ON EDGE $H$-IRREGULARITY STRENGTHS OF SOME GRAPHS

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Abstract

For a graph $G$ an edge-covering of $G$ is a family of subgraphs $H_1, H_2, \ldots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs $H_i$, $i = 1, 2, \ldots, t$. In this case we say that $G$ admits an $(H_1, H_2, \ldots, H_t)$-edge-covering. An $H$-covering of graph $G$ is an $(H_1, H_2, \ldots, H_t)$-edge-covering in which every subgraph $H_i$ is isomorphic to a given graph $H$.

Let $G$ be a graph admitting $H$-covering. An edge $k$-labeling $\alpha : E(G) \to \{1, 2, \ldots, k\}$ is called an $H$-irregular edge $k$-labeling of the graph $G$ if for every two different subgraphs $H'$ and $H''$ isomorphic to $H$ their weights...
wt_\alpha(H') and wt_\alpha(H'') are distinct. The weight of a subgraph H under an edge k-labeling \alpha is the sum of labels of edges belonging to H. The edge H-irregularity strength of a graph G, denoted by \text{ehs}(G, H), is the smallest integer k such that G has an H-irregular edge k-labeling.

In this paper we determine the exact values of \text{ehs}(G, H) for prisms, antiprisms, triangular ladders, diagonal ladders, wheels and gear graphs. Moreover the subgraph H is isomorphic to only \(C_4\), \(C_3\) and \(K_4\).

**Keywords:** H-irregular edge labeling, edge H-irregularity strength, prism, antiprism, triangular ladder, diagonal ladder, wheel, gear graph.

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1. Introduction

Consider a simple and finite graph \(G = (V, E)\) of order at least 2. An edge k-labeling is a function \(\alpha : E(G) \rightarrow \{1, 2, \ldots, k\}\), where k is a positive integer. Then the associated weight of a vertex \(x \in V(G)\) is \(w_\alpha(x) = \sum_{xy \in E(G)} \alpha(xy)\), where the sum is taken over all edges incident to x. Such a labeling \(\alpha\) is called irregular if the obtained weights of all vertices are different. The smallest positive integer k for which there exists an irregular labeling of \(G\) is called the irregularity strength of \(G\) and is denoted by \(s(G)\). If it does not exist, then we write \(s(G) = \infty\). One can easily see that \(s(G) < \infty\) if and only if \(G\) contains no isolated edges and has at most one isolated vertex.

The notion of the irregularity strength was firstly introduced by Chartrand et al. in [7]. Some results on the irregularity strength can be found in [2, 3, 5, 6, 8, 9, 11–14].

A vertex k-labeling \(\beta : V(G) \rightarrow \{1, 2, \ldots, k\}\) is called an edge irregular k-labeling of the graph G if the weights \(w_\beta(xy) \neq w_\beta(x'y')\) for every two distinct edges \(xy\) and \(x'y'\), where the weight of an edge \(xy \in E(G)\) is \(w_\beta(xy) = \beta(x) + \beta(y)\). The minimum k for which a graph G admits an edge irregular k-labeling is called the edge irregularity strength of G, denoted by \(\text{es}(G)\). The notion of the edge irregularity strength was defined by Ahmad et al. in [1].

A family of subgraphs \(H_1, H_2, \ldots, H_t\) is said to be an edge-covering of G if each edge of \(E(G)\) belongs to at least one of the subgraphs \(H_i\), \(i = 1, 2, \ldots, t\). In this case we say that G admits an \((H_1, H_2, \ldots, H_t)-(edge)\)-covering. If every subgraph \(H_i\), \(i = 1, 2, \ldots, t\), is isomorphic to a given graph H, then the graph G admits an H-covering.

Motivated by the irregularity strength and the edge irregularity strength of a graph G Ashraf et al. in [4] introduced a new parameter, edge H-irregularity strength, as a natural extension of the parameters \(s(G)\) and \(\text{es}(G)\). Let \(G\) be a graph admitting H-covering. An edge k-labeling \(\alpha\) is called an H-irregular
edge $k$-labeling of the graph $G$ if for every two different subgraphs $H'$ and $H''$ isomorphic to $H$ we have

$$wt_{\alpha}(H') = \sum_{e \in E(H')} \alpha(e) \neq \sum_{e \in E(H'')} \alpha(e) = wt_{\alpha}(H'').$$

The edge $H$-irregularity strength of a graph $G$, denoted by $ehs(G, H)$, is the smallest integer $k$ for which $G$ has an $H$-irregular edge $k$-labeling.

Next theorem proved in [4] gives the lower bound of the edge $H$-irregularity strength of a graph $G$.

