ON EDGE $H$-IRREGULARITY STRENGTHS OF SOME GRAPHS

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Abstract

For a graph $G$ an edge-covering of $G$ is a family of subgraphs $H_1, H_2, \ldots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs $H_i$, $i = 1, 2, \ldots, t$. In this case we say that $G$ admits an $(H_1, H_2, \ldots, H_t)$-edge covering. An $H$-covering of graph $G$ is an $(H_1, H_2, \ldots, H_t)$-edge covering in which every subgraph $H_i$ is isomorphic to a given graph $H$.

Let $G$ be a graph admitting $H$-covering. An edge $k$-labeling $\alpha : E(G) \to \{1, 2, \ldots, k\}$ is called an $H$-irregular edge $k$-labeling of the graph $G$ if for every two different subgraphs $H'$ and $H''$ isomorphic to $H$ their weights

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wt_α(H') and wt_α(H'') are distinct. The weight of a subgraph H under an edge k-labeling α is the sum of labels of edges belonging to H. The edge H-irregularity strength of a graph G, denoted by ehs(G, H), is the smallest integer k such that G has an H-irregular edge k-labeling.

In this paper we determine the exact values of ehs(G, H) for prisms, antiprisms, triangular ladders, diagonal ladders, wheels and gear graphs. Moreover the subgraph H is isomorphic to only \(C_4\), \(C_3\) and \(K_4\).

**Keywords:** H-irregular edge labeling, edge H-irregularity strength, prism, antiprism, triangular ladder, diagonal ladder, wheel, gear graph.

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1. Introduction

Consider a simple and finite graph \(G = (V, E)\) of order at least 2. An edge k-labeling is a function \(\alpha : E(G) \to \{1, 2, \ldots, k\}\), where \(k\) is a positive integer. Then the associated weight of a vertex \(x \in V(G)\) is \(w_\alpha(x) = \sum_{xy \in E(G)} \alpha(xy)\), where the sum is taken over all edges incident to \(x\). Such a labeling \(\alpha\) is called irregular if the obtained weights of all vertices are different. The smallest positive integer \(k\) for which there exists an irregular labeling of \(G\) is called the irregularity strength of \(G\) and is denoted by \(s(G)\). If it does not exist, then we write \(s(G) = \infty\).

One can easily see that \(s(G) < \infty\) if and only if \(G\) contains no isolated edges and has at most one isolated vertex.

The notion of the irregularity strength was firstly introduced by Chartrand et al. in [7]. Some results on the irregularity strength can be found in [2, 3, 5, 6, 8, 9, 11–14].

A vertex k-labeling \(\beta : V(G) \to \{1, 2, \ldots, k\}\) is called an edge irregular k-labeling of the graph \(G\) if the weights \(w_\beta(xy) \neq w_\beta(x'y')\) for every two distinct edges \(xy\) and \(x'y'\), where the weight of an edge \(xy \in E(G)\) is \(w_\beta(xy) = \beta(x) + \beta(y)\).

The minimum \(k\) for which a graph \(G\) admits an edge irregular k-labeling is called the edge irregularity strength of \(G\), denoted by \(es(G)\). The notion of the edge irregularity strength was defined by Ahmad et al. in [1].

A family of subgraphs \(H_1, H_2, \ldots, H_t\) is said to be an edge-covering of \(G\) if each edge of \(E(G)\) belongs to at least one of the subgraphs \(H_i\), \(i = 1, 2, \ldots, t\). In this case we say that \(G\) admits an \((H_1, H_2, \ldots, H_t)\)-(edge) covering. If every subgraph \(H_i\), \(i = 1, 2, \ldots, t\), is isomorphic to a given graph \(H\), then the graph \(G\) admits an \(H\)-covering.

