \mathcal{L}\).

Lemma 2.4. Let \(ab \in E(G)\) with \(deg_G(a) = deg_G(b) = 4\). Then \(N_G(a) - \{b\} \neq N_G(b) - \{a\}\).

Lemma 2.5. Let \(u, a, b, w\) be four distinct vertices with \(ua, ab, aw, bw \in E(G)\) and \(deg_G(a) = deg_G(b) = 4\), and write \(N_G(a) = \{u, b, w, x\}\) and \(N_G(b) = \{u, a, w, y\}\). Then \(x \neq y\), and we have \(ax, by \in E_c(G) \cup E_{in}(G)\).

Lemma 2.6. Under the notation of Lemma 2.5, suppose that \(deg_G(u)\), \(deg_G(w) \geq 5\). Then \(ax, by \in E_c(G)\).

Lemma 2.7. Under the notation of Lemma 2.5, suppose that \(deg_G(u) \geq 5\) and \(deg_G(w) = 4\). Then one of the following holds:

(i) \(xw \notin E(G)\) and \(ax \in E_c(G)\); or

(ii) \(yw \notin E(G)\) and \(by \in E_c(G)\).

Lemma 2.8. Let \((P, X) \in \mathcal{L}_0\) and \(u \in P\). Suppose that \(X\) is minimal, subject to the condition that \(u \in P\) (i.e., there is no \((R, Z) \in \mathcal{L}_0\) with \((P, X) \neq (R, Z)\) such that \(u \in R\) and \(Z \subseteq X\)). Then \(ua \in E_c(G) \cup E_{in}(G)\) for each \(a \in N_G(u) \cap X\).

Lemma 2.9. Let \((R, Z) \in \mathcal{L}_0\) and \(a \in R\). Suppose that \(|N_G(a) \cap Z| = 1\), and write \(N_G(a) \cap Z = \{x\}\). Then \(ax \in E_c(G) \cup E_{in}(G)\).

Lemma 2.10. Let \(u, a, b\) be three distinct vertices with \(ua, ub, ab \in E(G)\) and \(deg_G(a) = 4\), and write \(N_G(a) = \{u, b, x, y\}\). Suppose that there exists \((R, Z) \in \mathcal{L}_0\) such that \(u, a \in R\), \(b, y \in Z\) and \(x \in \overline{Z}\). Suppose further that \(Z\) is minimal, subject to the condition that \(u, a \in R\) and \(b \in Z\). Then the following hold.
(i) \( xy \notin E(G) \).
(ii) \( ax \in E_c(G) \cup E_{tn}(G) \).
(iii) \( ay \in E_c(G) \cup E_{tn}(G) \).

**Lemma 2.11.** Under the notation of Lemma 2.10, suppose that \( \deg_G(b) \geq 5 \). Then \( ax \in E_c(G) \) or \( ay \in E_c(G) \).

**Lemma 2.12.** Under the notation of Lemma 2.10, suppose that \( \deg_G(b) \), \( \deg_G(u) \geq 5 \). Then \( ax, ay \in E_c(G) \).

### 2.2. Vertices not contained in \( \tilde{V}(G) \)

Recall that \( F = E_n(G) - E_{tn}(G) - L \) and \( \tilde{V}(G) = \bigcup_{e \in F} V(e) \). In this subsection, we prove results concerning conditions for a vertex not to belong to \( \tilde{V}(G) \).

**Lemma 2.13.** Under the notation of Lemma 2.5, \( a, b \notin \tilde{V}(G) \).

**Proof.** In view of the symmetry of the roles of \( a \) and \( b \), it suffices to prove \( a \notin \tilde{V}(G) \). Suppose that \( a \in \tilde{V}(G) \). Then there exists \( e \in F \) such that \( e \) is incident with \( a \). Since \( au, aw \in E_{tn}(G) \) and \( ab \in L \), \( e \neq au, ab, aw \). Hence \( e = ax \). By Lemma 2.5, we get \( e \in E_c(G) \cup E_{tn}(G) \), a contradiction.

**Lemma 2.14.** Under the notation of Lemma 2.10, suppose that \( \deg_G(u) = 4 \) or \( \deg_G(b) = 4 \). Then \( a \notin \tilde{V}(G) \).

**Proof.** Suppose that \( a \in \tilde{V}(G) \). Then there exists \( e \in F \) such that \( e \) is incident with \( a \). Since \( au, ab \in E_{tn}(G) \cup L \), \( e \neq au, ab \). Consequently \( e = ax \) or \( ay \), which contradicts Lemma 2.10(ii) or (iii).

### 3. Proof of Theorem 1

In the rest of this paper, we establish Theorems 1 and 2 by proving several claims. The proofs of most of the claims in this paper are quite similar to the proofs of the claims in [1] having virtually the same statements. However, considering that we are dealing with \( E_n(G) - E_{tn}(G) - L \) instead of \( \tilde{E}(G) \cap E_n(G) \), we have decided to include the details of the proofs in this paper. In this section, we prove Theorem 1.

#### 3.1. Neighborhood of a vertex of degree 5

In this subsection, we prove that Statement B is true if \( \deg_G(u) \geq 5 \). Specifically, we prove the following proposition in a series of claims.
Proposition 3.1. Let \((P, X) \in \mathcal{L}_0\) and \(u \in P\), and suppose that \(\deg_G(u) \geq 5\). Then one of the following holds:

1. There exists \(a \in N_G(u) \cap X\) such that \(ua \in E_n(G)\); or
2. There exists \(a \in N_G(u) \cap (P \cup X) \cap V_4(G)\) for which there exists \(e \in E_n(G)\) such that \(e\) is incident with \(a\).