**Theorem 1** [4]. Let $G$ be a graph admitting an $H$-covering and $t$ is the number of all the subgraphs isomorphic with $H$. Then

$$ehs(G, H) \geq \left\lceil 1 + \frac{t - 1}{|E(H)|} \right\rceil.$$

Note that the parameter $t$ is the number of all subgraphs of $G$ isomorphic to $H$. In this paper we determine exact values of the edge $H$-irregularity strength for prisms, antiprisms, triangular ladders, diagonal ladders, wheels and gear graphs for some $H$. Moreover the subgraph $H$ is isomorphic to only $C_4$, $C_3$ and $K_4$.

2. **Prism and Antiprism**

The prism $D_n$ can be defined as the Cartesian product $C_n \square P_2$ of a cycle on $n$ vertices with a path on 2 vertices. Let $V(C_n \square P_2) = \{x_i, y_i : 1 \leq i \leq n\}$ be the vertex set and $E(C_n \square P_2) = \{x_ix_{i+1}, y_iy_{i+1} : 1 \leq i \leq n\} \cup \{x_iy_i : 1 \leq i \leq n\}$ be the edge set, where the indices are taken modulo $n$. Hence, the graph $D_n$ has $2n$ vertices and $3n$ edges.

**Theorem 2.** Let $D_n = C_n \square P_2$, $n \geq 3$, $n \neq 4$, be a prism. Then

$$ehs(D_n, C_4) = \left\lceil \frac{n + 3}{4} \right\rceil.$$

**Proof.** The prism $D_n$, $n \geq 3$, $n \neq 4$, admits a $C_4$-covering with exactly $n$ cycles $C_4$. We denote these 4-cycles by the symbols $C_4^i$, $i = 1, 2, \ldots, n$, such that the vertex set of $C_4^i$ is $V(C_4^i) = \{x_i, x_{i+1}, y_i, y_{i+1}\}$ and the edge set is $E(C_4^i) = \{x_ix_{i+1}, y_iy_{i+1}, x_iy_i, x_{i+1}y_{i+1}\}$.

From Theorem 1 it follows that $ehs(D_n, C_4) \geq \left\lceil \frac{n+3}{4} \right\rceil$. To show that $\left\lceil \frac{n+3}{4} \right\rceil$ is an upper bound for the edge $C_4$-irregularity strength of $D_n$ we define a $C_4$-irregular edge labeling $\alpha_1 : E(D_n) \rightarrow \{1, 2, \ldots, \left\lceil \frac{n+3}{4} \right\rceil\}$, in the following way. We distinguish to cases according to the parity of $n$. 
Case 1. When $n$ is odd, then

$$\alpha_1(x_i, x_{i+1}) = \begin{cases} \left\lfloor \frac{i}{2} \right\rfloor & \text{for } 1 \leq i \leq n + \frac{1}{2}, \\ \left\lfloor \frac{n + 1 - i}{2} \right\rfloor & \text{for } n + \frac{1}{2} + 1 \leq i \leq n, \end{cases}$$

$$\alpha_1(y_i, y_{i+1}) = \begin{cases} \left\lfloor \frac{i + 1}{2} \right\rfloor & \text{for } 1 \leq i \leq n + \frac{1}{2}, \\ \left\lfloor \frac{n + 2 - i}{2} \right\rfloor & \text{for } n + \frac{1}{2} + 1 \leq i \leq n, \end{cases}$$

$$\alpha_1(x_i, y_i) = \begin{cases} \left\lfloor \frac{i}{2} \right\rfloor & \text{for } 1 \leq i \leq n + \frac{1}{2}, \\ \left\lfloor \frac{n + 3 - i}{2} \right\rfloor & \text{for } n + \frac{1}{2} + 1 \leq i \leq n. \end{cases}$$

Case 2. When $n$ is even, then

$$\alpha_1(x_i, x_{i+1}) = \begin{cases} \left\lfloor \frac{i}{2} \right\rfloor & \text{for } 1 \leq i \leq \frac{n}{2}, \\ \left\lfloor \frac{n + 1 - i}{2} \right\rfloor & \text{for } \frac{n}{2} + 1 \leq i \leq n, \end{cases}$$

$$\alpha_1(y_i, y_{i+1}) = \begin{cases} \left\lfloor \frac{i + 1}{2} \right\rfloor & \text{for } 1 \leq i \leq \frac{n}{2}, \\ \left\lfloor \frac{n + 2}{2} \right\rfloor & \text{for } i = \frac{n}{2} + 1, \\ \left\lfloor \frac{n + 2 - i}{2} \right\rfloor & \text{for } \frac{n}{2} + 2 \leq i \leq n, \end{cases}$$

$$\alpha_1(x_i, y_i) = \begin{cases} \left\lfloor \frac{n}{4} + 1 \right\rfloor & \text{for } i = \frac{n}{2} + 1 \text{ and } n \equiv 0 \pmod{4}, \\ \left\lfloor \frac{n + 2}{4} \right\rfloor & \text{for } i = \frac{n}{2} + 1 \text{ and } n \equiv 2 \pmod{4}, \\ \left\lfloor \frac{n + 3 - i}{2} \right\rfloor & \text{for } \frac{n}{2} + 2 \leq i \leq n. \end{cases}$$