Motivated by the irregularity strength and the edge irregularity strength of a graph \(G\) Ashraf et al. in [4] introduced a new parameter, edge \(H\)-irregularity strength, as a natural extension of the parameters \(s(G)\) and \(es(G)\). Let \(G\) be a graph admitting \(H\)-covering. An edge k-labeling \(\alpha\) is called an \(H\)-irregular
edge $k$-labeling of the graph $G$ if for every two different subgraphs $H'$ and $H''$ isomorphic to $H$ we have

$$wt_\alpha(H') = \sum_{e \in E(H')} \alpha(e) \neq \sum_{e \in E(H'')} \alpha(e) = wt_\alpha(H'').$$

The edge $H$-irregularity strength of a graph $G$, denoted by $ehs(G, H)$, is the smallest integer $k$ for which $G$ has an $H$-irregular edge $k$-labeling.

Theorem 1 [4]. Let $G$ be a graph admitting an $H$-covering and $t$ is the number of all the subgraphs isomorphic with $H$. Then

$$ehs(G, H) \geq \left\lceil 1 + \frac{t - 1}{|E(H)|} \right\rceil.$$

Note that the parameter $t$ is the number of all subgraphs of $G$ isomorphic to $H$. In this paper we determine exact values of the edge $H$-irregularity strength for prisms, antiprisms, triangular ladders, diagonal ladders, wheels and gear graphs for some $H$. Moreover the subgraph $H$ is isomorphic to only $C_4$, $C_3$ and $K_4$.

2. Prism and Antiprism

The prism $D_n$ can be defined as the Cartesian product $C_n \square P_2$ of a cycle on $n$ vertices with a path on 2 vertices. Let $V(C_n \square P_2) = \{x_i, y_i : 1 \leq i \leq n\}$ be the vertex set and $E(C_n \square P_2) = \{x_ix_{i+1}, y_iy_{i+1} : 1 \leq i \leq n\} \cup \{x_iy_i : 1 \leq i \leq n\}$ be the edge set, where the indices are taken modulo $n$. Hence, the graph $D_n$ has $2n$ vertices and $3n$ edges.

Theorem 2. Let $D_n = C_n \square P_2$, $n \geq 3$, $n \neq 4$, be a prism. Then

$$ehs(D_n, C_4) = \left\lceil \frac{n + 3}{4} \right\rceil.$$

Proof. The prism $D_n$, $n \geq 3$, $n \neq 4$, admits a $C_4$-covering with exactly $n$ cycles $C_4$. We denote these 4-cycles by the symbols $C'_4$, $i = 1, 2, \ldots, n$, such that the vertex set of $C'_4$ is $V(C'_4) = \{x_i, x_{i+1}, y_i, y_{i+1}\}$ and the edge set is $E(C'_4) = \{x_ix_{i+1}, y_iy_{i+1}, x_iy_i, x_{i+1}y_{i+1}\}$.

From Theorem 1 it follows that $ehs(D_n, C_4) \geq \left\lceil \frac{n + 3}{4} \right\rceil$. To show that $\left\lceil \frac{n + 3}{4} \right\rceil$ is an upper bound for the edge $C_4$-irregularity strength of $D_n$ we define a $C_4$-irregular edge labeling $\alpha_1 : E(D_n) \to \{1, 2, \ldots, \left\lceil \frac{n + 3}{4} \right\rceil\}$, in the following way. We distinguish to cases according to the parity of $n$. 
Case 1. When \( n \) is odd, then

\[
\alpha_1(x_i x_{i+1}) = \begin{cases} 
\left\lceil \frac{i}{2} \right\rceil & \text{for } 1 \leq i \leq \frac{n+1}{2}, \\
\left\lceil \frac{n+1-i}{2} \right\rceil & \text{for } \frac{n+1}{2} + 1 \leq i \leq n,
\end{cases}
\]

\[
\alpha_1(y_i y_{i+1}) = \begin{cases} 
\left\lceil \frac{i+1}{2} \right\rceil & \text{for } 1 \leq i \leq \frac{n+1}{2}, \\
\left\lceil \frac{n+2-i}{2} \right\rceil & \text{for } \frac{n+1}{2} + 1 \leq i \leq n,
\end{cases}
\]

\[
\alpha_1(x_i y_i) = \begin{cases} 
\left\lceil \frac{i}{2} \right\rceil & \text{for } 1 \leq i \leq \frac{n+1}{2}, \\
\left\lceil \frac{n+3-i}{2} \right\rceil & \text{for } \frac{n+1}{2} + 1 \leq i \leq n.
\end{cases}
\]