Note that Proposition 3.1 implies that in Theorem 1, the assumption that \(S\) is cross free is not necessary for vertices \(u\) with \(\deg_G(u) \geq 5\). Throughout this subsection, let \((P, X), u\) be as in Proposition 3.1. We may assume that \(X\) is minimal, subject to the condition that \(u \in P\) (i.e., there is no \((R, Z) \in \mathcal{L}_0\) with \((R, Z) \neq (P, X)\) such that \(u \in R\) and \(Z \subseteq X\).

The following four claims are virtually the same as Claims 3.2 through 3.5 in [1].

Claim 3.2. Suppose that there exists an edge \(e\) joining a vertex in \(N_G(u) \cap X \cap V_4(G)\) and a vertex \(N_G(u) \cap (P \cup X) \cap V_4(G)\). Suppose that \(e \in E_n(G)\), and write \(e = \{a, b\}\). Then \(a\) or \(b\), say \(a\), satisfies the following conditions.

1. If we write \(N_G(a) = \{u, b, x, y\}\), then \(xy \notin E(G)\).
2. \(a \notin V_5(G)\).
3. There exists \(e' \in E_n(G)\) such that \(e'\) is incident with \(a\).

Proof. If \(ab \in E_{tn}(G)\), then there exists \(w \in V_4(G)\) such that \(wa, wb \in E(G)\), and hence the desired conclusions follows from Lemmas 2.7 and 2.13. Thus we may assume that \(ab \in E_n(G) - E_{tn}(G)\). Then there exists \((R, Z) \in \mathcal{L}_0\) with \(a, b \in R\). We first show that \(u \notin R\). Suppose that \(u \in R\). Then by Lemma 2.1, we may assume \(X \cap Z \neq \emptyset\) and \(X \cap Z \neq \emptyset\). Since \(a, b \in (P \cup X) \cap R\), it follows from Lemma 2.2 that \(((P \cap R) \cup (P \cap Z) \cup (X \cap R), X \cap Z) \in \mathcal{L}_0\), which contradicts the minimality of \(X\). Thus \(u \notin R\). We may assume \(u \in Z\). We may also assume that we have chosen \((R, Z)\) so that \(Z\) is minimal, subject to the condition that \(a, b \in R\) and \(u \in Z\). By Lemma 2.3(ii), we have \(N_G(a) \cap Z \neq \{u\}\) or \(N_G(b) \cap Z \neq \{u\}\).

We may assume \(N_G(a) \cap Z \neq \{u\}\). Since \(N_G(a) \cap Z \neq \emptyset\) by Lemma 2.3(ii), we have \(|N_G(a) \cap Z| = 2\) and \(|N_G(a) \cap Z| = 1\). Write \(N_G(a) \cap Z = \{u, y\}\) and \(N_G(a) \cap \{u\} = \{x\}\). Then \(b, a, u, x, y\) satisfy the assumptions of Lemmas 2.10, 2.11 and 2.14 with the roles of \(b\) and \(u\) replaced by each other. Consequently the desired conclusions follow from (i) of Lemma 2.10 and Lemmas 2.11 and 2.14. □

Claim 3.3. Let \(a \in X\), and suppose that \(ua \in E_n(G)\). Then \(ua \in E_{tn}(G)\).

Proof. This follows from Lemma 2.8. □

Claim 3.4. Suppose that each edge joining \(u\) and a vertex in \(X\) is 4-noncontractible, and that there is no edge which joins a vertex in \(N_G(u) \cap X \cap V_4(G)\) and a vertex in \(N_G(u) \cap (P \cup X) \cap V_4(G)\). Then \(N_G(u) \cap X \cap V_4(G) = \emptyset\).
Proof. Suppose that $N_G(u) \cap X \cap V_4(G) \neq \emptyset$, and take $a \in N_G(u) \cap X \cap V_4(G)$. We have $ua \in Etn(G)$ by Claim 3.3. Hence there exists $b \in V_4(G)$ such that $ub, ab \in E(G)$. From $a \in X$ and $ab \in E(G)$, it follows that $b \in P \cup X$. Thus $ab$ is an edge joining a vertex in $N_G(u) \cap X$ and a vertex in $N_G(u) \cap (P \cup X) \cap V_4(G)$, a contradiction.

Claim 3.5. Suppose that each edge joining $u$ and a vertex in $X$ is $4$-noncontractible, and that there is no edge which joins a vertex in $N_G(u) \cap X \cap V_4(G)$ and a vertex in $N_G(u) \cap (P \cup X) \cap V_4(G)$. Then there exists $a \in N_G(u) \cap P \cap V_4(G)$ and $b \in N_G(u) \cap X$ such that $ab \in E(G)$, $|N_G(a) \cap X| = 2$ and $|N_G(a) \cap X| = 1$.

Proof. Take $z \in N_G(u) \cap X$. Then $uz \in Etn(G)$ by Claim 3.3, and hence there exists $a_za \in V_4(G)$ such that $a_za, aza \in E(G)$. Since $N_G(u) \cap X \cap V_4(G) = \emptyset$ by Claim 3.4, $a_z \in P$. Since $deg_G(a_z) = 4$ and $u \in N_G(a_z) \cap P$, $|N_G(a_z) \cap X| + |N_G(a_z) \cap X| \leq 3$, and hence it follows from Lemma 2.3(ii) that $1 \leq |N_G(a_z) \cap X| \leq 2$. Now by way of contradiction, suppose that the claim is false. Then $|N_G(a_z) \cap X| = 1$, i.e., $N_G(a_z) \cap X = \{z\}$. Since $z \in N_G(u) \cap X$ is arbitrary, this means that $a_y \neq a_z$ for any $y, z \in N_G(u) \cap X$ with $y \neq z$ and if we set $W = \{a_z \mid E \subseteq N_G(u) \cap X\}$, then we have $|W| = |N_G(u) \cap X|$ and $|N_G\{\{u\} \cup W\} \cap X = N_G(u) \cap X$, and hence $|N(\{u\} \cup W) \cap X| = |W| = |\{u\} \cup W| - 1$. In view of Lemma 2.3(i), this implies $|\{u\} \cup W| \geq |X| + 1$, i.e., $|W| \geq |X|$. Again fix $z \in N_G(u) \cap X$. Since $N_G(a_y) \cap X = \{y\}$ for each $y \in (N_G(u) \cap X) - \{z\}$, $N_G(z) \subseteq (P - (W - \{a_z\})) \cup (X - \{z\})$. Consequently $deg_G(z) \leq |P| - |W| + |X| \leq |P| = 4$, which implies $z \in N_G(u) \cap X \cap V_4(G)$. But this contradicts Claim 3.4, completing the proof.