It is easy to see that under the labeling $\alpha_1$ all edge labels are at most $\lfloor \frac{n + 3}{4} \rfloor$. The $C_4$-weights of the cycles $C_4^i$, $i = 1, 2, \ldots, n$, under the edge labeling $\alpha_1$, are given by

$$wt_{\alpha_1}(C_4^i) = \sum_{e \in E(C_4^i)} \alpha_1(e) = \alpha_1(x_i, x_{i+1}) + \alpha_1(y_i, y_{i+1}) + \alpha_1(x_i, y_i) + \alpha_1(x_{i+1}, y_{i+1}).$$

Case 1. When $n$ is odd, then

$$\begin{align*}
wt_{\alpha_1}(C_4^i) &= \left\lfloor \frac{i}{2} \right\rfloor + \left\lfloor \frac{i + 1}{2} \right\rfloor + \left\lfloor \frac{i}{2} \right\rfloor + \left\lfloor \frac{i + 1}{2} \right\rfloor = 2i + 2 & \text{for } 1 \leq i \leq \frac{n - 1}{2}, \\
wt_{\alpha_1}\left(\frac{n + 1}{4}\right) &= \left\lfloor \frac{n + 1}{2} \right\rfloor + \left\lfloor \frac{n + 3}{2} \right\rfloor + \left\lfloor \frac{n + 1}{4} \right\rfloor + \left\lfloor \frac{n + 3}{4} \right\rfloor = n + 3,
\end{align*}$$

$$\begin{align*}
wt_{\alpha_1}(C_4^i) &= \left\lfloor \frac{n + 1 - i}{2} \right\rfloor + \left\lfloor \frac{n + 2 - i}{2} \right\rfloor + \left\lfloor \frac{n + 3 - i}{2} \right\rfloor + \left\lfloor \frac{n + 2 - i}{2} \right\rfloor = 2n + 5 - 2i & \text{for } \frac{n + 1}{2} + 1 \leq i \leq n - 1,
\end{align*}$$

$$\begin{align*}
wt_{\alpha_1}(C_4^i) &= \left\lfloor \frac{i}{2} \right\rfloor + \left\lfloor \frac{2}{2} \right\rfloor + \left\lfloor \frac{3}{2} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor = 5.
\end{align*}$$
Case 2. When \( n \) is even, then
\[
wt_{\alpha_1}(C_i^n) = \left\lceil \frac{i}{2} \right\rceil + \left\lceil \frac{i+1}{2} \right\rceil + \left\lceil \frac{i}{2} \right\rceil + \left\lceil \frac{i+1}{2} \right\rceil = 2i + 2 \quad \text{for} \quad 1 \leq i \leq \frac{n}{2} - 1,
\]
where \( C_i^n \) is defined as:
\[
\{x_i, x_{i+1}, y_i : 1 \leq i \leq n\} \quad \text{and} \quad E(C_i^n) = \{x_ix_{i+1}, x_iy_i, y_ix_{i+1} : 1 \leq i \leq n\}.
\]
The second type has the vertex set \( A_4^n \), with 4 vertices and 4 edges. The vertex and edge set of \( A_4^n \) is defined as:
\[
\{x_i, x_{i+1}, x_iy_i, y_ix_{i+1} : 1 \leq i \leq n\}.
\]

The antiprism \( A_n \) [10], \( n \geq 3 \), is a 4-regular graph (Archimedean convex polytope), consisting of \( 2n \) vertices and \( 4n \) edges. The vertex and edge set of \( A_n \) are defined as: \( V(A_n) = \{x_i, y_i : 1 \leq i \leq n\} \), \( E(A_n) = \{x_iy_i : 1 \leq i \leq n\} \cup \{x_ix_{i+1} : 1 \leq i \leq n\} \cup \{y_ix_{i+1} : 1 \leq i \leq n\} \cup \{y_{i+1}y_i : 1 \leq i \leq n\} \), with indices taken modulo \( n \).

**Theorem 3.** Let \( A_n, n \geq 4 \), be an antiprism. Then
\[
\text{ehs}(A_n, C_3) = \left\lceil \frac{2n + 2}{3} \right\rceil.
\]

**Proof.** The antiprism \( A_n, n \geq 4 \), admits a \( C_3 \)-covering with exactly \( 2n \) cycles \( C_3 \). The first type of the cycle \( C_3 \) has the vertex set \( V(C_3^n) = \{x_i, x_{i+1}, y_i : 1 \leq i \leq n\} \) and the edge set \( E(C_3^n) = \{x_ix_{i+1}, x_iy_i, y_ix_{i+1} : 1 \leq i \leq n\} \). The second type...
of the cycle $C_3$ has the vertex set $V(C_3^i) = \{y_i, y_{i+1}, x_{i+1} : 1 \leq i \leq n\}$ and the edge set $E(C_3^i) = \{y_iy_{i+1}, y_ix_{i+1}, y_{i+1}x_{i+1} : 1 \leq i \leq n\}$. Note that the indices are taken modulo $n$.