Case 2. When \( n \) is even, then

\[
\alpha_1(x_i x_{i+1}) = \begin{cases} 
\left\lceil \frac{i}{2} \right\rceil & \text{for } 1 \leq i \leq \frac{n}{2}, \\
\left\lceil \frac{n+1-i}{2} \right\rceil & \text{for } \frac{n}{2} + 1 \leq i \leq n,
\end{cases}
\]

\[
\alpha_1(y_i y_{i+1}) = \begin{cases} 
\left\lceil \frac{i+1}{2} \right\rceil & \text{for } 1 \leq i \leq \frac{n}{2}, \\
\left\lceil \frac{n+2-i}{2} \right\rceil & \text{for } \frac{n}{2} + 1 \leq i \leq n,
\end{cases}
\]

\[
\alpha_1(x_i y_i) = \begin{cases} 
\frac{n}{4} + 1 & \text{for } i = \frac{n}{2} + 1 \text{ and } n \equiv 0 \pmod{4}, \\
\frac{n+2}{4} & \text{for } i = \frac{n}{2} + 1 \text{ and } n \equiv 2 \pmod{4}, \\
\left\lceil \frac{n+3-i}{2} \right\rceil & \text{for } \frac{n}{2} + 2 \leq i \leq n.
\end{cases}
\]

It is easy to see that under the labeling \( \alpha_1 \) all edge labels are at most \( \left\lceil \frac{n+3}{4} \right\rceil \).

The \( C_4 \)-weights of the cycles \( C_4^i, i = 1, 2, \ldots, n \), under the edge labeling \( \alpha_1 \), are given by

\[
wt_{\alpha_1}(C_4^i) = \sum_{e \in E(C_4^i)} \alpha_1(e) = \alpha_1(x_i x_{i+1}) + \alpha_1(y_i y_{i+1}) + \alpha_1(x_i y_i) + \alpha_1(x_{i+1} y_{i+1}).
\]

Case 1. When \( n \) is odd, then

\[
wt_{\alpha_1}(C_4^i) = \left\lceil \frac{i}{2} \right\rceil + \left\lceil \frac{i+1}{2} \right\rceil + \left\lceil \frac{i}{2} \right\rceil + \left\lceil \frac{i+1}{2} \right\rceil = 2i + 2 \quad \text{for } 1 \leq i \leq \frac{n-1}{2},
\]

\[
wt_{\alpha_1}\left(C_4^\frac{n+1}{2}\right) = \left\lceil \frac{n+1}{2} \right\rceil + \left\lceil \frac{n+3}{2} \right\rceil + \left\lceil \frac{n+1}{2} \right\rceil + \left\lceil \frac{n+3}{2} \right\rceil = n + 3,
\]

\[
wt_{\alpha_1}(C_4^i) = \left\lceil \frac{n+1-i}{2} \right\rceil + \left\lceil \frac{n+2-i}{2} \right\rceil + \left\lceil \frac{n+3-i}{2} \right\rceil + \left\lceil \frac{n+2-i}{2} \right\rceil = 2n + 5 - 2i \quad \text{for } \frac{n+1}{2} + 1 \leq i \leq n - 1,
\]

\[
wt_{\alpha_1}(C_4^n) = \left\lceil \frac{1}{2} \right\rceil + \left\lceil \frac{3}{2} \right\rceil + \left\lceil \frac{1}{2} \right\rceil = 5.
\]
Case 2. When \( n \) is even, then