The following claim corresponds to Claim 3.6 in [1].

Claim 3.6. Suppose that each edge joining $u$ and a vertex in $X$ is $4$-noncontractible, and that there is no edge which joins a vertex in $N_G(u) \cap X \cap V_4(G)$ and a vertex in $N_G(u) \cap (P \cup X) \cap V_4(G)$. Further let $a, b$ be as in Claim 3.5, and write $N_G(a) \cap X = \{b, y\}$ and $N_G(a) \cap X = \{x\}$. Then $xy \notin E(G)$, and $ax, ay \in E_G(G)$.

Proof. Note that $deg_G(b) \geq 5$ by Claim 3.4, and $deg_G(u) \geq 5$ by the assumption of Proposition 3.1. Thus the desired conclusions follows from (i) of Lemma 2.10 and Lemma 2.12.

Proposition 3.1 now follows from Claims 3.2 and 3.6.

3.2. Non-crossing 4-cutsets

In this subsection, we complete the proof of Theorem 1. Throughout the rest of this paper, we let $S, \mathcal{K}, \mathcal{K}^*$ and $\mathcal{K}_0$ be as in the paragraph preceding Statement B with $F = E_n(G) - E_{tn}(G) - L$, and suppose that $S$ is cross free.

The following claim immediately follows from the definition of $\mathcal{K}^*$. 
Claim 3.7. Let $u \in \tilde{V}(G)$. Then for each $(u, S, A) \in K$, there exists a member $(v, T, B)$ of $K^*$ with $v = u$ and $B \subseteq A$. In particular, there exist at least two members $(v, T, B)$ of $K^*$ with $v = u$.

The following claim is virtually the same as Claim 4.3 in [1].

Claim 3.8. Let $(u, S, A), (v, T, B) \in K^* \cap \tilde{V}(G)$ with $v = u$ and $(S, A) \neq (T, B)$. Then $(S \cup A) \cap B = A \cap (T \cup B) = \emptyset$.

Proof. If $S = T$, the desired conclusion clearly holds. Thus we may assume that $S \neq T$. Since $S$ is cross-free, we have that $S \cap B = T \cap A = \emptyset$, $S \cap B = T \cap A = \emptyset$, $S \cap B = T \cap A = \emptyset$, or $S \cap B = T \cap A = \emptyset$. Suppose that $S \cap B = T \cap A = \emptyset$. Then since $S \neq T$, we have $A \cap T \neq \emptyset$ and $|(S \cap T) \cup (A \cap T) \cup (S \cap B)| = |T| = |A \cap T| < 4$, and hence $A \cap B = \emptyset$. Since $S \cap B = \emptyset$ and $A \cap T \neq \emptyset$, this implies $B$ is a proper subset of $A$. But since $(u, T, B) \in K$ and $(u, S, A) \in K^*$, this contradicts the definition of $K^*$. If $S \cap B = T \cap A = \emptyset$ or $S \cap B = T \cap A = \emptyset$, then we obtain $B \subseteq A$ or $A \subseteq B$, respectively, and hence we similarly get a contradiction. Thus $S \cap B = T \cap A = \emptyset$. Since $S \neq T$, this also implies $A \cap B = \emptyset$, as desired. 

Recall that $\tilde{G} = (V(G), F)$. The following claim corresponds to Claim 4.4 in [1].

Claim 3.9. Let $u \in \tilde{V}(G)$. Then the following hold.

(i) There exists a member $(v, T, B)$ of $K_0$ with $v = u$.

(ii) Suppose that $\deg_G(u) \geq 5$, or $\deg_{\tilde{G}}(u) \geq 2$, or there exist three members $(v, T, B)$ of $K^*$ with $v = u$. Then for each $(u, S, A) \in K^*$, we have $(u, S, A) \in K_0$. In particular, if $\deg_G(u) = 4$ and $\deg_{\tilde{G}}(u) \geq 2$, then $\deg_G(u) = 2$ and there exist precisely two members $(v, T, B)$ of $K_0$ with $v = u$.