From Theorem 1 it follows that $ehs(A_n, C_3) \geq \left\lceil \frac{2n+2}{3} \right\rceil$. To show that $\left\lceil \frac{2n+2}{3} \right\rceil$ is an upper bound for the edge $C_3$-irregularity strength of $A_n$ we define a $C_3$-irregular edge labeling $\alpha_2 : E(A_n) \to \{1, 2, \ldots, \left\lceil \frac{2n+2}{3} \right\rceil\}$, in the following way. We distinguish two cases.

Case 1. When $n \equiv 0, 4, 5$ (mod 6), then

$$
\alpha_2(x_i, x_{i+1}) = \begin{cases}
  i & \text{for } i = 1, 2, \\
  i + \left\lfloor \frac{i}{3} \right\rfloor & \text{for } 3 \leq i \leq t-1, \\
  \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = t, \\
  n-i+3 + \left\lfloor \frac{n-i-2}{3} \right\rfloor & \text{for } t+1 \leq i \leq n-2, \\
  n-i+2 & \text{for } i = n-1, n,
\end{cases}
$$

where $t = \begin{cases}
  \frac{n+1}{2} & \text{if } n \equiv 5 \text{ (mod 6)}, \\
  \frac{n}{2} + 1 & \text{if } n \equiv 0, 4 \text{ (mod 6)}.
\end{cases}$

$$
\alpha_2(y_i, y_{i+1}) = \begin{cases}
  i + 1 + \left\lfloor \frac{i-1}{3} \right\rfloor & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\
  \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \equiv 5 \text{ (mod 6)}, \\
  \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \equiv 0, 4 \text{ (mod 6)}, \\
  n-i+2 + \left\lfloor \frac{n-i-1}{3} \right\rfloor & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n-1, \\
  1 & \text{for } i = n,
\end{cases}
$$

$$
\alpha_2(x_i, y_i) = \begin{cases}
  i + \left\lfloor \frac{i}{3} \right\rfloor & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
  \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \equiv 0, 5 \text{ (mod 6)}, \\
  \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \equiv 4 \text{ (mod 6)}, \\
  n-i+3 + \left\lfloor \frac{n-i-1}{3} \right\rfloor & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n-1, \\
  2 & \text{for } i = n,
\end{cases}
$$

$$
\alpha_2(y_i, x_{i+1}) = \begin{cases}
  1 & \text{for } i = 1, \\
  i + 1 + \left\lfloor \frac{i-2}{3} \right\rfloor & \text{for } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\
  \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \equiv 0, 4 \text{ (mod 6)}, \\
  \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \equiv 5 \text{ (mod 6)}, \\
  n-i+2 + \left\lfloor \frac{n-i-1}{3} \right\rfloor & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n.
\end{cases}
$$
Case 2. When \( n \equiv 1, 2, 3 \pmod{6} \), then

\[
\alpha_2(x_i x_{i+1}) = \begin{cases} 
  i & \text{for } i = 1, 2, \\
  i + \left\lfloor \frac{i}{3} \right\rfloor & \text{for } 3 \leq i \leq t - 1, \\
  \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = t \text{ and } n \equiv 2 \pmod{6}, \\
  n - i + 3 + \left\lceil \frac{n-i-2}{3} \right\rceil & \text{for } t + 1 \leq i \leq n - 2, \\
  n - i + 2 & \text{for } i = n - 1, n,
\end{cases}
\]

where \( t = \begin{cases} 
  \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod{6}, \\
  \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod{6}.
\end{cases} \)

\[
\alpha_2(y_i y_{i+1}) = \begin{cases} 
  i + 1 + \left\lfloor \frac{i-1}{3} \right\rfloor & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\
  \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1 \text{ and } n \equiv 2 \pmod{6}, \\
  n - i + 2 + \left\lceil \frac{n-i-1}{3} \right\rceil & \text{for } \left\lceil \frac{n}{2} \right\rceil + 2 \leq i \leq n - 1, \\
  1 & \text{for } i = n,
\end{cases}
\]

\[
\alpha_2(x_i y_i) = \begin{cases} 
  i + 1 + \left\lfloor \frac{i-1}{3} \right\rfloor & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\
  \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1 \text{ and } n \equiv 1, 3 \pmod{6}, \\
  n - i + 3 + \left\lceil \frac{n-i-1}{3} \right\rceil & \text{for } \left\lceil \frac{n}{2} \right\rceil + 2 \leq i \leq n - 1, \\
  1 & \text{for } i = n,
\end{cases}
\]