\[
wt_{\alpha_1}(C^i_4) = \left\lceil \frac{i}{2} \right\rceil + \left\lceil \frac{i+1}{2} \right\rceil + \left\lceil \frac{i}{2} \right\rceil + \left\lceil \frac{i+1}{2} \right\rceil = 2i + 2 \quad \text{for} \quad 1 \leq i \leq \frac{n}{2} - 1,
\]

\[
wt_{\alpha_1}\left(C^\frac{n}{4}_4\right) = \left\lceil \frac{n}{4} \right\rceil + \left\lceil \frac{n+2}{4} \right\rceil + \left\lceil \frac{n}{4} \right\rceil + \frac{n}{4} + 1 = n + 2 \quad \text{for} \quad n \equiv 0 \pmod{4},
\]

\[
wt_{\alpha_1}\left(C^\frac{n}{4}_4\right) = \left\lceil \frac{n}{4} \right\rceil + \left\lceil \frac{n+2}{4} \right\rceil + \left\lceil \frac{n}{4} \right\rceil + \frac{n+2}{4} = n + 2 \quad \text{for} \quad n \equiv 2 \pmod{4},
\]

\[
wt_{\alpha_1}\left(C^{\frac{n}{4}+1}_4\right) = \left\lceil \frac{n}{4} \right\rceil + \left\lceil \frac{n+3}{4} \right\rceil + \left\lceil \frac{n}{4} \right\rceil + \frac{n+2}{4} = n + 3 \quad \text{for} \quad n \equiv 2 \pmod{4},
\]

\[
wt_{\alpha_1}(C^i_4) = \left\lceil \frac{n+1-i}{2} \right\rceil + \left\lceil \frac{n+2-i}{2} \right\rceil + \left\lceil \frac{n+3-i}{2} \right\rceil + \left\lceil \frac{n+4-i}{2} \right\rceil = 2n - 2i
\quad \text{for} \quad \frac{n}{2} + 2 \leq i \leq n - 1,
\]

\[
wt_{\alpha_1}(C^n_4) = \left\lceil \frac{1}{2} \right\rceil + \left\lceil \frac{3}{2} \right\rceil + \left\lceil \frac{5}{2} \right\rceil + \left\lceil \frac{7}{2} \right\rceil = 5.
\]

Combining the previous we get that

\[
wt_{\alpha_1}(C^i_4) = \begin{cases} 
2(1+i) & \text{for} \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
1 + 2(n + 2 - i) & \text{for} \quad \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n.
\end{cases}
\]

One can see that the weights of cycles \( C^i_4 \), for \( i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \), are even and in increasing order, therefore \( wt_{\alpha_1}(C^{i+1}_4) > wt_{\alpha_1}(C^i_4) \).

On the other hand, the weights of cycles \( C^i_4 \), for \( i = \left\lfloor \frac{n}{2} \right\rfloor + 1, \ldots, n \), are odd and in decreasing order, therefore \( wt_{\alpha_1}(C^{i+1}_4) < wt_{\alpha_1}(C^i_4) \).

Thus the edge weights are distinct numbers from the set \( \{4, 5, \ldots, n + 3\} \). This shows that \( \text{ehs}(D_n, C_4) \leq \left\lfloor \frac{n+3}{4} \right\rfloor \). Hence the proof is concluded. \( \blacksquare \)

The antiprism \( A_n \) [10], \( n \geq 3 \), is a 4-regular graph (Archimedean convex polytope), consisting of \( 2n \) vertices and \( 4n \) edges. The vertex set and edge set of \( A_n \) are defined as: \( V(A_n) = \{x_i, y_i : 1 \leq i \leq n\} \) and \( E(A_n) = \{x_iy_i : 1 \leq i \leq n\} \cup \{x_ix_{i+1} : 1 \leq i \leq n\} \cup \{y_{i+1}y_i : 1 \leq i \leq n\} \cup \{y_{i+1}y_{i+1} : 1 \leq i \leq n\} \), with indices taken modulo \( n \).