Proof. If $\deg_G(u) \geq 5$, the desired conclusion immediately follows from Claim 3.7 and the definition of $K_0$. Thus we may assume that $\deg_G(u) = 4$. We first prove (i). Thus let $u$ be as in (ii) with $\deg_G(u) = 4$. Then by Lemma 2.3(ii) and Claim 3.8, it follows that $|N_G(u) \cap A| = 1$ for each $(u, S, A) \in K^*$, and that for each $a \in N_G(u) - N_{\tilde{G}}(u)$, there exists $(u, S, A) \in K^*$ such that $a \in A$. Again by Claim 3.8, this implies that for each $(u, S, A) \in K^*$, $N_G(u) \cap S = N_{\tilde{G}}(u) \cap S$. Note that this also implies that if $\deg_G(u) \geq 2$, then we have $\deg_{\tilde{G}}(u) = 2$ and there exist precisely two members $(v, T, B)$ of $K^*$ with $v = u$. Now let $(u, S, A) \in K^*$, and write $N_G(u) \cap A = \{a\}$. To complete the proof of (ii), it suffices to show that $(u, S, A) \in K_0$. Suppose that $(u, S, A) \notin K_0$. Then $ua \in E_{\tilde{G}}(G)$, and hence $ua \in E_{in}(G)$ by Lemma 2.9, which implies that there exists $c \in V_4(G)$ such that $cu, ca \in E(G)$. Since $N_G(u) \cap A = \{a\}$, this forces $c \in S$. But since $uc \in L$, $c \notin N_{\tilde{G}}(u)$, which contradicts the earlier assertion that $N_G(u) \cap S = N_{\tilde{G}}(u) \cap S$. Thus (ii) is proved.
We now prove (i). We may assume that there exists \((u, S, A) \in \mathcal{K}^*\) such that \((u, S, A) \notin \mathcal{K}_0\). Then arguing as above, we see that \(|N_G(u) \cap (S \cup A)| \geq \frac{5}{2}\). We may assume that there exists \((u, T, B) \in \mathcal{K}^*\) with \(B \subseteq \overline{A}\). Then \(|N_G(u) \cap B| = 1\). Write \(N_G(u) \cap B = \{b\}\). Suppose that \((u, T, B) \notin \mathcal{K}_0\). Then there exists \(c' \in V_4(G)\) such that \(c'u, c'b \in E(G)\). This in turn implies \(|N_G(u) \cap A| = 1\). Write \(N_G(u) \cap A = \{a\}\). Then there exists \(c \in V_4(G)\) such that \(cu, ca \in E(G)\). Since \(\deg_G(u) = 4\), \(\deg_G(c) \geq 1\) and \(ab \notin E(G)\), this forces \(c = c'\). But then applying Lemma 2.13 with \(a, b\) replaced by \(u, c\), we obtain \(u \notin \overline{V}(G)\), which contradicts the assumption that \(u \in \overline{V}(G)\). Thus (i) is also proved.

We are now in a position to complete the proof of Theorem 1.

Let \(u, S, A\) be as in Statement B. Then \((S, A) \in \mathcal{L}_0\). Hence if \(\deg_G(u) \geq 5\), then the desired conclusion follows from Proposition 3.1. Thus we may assume \(\deg_G(u) = 4\). But then from Claim 3.9(i) and the definition of \(\mathcal{K}_0\), we see that there exists \(e \in E_G(G)\) such that \(e\) is incident with \(u\). Consequently \(F = E_n(G) - E_{tn}(G) - L\) is admissible, as desired.

4. Proof of Theorem 2

In this section, we prove Theorem 2. We continue with the notation of Subsection 3.2. In particular, we suppose that \(S\) is cross free, which is the assumption of Theorem 2.

4.1. Definition of \(\lambda(u, S, A), \alpha(u, S, A)\) and \(\varphi(u, S, A)\)

In this subsection, to each \((u, S, A) \in \mathcal{K}_0\), we assign an edge \(\lambda(u, S, A)\), and an endvertex \(\alpha(u, S, A)\) of \(\lambda(u, S, A)\), and a 4-contractible edge \(\varphi(u, S, A)\) incident with \(\alpha(u, S, A)\). The following claim corresponds to Claim 5.1 in [1].

Claim 4.1. Let \((u, S, A) \in \mathcal{K}_0\), and set \(W = \{z \in S - \{u\} \cap N_G(u) \mid |N_G(z) \cap A| = 1\}\). Then \((S - W) \cup (N_G(W) \cap A), A - (N_G(W) \cap A)) \in \mathcal{L}_0\).

Proof. By the definition of \(\mathcal{K}\), there exists \(e \in F\) such that \(u \in V(e) \subseteq S\). Hence \(W \subseteq S - V(e)\), which implies \(|W| \leq 2\). On the other hand, since \((S, A) \in \mathcal{L}_0\), \(|A| \geq 2\). Thus \(|W| \leq |A|\). Suppose that \(|W| = |A|\). Then \(|W| = |A| = 2\).

By Lemma 2.3(i), \(N_G(\{x, z\}) \cap A = A\) for each \(x \in V(e)\) and \(z \in W\). Since we also have \(N_G(W) \cap A = A\) by Lemma 2.3(i) and since \(|N_G(z) \cap A| = 1\) for each \(z \in W\), this means that \(N_G(x) \cap A = A\) for each \(x \in V(e)\). Consequently \(\deg_G(a) = 4\) and \(V(e) \subseteq N_G(a)\) for each \(a \in A\), which implies \(e \in E_{tn}(G)\), a contradiction. Thus \(|W| < |A|\). Therefore it follows from Lemma 2.3(i) that \(((S - W) \cup (N_G(W) \cap A), A - (N_G(W) \cap A)) \in \mathcal{L}\), which implies the desired conclusion because \(V(e) \subseteq S - W\).
Now let \((u, S, A) \in K_0\), and let \(W\) be as in Claim 4.1. We let \((P_{u,S,A}, X_{u,S,A})\) be a member of \(L_0\) with \(u \in P_{u,S,A}\) and \(X_{u,S,A} \subseteq A - (N_G(W) \cap A)\) such that \(X_{u,S,A}\) is minimal, i.e., there is no \((R, Z) \in L_0\) with \((R, Z) \neq (P_{u,S,A}, X_{u,S,A})\) such that \(u \in R\) and \(Z \subseteq X_{u,S,A}\). We remark that we do not require that there should exist an edge \(e \in E_u(G)\) with \(u \in V(e) \subseteq P_{u,S,A}\). The following claim immediately follows from the definition of \((P_{u,S,A}, X_{u,S,A})\).