Now we compute the \( C_3 \)-weights under the edge labeling \( \alpha_2 \) as follows. For the weights of 3-cycles of the first type we get

\[
wt_{\alpha_2}(C_3^i) = \sum_{e \in \mathcal{E}(C_3^i)} \alpha_2(e) = \alpha_2(x_i x_{i+1}) + \alpha_2(x_i y_i) + \alpha_2(y_i x_{i+1})
\]

\[
= \begin{cases} 
  4i - 1 & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
  4n - 4i + 6 & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n,
\end{cases}
\]
and for the weights of 3-cycles of the second type we have

\[ \text{wt}_{\alpha_2}(C^i_1) = \sum_{e \in E(C^i_1)} \alpha_2(e) = \alpha_2(y_i y_{i+1}) + \alpha_2(y_i x_{i+1}) + \alpha_2(y_{i+1} x_{i+1}) \]

\[ = \begin{cases} 4i + 1 & \text{for } 1 \leq i \leq \left\lceil \frac{n+1}{2} \right\rceil - 1, \\ 4n - 4i + 4 & \text{for } \left\lceil \frac{n+1}{2} \right\rceil \leq i \leq n. \end{cases} \]

Combining these two cases one can see that the weights of the cycles \( C^i_1 \) are different to the weights of the cycles \( C^i_2 \). This shows that \( \alpha_2 \) is an edge \( C_3 \)-irregular labeling of \( A_n \). Therefore, \( \text{ehs}(A_n, C_4) \leq \left\lceil \frac{2n+2}{3} \right\rceil \) and we arrive at the desired result. \( \blacksquare \)

3. Triangular Ladder and Diagonal Ladder

Let \( L_n \cong P_n \square P_2, n \geq 2 \), be a ladder with the vertex set \( V(L_n) = \{x_i, y_i : i = 1,2,\ldots,n\} \) and the edge set \( E(L_n) = \{x_i x_{i+1}, y_i y_{i+1} : i = 1,2,\ldots,n-1\} \cup \{x_i y_i : i = 1,2,\ldots,n\} \). The triangular ladder \( TL_n, n \geq 2 \), is obtained from a ladder \( L_n \) by adding the edges \( y_i x_{i+1} \) for \( i = 1,2,\ldots,n-1 \).

**Theorem 4.** Let \( TL_n, n \geq 2 \), be a triangular ladder. Then

\[ \text{ehs}(TL_n, C_3) = \left\lceil \frac{2n}{3} \right\rceil. \]

**Proof.** The triangular ladder \( TL_n, n \geq 2 \), admits a \( C_3 \)-covering with exactly \( 2(n-1) \) cycles \( C_3 \). There are two types of cycles \( C_3 \) that cover \( TL_n \). The first type of cycles \( C_3 \) has the vertex set \( V(C^i_3) = \{x_i, x_{i+1}, y_i : 1 \leq i \leq n-1\} \) and the edge set \( E(C^i_3) = \{x_i x_{i+1}, x_i y_i, y_i x_{i+1} : 1 \leq i \leq n-1\} \). The second type of cycles \( C_3 \) has the vertex set \( V(C^i_3) = \{y_i, y_{i+1}, x_{i+1} : 1 \leq i \leq n-1\} \) and the edge set \( E(C^i_3) = \{y_i y_{i+1}, y_i x_{i+1}, y_i x_{i+1} : 1 \leq i \leq n-1\} \).

According to Theorem 1 it follows that \( \text{ehs}(TL_n, C_3) \geq \left\lceil \frac{2n}{3} \right\rceil \). To show that \( \left\lceil \frac{2n}{3} \right\rceil \) is an upper bound for the edge \( C_3 \)-irregularity strength of \( TL_n \) we define a \( C_3 \)-irregular edge labeling \( \alpha_3 : E(TL_n) \to \{1,2,\ldots,\left\lceil \frac{2n}{3} \right\rceil\} \) as follows. Let us consider three cases.