**Claim 4.2.** Let \((u, S, A) \in K_0\). Let \(z \in S - \{u\} - N_G(u)\) and suppose that \(|N_G(z) \cap A| = 1\). Then \(z \notin P_{u,S,A}\).

Let again \((u, S, A) \in K_0\), and let \((P, X) = (P_{u,S,A}, X_{u,S,A})\) be as above. We define the type of \((u, S, A)\) as follows: \((u, S, A)\) is of type 1 if there exists a 4-contractible edge joining \(u\) and a vertex in \(X\); \((u, S, A)\) is of type 2 if it is not of type 1 and there exists a 4-contractible edge joining a vertex in \(N_G(u) \cap X \cap V_i(G)\) and a vertex in \(N_G(u) \cap (P \cup X) \cap V_i(G)\); \((u, S, A)\) is of type 3 if it is not of type 1 or 2 but there exists an edge joining a vertex in \(N_G(u) \cap X \cap V_i(G)\) and a vertex in \(N_G(u) \cap (P \cup X) \cap V_i(G)\); \((u, S, A)\) is of type 4 if it is not of type \(i\) for any \(i = 1, 2, 3\). We let \(K_i\) denote the set of those members of \(K_0\) which are the type \(i\) \((i = 1, 2, 3, 4)\). The following claim, which will be used implicitly throughout the rest of this paper, is virtually the same as Claim 5.3 in [1].

**Claim 4.3.** Let \((u, S, A) \in K_0 - K_1\). Then \(deg_G(u) \geq 5\).

**Proof.** Suppose that \(deg_G(u) = 4\). Then by the definition of \(K_0\), \(|N_G(u) \cap A| = 1\) and, if we write \(N_G(u) \cap A = \{a\}\), then \(ua \in E_u(G)\). By Lemma 2.3(ii), \(a \in X\). Consequently \((u, S, A) \in K_1\) by definition, which contradicts the assumption that \((u, S, A) \in K_0 - K_1\).

We first define \(\lambda(u, S, A)\). If \((u, S, A) \in K_1\), let \(\lambda(u, S, A)\) be a 4-contractible edge joining \(u\) and a vertex in \(X\); if \((u, S, A) \in K_2\), let \(\lambda(u, S, A)\) be a 4-contractible edge joining a vertex in \(N_G(u) \cap X \cap V_i(G)\) and a vertex in \(N_G(u) \cap (P \cup X) \cap V_i(G)\); if \((u, S, A) \in K_3\), let \(\lambda(u, S, A)\) be an edge joining a vertex in \(N_G(u) \cap X \cap V_i(G)\) and a vertex in \(N_G(u) \cap (P \cup X) \cap V_i(G)\); if \((u, S, A) \in K_4\), let \(\lambda(u, S, A) = ab\) where \(a, b\) are as in Claim 3.5. The following claim follows from the definition of \(\lambda(u, S, A)\).

**Claim 4.4.** Let \(2 \leq i, j \leq 4\) with \(i \neq j\), and let \((u_1, S_1, A_1) \in K_i\) and \((u_2, S_2, A_2) \in K_j\). Then \(\lambda(u_1, S_1, A_1) \neq \lambda(u_2, S_2, A_2)\).

The following claims are virtually the same as Claims 5.5 and 5.6, respectively, in [1].

**Claim 4.5.** Let \((u_1, S_1, A_1), (u_2, S_2, A_2) \in K_0\) with \(u_1 = u_2\) and \((S_1, A_1) \neq (S_2, A_2)\). Then \(\lambda(u_1, S_1, A_1) \neq \lambda(u_2, S_2, A_2)\).
**Proof.** By Claim 3.8, $A_1 \cap A_2 = \emptyset$. Hence $X_{u_1,S_1,A_1} \cap X_{u_2,S_2,A_2} \subseteq A_1 \cap A_2 = \emptyset$. Since at least one of the endvertices of $\lambda(u_j, S_j, A_j)$ is in $X_{u_1,S_1,A_1}$, this implies $\lambda(u_1, S_1, A_1) \neq \lambda(u_2, S_2, A_2).$

**Claim 4.6.** Let $e$ be an edge joining two vertices of degree 4. Then there exist at most two members $(u, S, A)$ of $K_2 \cup K_3$ for which $\lambda(u, S, A) = e$.

**Proof.** Suppose that there exist three members $(u_j, S_j, A_j)$ ($1 \leq j \leq 3$) of $K_2 \cup K_3$ such that $\lambda(u_j, S_j, A_j) = e$. By Claim 4.5, the $u_j$ are all distinct. But this contradicts Lemma 2.4.

We prove two more claims concerning properties of $\lambda(u, S, A)$. The following claim corresponds to Claim 6.1 in [1].

**Claim 4.7.** Let $(u, S, A), (v, T, B) \in K_0 - K_1$ with $u = v$ and $(S, A) \neq (T, B)$. Then $V(\lambda(u, S, A)) \cap V(\lambda(v, T, B)) \cap V_4(G) = \emptyset$.

**Proof.** Suppose that $V(\lambda(u, S, A)) \cap V(\lambda(v, T, B)) \cap V_4(G) \neq \emptyset$, and let $a \in V(\lambda(u, S, A)) \cap V(\lambda(v, T, B)) \cap V_4(G)$, and let $(P, X) = (P_{u,S,A}, X_{u,S,A})$. Then $a \in P \cup X \subseteq S \cup A$. Similarly $a \in T \cup B$. Hence $a \in (S \cup A) \cap (T \cup B) \subseteq S \cap T$ by Claim 3.8. Since $\deg_G(a) = 4$ and $u \in N_G(a) \cap S \cap T$, $|N_G(a) \cap (A \cup B)| \leq 3$. Since $A \cap B = \emptyset$ by Claim 3.8, this together with Lemma 2.3(ii) implies that we have $|N_G(a) \cap A| = 1$ or $|N_G(a) \cap B| = 1$. We may assume $|N_G(a) \cap A| = 1$. If $(u, S, A) \in K_4$, then by the definition of $\lambda(u, S, A)$, $a$ coincides with the vertex $a$ in Claim 3.5, and hence $|N_G(a) \cap A| \geq |N_G(a) \cap X| = 2$ by Claim 3.5, a contradiction. Thus $(u, S, A) \in K_2 \cup K_3$. Consequently $ua \in E_{tn}(G)$ by the definition of types 2 and 3, and hence $a \notin N_G(u)$. By Claim 4.2, this implies $a \notin P$, which contradicts the fact that $a \in (P \cup X) \cap S \subseteq P$.

The following claim is virtually the same as Claim 6.2 in [1].

**Claim 4.8.** Let $(u, S, A), (v, T, B) \in K_4$ with $(u, S, A) \neq (v, T, B)$. Then $\lambda(u, S, A) \neq \lambda(v, T, B)$.

**Proof.** Suppose that $\lambda(u, S, A) = \lambda(v, T, B)$. Let $(P, X) = (P_{u,S,A}, X_{u,S,A})$, and let $a, b, x, y$ be as in Claims 3.5 and 3.6. Then $\lambda(u, S, A) = \lambda(v, T, B) = ab$, and hence $v \in N_G(a) \cap N_G(b)$. In particular, $v \in N_G(a) - \{b\} = \{u, x, y\}$. Since we get $xb \notin E(G)$ from $x \in X$ and $b \in X$, $v \neq x$. We also have $v \neq u$ by Claim 4.5. Thus $v = y$, and hence $y, a \in P_{v,T,B}$. Consequently $ya \in E_{tn}(G)$, which contradicts Claim 3.6.

We now define $\alpha(u, S, A)$. If $(u, S, A) \in K_1$, let $\alpha(u, S, A) = u$. Now assume $(u, S, A) \in K_2$. In this case, we let $\alpha(u, S, A)$ be an endvertex of $\lambda(u, S, A)$. If $\lambda(u, S, A)$ has an endvertex in $P$ and there is no $(w, R, Z) \in K_2$ with $(w, R, Z) \neq (u, S, A)$ such that $\lambda(w, R, Z) = \lambda(u, S, A)$, then we let $\alpha(u, S, A)$ be the endvertex
of $\lambda(u, S, A)$ in $X$. Next assume $(u, S, A) \in K_3$. In this case, we let $\alpha(u, S, A)$ be an endvertex of $\lambda(u, S, A)$ which satisfies (ii) and (iii) of Claim 3.2. If there is no $(w, R, Z) \in K_3$ with $(w, R, Z) \neq (u, S, A)$ such that $\lambda(w, R, Z) = \lambda(u, S, A)$, then we choose $\alpha(u, S, A)$ so that it also satisfies (i) of Claim 3.2. Finally, if $(u, S, A) \in K_4$, let $\alpha(u, S, A) = a$, where $a$ is as in Claim 3.5. Note that if $(u_1, S_1, A_1), (u_2, S_2, A_2) \in K_3$ with $(u_1, S_1, A_1) \neq (u_2, S_2, A_2)$ and $\lambda(u_1, S_1, A_1) = \lambda(u_2, S_2, A_2)$, then $u_1 \neq u_2$ by Claim 4.5, and hence it follows from Lemmas 2.6 and 2.13 that both endvertices of $\lambda(u_1, S_1, A_1)$ satisfy (ii) and (iii) of Claim 3.2. Thus in view of Claim 4.6, we can define $\alpha(u, S, A)$ so that the following claim holds.

Claim 4.9. Let $(u_1, S_1, A_1), (u_2, S_2, A_2) \in K_2 \cup K_3$ with $(u_1, S_1, A_1) \neq (u_2, S_2, A_2)$ and $\lambda(u_1, S_1, A_1) = \lambda(u_2, S_2, A_2)$. Then $\alpha(u_1, S_1, A_1) \neq \alpha(u_2, S_2, A_2)$.

Finally we define $\varphi(u, S, A)$. If $(u, S, A) \in K_1 \cup K_2$, simply let $\varphi(u, S, A) = \lambda(u, S, A)$; if $(u, S, A) \in K_3$, let $\varphi(u, S, A)$ be a 4-contractible edge incident with $\alpha(u, S, A)$, whose existence is guaranteed by Claim 3.2(iii) or Lemma 2.6 (it is possible that the other endvertex of $\varphi(u, S, A)$ lies in $\overline{X}$); if $(u, S, A) \in K_4$, let $\varphi(u, S, A) = ax$, where $a, x$ are as in Claim 3.6.

4.2. Properties of $\varphi(u, S, A)$

In this subsection, we complete the proof of Theorem 2 by showing that for any pair $(e, a)$ of a 4-contractible edge $e$ and an endvertex $a$ of $e$, there are at most two members $(u, S, A)$ of $K_0$ for which $(\varphi(u, S, A), \alpha(u, S, A)) = (e, a)$. The first two claims immediately follow from Claims 4.5 and 4.9, respectively.

Claim 4.10. Let $(u, S, A), (v, T, B) \in K_1$ with $(u, S, A) \neq (v, T, B)$. Then $(\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))$.

Claim 4.11. Let $(u, S, A), (v, T, B) \in K_2$ with $(u, S, A) \neq (v, T, B)$. Then $(\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))$.

The following claims are virtually the same as Claims 7.3 and 7.4, respectively, in [1].