**Case 1.** When \( i \equiv 0 \pmod{3} \), then

\[ \alpha_3(x_i x_{i+1}) = \alpha_3(y_i y_{i+1}) = \frac{2i}{3} \quad \text{for } i = 3,6,\ldots,n-1, \]

\[ \alpha_3(y_i x_{i+1}) = \frac{2i+3}{3} \quad \text{for } i = 3,6,\ldots,n-1, \]

\[ \alpha_3(x_i y_i) = \frac{2i}{3} \quad \text{for } i = 3,6,\ldots,n. \]
Case 2. When \( i \equiv 1 \pmod{3} \), then
\[
\alpha_3(x_{i}x_{i+1}) = \alpha_3(y_{i}y_{i+1}) = \alpha_3(y_{i}x_{i+1}) = \frac{2i+1}{3} \quad \text{for } i = 1, 4, \ldots, n - 1,
\]
\[
\alpha_3(x_{i}y_{i}) = \frac{2i+1}{3} \quad \text{for } i = 1, 4, \ldots, n.
\]

Case 3. When \( i \equiv 2 \pmod{3} \), then
\[
\alpha_3(x_{i}x_{i+1}) = \frac{2i-1}{3} \quad \text{for } i = 2, 5, \ldots, n - 1,
\]
\[
\alpha_3(y_{i}y_{i+1}) = \frac{2i+2}{3} \quad \text{for } i = 2, 5, \ldots, n - 1,
\]
\[
\alpha_3(x_{i}y_{i}) = \frac{2i+2}{3} \quad \text{for } i = 2, 5, \ldots, n.
\]

It is a routine matter to verify that under the labeling \( \alpha_3 \) all edge labels are at most \( \left\lfloor \frac{2n}{3} \right\rfloor \). It is not difficult to see that under the edge labeling \( \alpha_3 \) the weights of the cycles \( C_3^i, 1 \leq i \leq n - 1 \), are of the form
\[
wt_{\alpha_3}(C_3^i) = \alpha_3(x_{i}x_{i+1}) + \alpha_3(x_{i}y_{i}) + \alpha_3(y_{i}x_{i+1}) = 2i + 1.
\]

The weights of the cycles \( C_3^i, 1 \leq i \leq n - 1 \), are of the form
\[
wt_{\alpha_3}(C_3^i) = \alpha_3(y_{i}y_{i+1}) + \alpha_3(x_{i+1}y_{i+1}) + \alpha_3(y_{i}x_{i+1}) = 2(i + 1).
\]

Combining these two cases we obtained that the weights are different for any two distinct cycles \( C_3 \). Thus \( \text{ehs}(\mathcal{L}_n, C_3) \leq \left\lfloor \frac{2n}{3} \right\rfloor \). This completes the proof. \( \blacksquare \)

The diagonal ladder \( DL_n \) is obtained from a ladder \( L_n \) by adding the edges \( \{x_{i}y_{i+1}, x_{i+1}y_{i}: 1 \leq i \leq n - 1\} \). So the diagonal ladder \( DL_n \) contains 2n vertices and 5n - 4 edges.

**Theorem 5.** Let \( DL_n, n \geq 2 \), be a diagonal ladder. Then
\[
\text{ehs}(DL_n, K_4) = \left\lfloor \frac{n + 4}{6} \right\rfloor.
\]

**Proof.** The diagonal ladder \( DL_n, n \geq 2 \), admits a \( K_4 \)-covering with exactly \( (n - 1) \) complete graphs \( K_4 \). The \( K_4^i \) has the vertex set \( V(K_4^i) = \{x_{i}, y_{i}, x_{i+1}, y_{i+1}: 1 \leq i \leq n - 1\} \) and the edge set \( E(K_4^i) = \{x_{i}x_{i+1}, x_{i}y_{i}, x_{i+1}y_{i}, y_{i+1}y_{i}, y_{i+1}x_{i+1}, x_{i+1}y_{i+1}: 1 \leq i \leq n - 1\} \).

With respect to Theorem 1 it follows that \( \text{ehs}(DL_n, K_4) \geq \left\lfloor \frac{n+4}{6} \right\rfloor \). To show that \( \text{ehs}(DL_n, K_4) \leq \left\lfloor \frac{n+4}{6} \right\rfloor \) we define a \( K_4 \)-irregular edge labeling \( \alpha_4 : \)
$E(DL_n) \to \{1, 2, \ldots, \left\lceil \frac{n+4}{6} \right\rceil \}$, in the following way.

$$
\alpha_4(y_iy_{i+1}) = \left\lceil \frac{i}{6} \right\rceil \quad \text{for } 1 \leq i \leq n - 1,
$$

$$
\alpha_4(x_ix_{i+1}) = \begin{cases} \\
\left\lceil \frac{i}{6} \right\rceil & \text{for } 1 \leq i \leq 5, \\
\left\lceil \frac{i+1}{6} \right\rceil & \text{for } 6 \leq i \leq n - 1,
\end{cases}
$$

$$
\alpha_4(x_iy_i) = \begin{cases} \\
\left\lceil \frac{i}{6} \right\rceil & \text{for } 1 \leq i \leq 4, \\
\left\lceil \frac{i+2}{6} \right\rceil & \text{for } 5 \leq i \leq n,
\end{cases}
$$

$$
\alpha_4(x_iy_{i+1}) = \begin{cases} \\
1 & \text{for } i = 1, \\
\left\lceil \frac{i+5}{6} \right\rceil & \text{for } 2 \leq i \leq n - 1,
\end{cases}
$$

$$
\alpha_4(x_{i+1}y_i) = \begin{cases} \\
1 & \text{for } i = 1, 2, \\
\left\lceil \frac{i+4}{6} \right\rceil & \text{for } 3 \leq i \leq n - 1.
\end{cases}
$$