Claim 4.12. Let $(u, S, A) \in K_2$ and $(v, T, B) \in K_1$, and suppose that $\varphi(u, S, A) = \varphi(v, T, B)$. Then $v \in P_{u, S, A}$, and there is no $(w, R, Z) \in K_2$ with $(w, R, Z) \neq (u, S, A)$ such that $\varphi(w, R, Z) = \varphi(u, S, A)$.

Proof. Write $\varphi(u, S, A) = \varphi(v, T, B) = vb$. Also let $vz$ be an edge in $F$ such that $v, z \in T$. Let $(P, X) = (P_{u, S, A}, X_{u, S, A})$. Suppose that $v \in X$. Then since $vz \in E(G)$, we have $z \in P \cup X$, and hence $z \in (P \cup X) \cap T$. Since $\deg_G(v) = 4$, it follows from the definition of $K_0$ that $N_G(v) \cap B = \{b\}$. Since $u \in N_G(v) \cap N_G(b)$, this implies $u \in T$, and hence $u \in P \cap T$. Thus by Lemmas 2.1 and 2.2, there
exists a 4-cutset $U$ with $U \supseteq (P \cup X) \cap T$ such that $G - U$ has a component $H$ with $V(H) \subseteq X - (X \cap T) \subseteq X - \{v\}$. But then since $v \in X \cap T \subseteq U$, $z \in (P \cup X) \cap T \subseteq U$ and $vz \in F \subseteq E_\alpha(G) - E_{\alpha}(G)$, $U$ is a nontrivial 4-cutset, which contradicts the minimality of $X$ because $u \in P \cap T \subseteq U$ (see the remark made in the paragraph preceding Claim 4.2). Thus $v \in P$. Now suppose that there exists $(w, R, Z) \in K_2$ with $(w, R, Z) \neq (u, S, A)$ such that $\varphi(w, R, Z) = \varphi(u, S, A)$. Then $w \neq u$ by Claim 4.5. Hence applying Lemma 2.13 with $a = v$, we see that $v \notin \tilde{\varphi}(G)$. But this contradicts the assumption that $(v, T, B) \in K_1$. Thus there no such $(w, R, Z)$.

**Claim 4.13.** Let $(u, S, A) \in K_2$ and $(v, T, B) \in K_1$. Then $(\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))$.

**Proof.** We may assume $\varphi(u, S, A) = \varphi(v, T, B)$. Write $\varphi(u, S, A) = vb$. We have $\alpha(v, T, B) = v$ by definition. On the other hand, in view of Claim 4.12, $\alpha(u, S, A) = b$ by the choice of $\alpha(u, S, A)$ described in Subsection 4.1. Thus $\alpha(u, S, A) \neq \alpha(v, T, B)$.

The following claim corresponds to Claim 7.5 in [1].

**Claim 4.14.** Let $(u, S, A) \in K_3$ and $(v, T, B) \in K_1$. Then $\alpha(u, S, A) \neq \alpha(v, T, B)$.

**Proof.** By Lemma 2.13 and Claim 3.2, $\alpha(u, S, A) \notin \tilde{\varphi}(G)$. On the other hand, $\alpha(v, T, B) = v \in \tilde{\varphi}(G)$. Thus $\alpha(u, S, A) \neq \alpha(v, T, B)$.

The following claims are virtually the same as Claims 7.6 and 7.7, respectively, in [1].

**Claim 4.15.** Let $(u, S, A) \in K_3 \cup K_4$ and $(v, T, B) \in K_2$. Then $\varphi(u, S, A) \neq \varphi(v, T, B)$.

**Proof.** Suppose that $\varphi(u, S, A) = \varphi(v, T, B)$. Write $\lambda(u, S, A) = ab$ with $\alpha(u, S, A) = a$. Then $\deg_G(a) = 4$. Also write $\varphi(v, T, B) = ax$. Then $v \in N_G(a) \cap N_G(x)$. First assume that there exists $(w, R, Z) \in K_3$ with $(w, R, Z) \neq (u, S, A)$ such that $\lambda(w, R, Z) = \lambda(u, S, A)$. Then $\deg_G(b) = 4$. By Claim 4.5, $w \neq u$. Thus $N_G(a) = \{u, b, w, x\}$. Since $\deg_G(v) \geq 5$ and $\deg_G(b) = 4$, $v \neq b$. Since $v \in N_G(a) \cap N_G(x) \subseteq N_G(a) - \{x\}$, this implies $v = u$ or $w$. On the other hand, $\deg_G(a) = 4$ and $a$ is a common endvertex of $\varphi(v, T, B)$ and $\lambda(u, S, A) = \lambda(w, R, Z)$. Thus $\varphi(v, T, B) = \lambda(w, T, B)$, this contradicts Claim 4.7. Next assume that there is no such $(w, R, Z)$. Write $N_G(a) = \{u, b, x, y\}$. Suppose that $(u, S, A) \in K_3$. Then $xy \notin E(G)$ by the choice of $\alpha(u, S, A)$, which implies $v \neq y$. Also we have $\deg_G(b) = 4$ by the definition of $\lambda(u, S, A)$, which implies $v \neq b$. Consequently, $v = u$, which contradicts Claim 4.7. Suppose that $(u, S, A) \in K_4$. By Claim 3.6, $xy \notin E(G)$, which implies $v \neq y$. Again by
Claim 3.6, \( xb \notin E(G) \), and hence \( v \neq b \). Thus \( v = u \), which again contradicts Claim 4.7.

**Claim 4.16.** Let \((u, S, A), (v, T, B) \in K_3\) with \((u, S, A) \neq (v, T, B)\). Then \((\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))\).

**Proof.** Suppose that \((\varphi(u, S, A), \alpha(u, S, A)) = (\varphi(v, T, B), \alpha(v, T, B))\). Write \( \lambda(u, S, A) = ab \), \( \varphi(u, S, A) = \varphi(v, T, B) = ax \), and \( N_G(a) = \{u, b, x, y\} \). Then \( \alpha(u, S, A) = \alpha(v, T, B) = a \), and \( v \in N_G(a) - \{x\} \). Since \( \deg_G(a) = 4 \), \( v \neq b \). Thus \( v = y \), and hence \( \lambda(v, T, B) = au \) or \( ab \). On the other hand, since \( \deg_G(u) \geq 5 \), \( \lambda(v, T, B) \neq au \). Consequently \( \lambda(v, T, B) = ab \), which contradicts Claim 4.9.

The following claim shows that in most cases, we have \((\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))\) for \((u, S, A), (v, T, B) \in K_0\) with \((u, S, A) \neq (v, T, B)\).

**Claim 4.17.** The following hold.

(i) Let \((u, S, A), (v, T, B) \in K_0 - K_4\) with \((u, S, A) \neq (v, T, B)\). Then \((\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))\).

(ii) Let \((u, S, A) \in K_4\), \((v, T, B) \in K_0 - K_1\) with \((u, S, A) \neq (v, T, B)\). Then \((\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))\).

**Proof.** Statement (i) follows from Claims 4.10, 4.11 and 4.13 through 4.16. Thus we prove (ii). By Claim 4.15, we may assume that \((v, T, B) \in K_3 \cup K_4\). Suppose that \((\varphi(u, S, A), \alpha(u, S, A)) = (\varphi(v, T, B), \alpha(v, T, B))\). Let \((P, X) = (P_{u,S,A}, X_{u,S,A})\) and let \(a, b, x, y\) be as in Claims 3.5 and 3.6. Also let \((Q, Y) = (P_{v,T,B}, X_{v,T,B})\). Note that \( N_G(a) = \{u, b, x, y\} \) and \( v \in N_G(a) - \{x\} \). If \( v = y \), then \( a, y \in Q \), and hence \( ay \in E_n(G) \), which contradicts Claim 3.6. Thus \( v \neq y \). We also have \( v \neq u \) by Claim 4.7. Consequendy \( v = b \), which implies \( \lambda(v, T, B) = au \) or \( ay \). Suppose that \((v, T, B) \in K_3\). Then since \( V(\lambda(v, T, B)) \subseteq V_4(G) \), \( \lambda(v, T, B) = ay \). But then \( ay \in E_n(G) \) by the definition of \( K_3 \), which contradicts Claim 3.6. Thus we have \((v, T, B) \in K_4\). Applying Claim 3.6 to \((Q, Y)\), we now obtain \( b, a \in Q \), \( x \in \overline{Y} \) and \( y, u \in Y \). In particular, \( xu \notin E(G) \). Set \( U = (P \cap Q) \cup (P \cap Y) \cup (X \cap Q) \). Since \( y \in X \cap Y \) and \( x \in \overline{X} \in \overline{Y} \), it follows from Lemma 2.2 that \((U, X \cap Y) \in \mathcal{L} \). Since \( u \in P \cap Y \subseteq U \), it follows from the minimality of \( X \) that \((U, X \cap Y) \notin \mathcal{L}_0 \), i.e., \( U \) is a trivial 4-cutset. There exists \( c \in V_4(G) \) such that \( N_G(c) = U \). Since \( a, b, u \in U \), \( c \in N_G(a) - \{b, u\} = \{x, y\} \). On the other hand, since \( xu \notin E(G) \), \( c \neq x \). Consequently \( c = y \), which implies \( y \in N_G(u) \cap X \in V_4(G) \). But since \((u, S, A) \in K_4\), this contradicts Claim 3.4.

The following claim, together with Claim 4.17, shows that for each \( e \in E_n(G) \) and for each endvertex \( a \) of \( e \), there are at most two members \((u, S, A)\) of \( K_0 \) such that \((\varphi(u, S, A), \alpha(u, S, A)) = (e, a)\).
Claim 4.18. Let $(u, S, A) \in K_4$, $(v, T, B) \in K_1$ with $(\varphi(u, S, A), \alpha(u, S, A)) = (\varphi(v, T, B), \alpha(v, T, B))$. Then $(\varphi(w, R, Z), \alpha(w, R, Z)) \neq (\varphi(u, S, A), \alpha(u, S, A))$ for $(w, R, Z) \in K_0 - \{(u, S, A), (v, T, B)\}$.

Proof. Suppose that there exists $(w, R, Z) \in K_0 - \{(u, S, A), (v, T, B)\}$ such that $(\varphi(w, R, Z), \alpha(w, R, Z)) = (\varphi(u, S, A), \alpha(u, S, A))$.

By Claim 4.17(ii), we have $(w, R, Z) \in K_1 - \{(v, T, B)\}$. On the other hand, since $(\varphi(v, T, B), \alpha(v, T, B)) = (\varphi(u, S, A), \alpha(u, S, A)) = (\varphi(w, R, Z), \alpha(w, R, Z))$, it follows from Claim 4.17(i) that $(w, R, Z) \in K_4 - \{(u, S, A)\}$, which is a contradiction.

In view of the remark made before the statement of Claim 4.18, it follows from Claims 4.17 and 4.18 that for each $e \in E_c(G)$, there are at most four members $(u, S, A)$ of $K_0$ such that $\varphi(u, S, A) = e$. This completes the proof of Theorem 2.

Acknowledgments

I would like to thank Professor Yoshimi Egawa and Professor Keiko Kotani for the help they gave to me during the preparation of this paper. I would also like to thank the referees for helpful suggestions that have greatly improved the accuracy and presentation of this paper.

References


Received 20 March 2018
Revised 4 February 2019
Accepted 6 May 2019