One can verify that under the labeling $\alpha_4$ all edge labels are at least 1 and at most $\left\lceil \frac{n+4}{6} \right\rceil$. To show that $\alpha_4$ is edge $K_4$-irregular labeling it will be enough to show that $wt_{\alpha_4}(K_4^i) < wt_{\alpha_4}(K_4^{i+1})$. It is a simple mathematical exercise that the weights of the subgraphs $K_4^i$, $i = 1, 2, 3, 4, 5$ are $wt_{\alpha_4}(K_4^i) = 5 + i$.

For $i = 6, 7, \ldots, n - 1$ we get

$$
wt_{\alpha_4}(K_4^i) = \sum_{e \in E(K_4^i)} \alpha_4(e) = \alpha_4(x_ix_{i+1}) + \alpha_4(y_iy_{i+1}) + \alpha_4(x_iy_i) + \alpha_4(x_{i+1}y_{i+1})
$$

$$
+ \alpha_4(x_iy_{i+1}) + \alpha_4(x_{i+1}y_i) = \left\lceil \frac{i+1}{6} \right\rceil + \left\lceil \frac{i}{6} \right\rceil + \left\lceil \frac{i+2}{6} \right\rceil + \left\lceil \frac{i+3}{6} \right\rceil + \left\lceil \frac{i+4}{6} \right\rceil
$$

$$
+ \left\lceil \frac{i+5}{6} \right\rceil = 5 + i.
$$

This proves that $wt_{\alpha_4}(K_4^{i+1}) = wt_{\alpha_4}(K_4^{i}) + 1$ for $i = 1, 2, \ldots, n - 1$. Therefore, $\alpha_4$ is an edge $K_4$-irregular labeling of $DL_n$. Thus $ehs(DL_n, K_4) \leq \left\lceil \frac{n+4}{6} \right\rceil$. This concludes the proof.

4. Wheel and Gear Graph

A wheel $W_n$, $n \geq 3$, is a graph obtained by joining all vertices of cycle $C_n$ to a further vertex $c$, called the center. Thus $W_n$ contains $n + 1$ vertices, say, $c, x_1, x_2, \ldots, x_n$ and $2n$ edges, say, $cx_i, x_ix_{i+1}$, $1 \leq i \leq n$, where the indices are taken modulo $n$.

**Theorem 6.** Let $W_n$, $n \geq 4$, be a wheel. Then

$$
ehs(W_n, C_3) = \left\lceil \frac{n + 2}{3} \right\rceil.
$$
Proof. The wheel $W_n$, $n \geq 4$, admits a $C_3$-covering with exactly $n$ cycles $C_3$. Every cycle $C_3$ in $W_n$ is of the form $C_3^i = cx_ix_{i+1}$, where $i = 1, 2, \ldots, n$ with indices taken modulo $n$.

According to Theorem 1 we have that $\text{ehs}(W_n, C_3) \geq \lceil \frac{n+2}{3} \rceil$. To show that $\text{ehs}(W_n, C_4)$ is an upper bound for the edge $C_3$-irregularity strength of $W_n$ we define a $C_3$-irregular edge labeling $\alpha_5 : E(W_n) \to \{1, 2, \ldots, \lceil \frac{n+2}{3} \rceil\}$ as follows.

$$\alpha_5(x_ix_{i+1}) = \begin{cases} 
  i & \text{for } i = 1, \ 2, \\
  i - \left\lfloor \frac{i}{3} \right\rfloor & \text{for } 3 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor - 1, \\
  \left\lceil \frac{n+2}{3} \right\rceil & \text{for } i = \left\lceil \frac{n+1}{2} \right\rceil, \\
  n - i + 1 - \left\lfloor \frac{n-i}{3} \right\rfloor & \text{for } \left\lceil \frac{n+1}{2} \right\rceil + 1 \leq i \leq n,
\end{cases}$$

$$\alpha_5(cx_i) = \begin{cases} 
  1 & \text{for } i = 1, \\
  i - 1 - \left\lfloor \frac{i-2}{3} \right\rfloor & \text{for } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
  \left\lceil \frac{n+2}{3} \right\rceil & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1 \text{ and } n \equiv 0, 1 \pmod{3}, \\
  \left\lceil \frac{n+2}{3} \right\rceil - 1 & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1 \text{ and } n \equiv 2 \pmod{3}, \\
  n - i + 1 - \left\lfloor \frac{n-i-1}{3} \right\rfloor & \text{for } \left\lceil \frac{n}{2} \right\rceil + 2 \leq i \leq n - 1, \\
  2 & \text{for } i = n.
\end{cases}$$

It is a matter of routine checking that under the labeling $\alpha_5$ all edge labels are at most $\lceil \frac{n+2}{3} \rceil$. For the $C_3$-weight of the cycle $C_3^i$ we get

$$\omega_{\alpha_5}(C_3^i) = \alpha_5(cx_i) + \alpha_5(cx_{i+1}) + \alpha_5(x_ix_{i+1}) = \begin{cases} 
  2i + 1 & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\
  2(n+2-i) & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n.
\end{cases}$$

Clearly, the weights of $C_3^i$ for $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$ are odd and increasing. On the other hand the weights of $C_3^i$ for $\left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n$ are even and decreasing. So, it concludes that all the weights of $C_3^i$ are different. Thus $\alpha_5$ is an edge $C_3$-irregular labeling of $W_n$. ehs$(W_n, C_3) \leq \lceil \frac{n+2}{3} \rceil$. This completes the proof.  

A gear graph $G_n$ is obtained from $W_n$ by inserting a vertex to each edge on the cycle $C_n$. Then the vertex set of $G_n$ is $V(G_n) = \{c, x_i, y_i \mid 1 \leq i \leq n\}$ and the edge set is $E(W_n) = \{x_iy_i, y_ix_{i+1}, cx_i \mid 1 \leq i \leq n\}$ with indices taken modulo $n$.

Theorem 7. Let $G_n$, $n \geq 3$, be a gear graph. Then

$$\text{ehs}(G_n, C_4) = \left\lceil \frac{n+3}{4} \right\rceil.$$

Proof. The gear $G_n$, $n \geq 3$, admits a $C_4$-covering with exactly $n$ cycles $C_4$. According to Theorem 1 we obtain that $\text{ehs}(G_n, C_4) \geq \left\lceil \frac{n+3}{4} \right\rceil$. To show that
$\left\lceil \frac{n+3}{4} \right\rceil$ is an upper bound for the edge $C_4$-irregularity strength of $G_n$ we define a $C_4$-irregular edge labeling $\alpha_6 : E(G_n) \to \{1, 2, \ldots, \left\lceil \frac{n+3}{4} \right\rceil \}$, in the following way.

$$\alpha_6(x_i y_i) = \begin{cases} \left\lceil \frac{i}{2} \right\rceil & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1, \\ \left\lceil \frac{n}{4} \right\rceil & \text{for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and } n \not\equiv 2 \pmod{4}, \\ \left\lceil \frac{n-i+2}{2} \right\rceil & \text{for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and } n \equiv 2 \pmod{4}, \\ \left\lceil \frac{n-i+2}{2} \right\rceil & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\ \left\lceil \frac{n-i+1}{2} \right\rceil & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n, \\ \left\lceil \frac{n-i+3}{2} \right\rceil & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n. \end{cases}$$

$$\alpha_6(y_i x_{i+1}) = \begin{cases} \left\lceil \frac{i+1}{2} \right\rceil & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\ \left\lceil \frac{n-i+1}{2} \right\rceil & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n. \end{cases}$$

$$\alpha_6(cx_i) = \begin{cases} \left\lceil \frac{i}{2} \right\rceil & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\ \left\lceil \frac{n-i+3}{2} \right\rceil & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n. \end{cases}$$

It is easy to verify that under the labeling $\alpha_6$ all edge labels are at most $\left\lceil \frac{n+3}{4} \right\rceil$. For the $C_4$-weight of the cycle $C_i^4$, $i = 1, 2, \ldots, n$, under the edge labeling $\alpha_6$, we get

$$\text{wt}_{\alpha_6}(C_i^4) = \sum_{e \in E(C_i^4)} \alpha_6(e) = \alpha_6(cx_i) + \alpha_6(cx_{i+1}) + \alpha_6(x_i y_i) + \alpha_6(y_i x_{i+1})$$

$$= \begin{cases} 2i + 2 & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\ 2n + 5 - 2i & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n. \end{cases}$$

Clearly, the weights of $C_i^4$ for $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$ are even and increasing. On the other hand the weights of $C_i^4$ for $\left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n$ are odd and decreasing. So, it concluded that all the weights of $C_i^4$ are different. Thus $\alpha_6$ is an edge $C_4$-irregular labeling of $G_n$. Hence $\text{ehs}(G_n, C_4) \leq \left\lceil \frac{n+2}{3} \right\rceil$. This completes the proof of theorem.

5. Conclusion

In this paper we have investigated the edge $H$-irregularity strength of some graphs. We have found the exact values of this parameter for several families of graphs namely, prisms, antiprisms, triangular ladders, diagonal ladders, wheels and gear graphs.

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