**Theorem 3.** Let \( A_n, n \geq 4 \), be an antiprism. Then

\[
\text{ehs}(A_n, C_3) = \left\lceil \frac{2n + 2}{3} \right\rceil.
\]

**Proof.** The antiprism \( A_n, n \geq 4 \), admits a \( C_3 \)-covering with exactly \( 2n \) cycles \( C_3 \). The first type of the cycle \( C_3 \) has the vertex set \( V(C^i_3) = \{x_i, x_{i+1}, y_i : 1 \leq i \leq n\} \) and the edge set \( E(C^i_3) = \{x_ix_{i+1}, x_iy_i, y_ix_{i+1} : 1 \leq i \leq n\} \). The second type
of the cycle $C_3$ has the vertex set \( V(C_3) = \{y_i, y_{i+1}, x_{i+1} : 1 \leq i \leq n \} \) and the edge set \( E(C_3) = \{y_i y_{i+1}, y_i x_{i+1}, y_{i+1} x_{i+1} : 1 \leq i \leq n \} \). Note that the indices are taken modulo \( n \).

From Theorem 1 it follows that \( \text{ehs}(A_n, C_3) \geq \left\lceil \frac{2n+2}{3} \right\rceil \). To show that \( \left\lceil \frac{2n+2}{3} \right\rceil \) is an upper bound for the edge \( C_3 \)-irregularity strength of \( A_n \) we define a \( C_3 \)-irregular edge labeling \( \alpha_2 : E(A_n) \to \{1, 2, \ldots, \left\lceil \frac{2n+2}{3} \right\rceil \} \), in the following way. We distinguish two cases.

**Case 1.** When \( n \equiv 0, 4, 5 \pmod{6} \), then

\[
\alpha_2(x_i, x_{i+1}) = \begin{cases} 
  i & \text{for } i = 1, 2, \\
  i + \left\lfloor \frac{i}{3} \right\rfloor & \text{for } 3 \leq i \leq t - 1, \\
  \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = t, \\
  n - i + 3 + \left\lfloor \frac{n-i-2}{3} \right\rfloor & \text{for } t + 1 \leq i \leq n - 2, \\
  n - i + 2 & \text{for } i = n - 1, n,
\end{cases}
\]

where \( t = \begin{cases} 
  \frac{n+1}{2} & \text{if } n \equiv 5 \pmod{6}, \\
  \frac{n}{2} + 1 & \text{if } n \equiv 0, 4 \pmod{6}.
\end{cases} \)

\[
\alpha_2(y_i, y_{i+1}) = \begin{cases} 
  i + 1 + \left\lfloor \frac{i-1}{3} \right\rfloor & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\
  \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \equiv 5 \pmod{6}, \\
  \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and } n \equiv 0, 4 \pmod{6}, \\
  n - i + 2 + \left\lfloor \frac{n-i-1}{3} \right\rfloor & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n - 1, \\
  1 & \text{for } i = n,
\end{cases}
\]

\[
\alpha_2(x_i, y_i) = \begin{cases} 
  i + \left\lfloor \frac{i-1}{3} \right\rfloor & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
  \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1 \text{ and } n \equiv 0, 5 \pmod{6}, \\
  \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1 \text{ and } n \equiv 4 \pmod{6}, \\
  n - i + 3 + \left\lfloor \frac{n-i-1}{3} \right\rfloor & \text{for } \left\lceil \frac{n}{2} \right\rceil + 2 \leq i \leq n - 1, \\
  2 & \text{for } i = n,
\end{cases}
\]

\[
\alpha_2(y_i, x_{i+1}) = \begin{cases} 
  1 & \text{for } i = 1, \\
  i + 1 + \left\lfloor \frac{i-2}{3} \right\rfloor & \text{for } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\
  \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \equiv 0, 4 \pmod{6}, \\
  \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = \left\lfloor \frac{n}{2} \right\rceil \text{ and } n \equiv 5 \pmod{6}, \\
  n - i + 2 + \left\lfloor \frac{n-i-1}{3} \right\rfloor & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n.
\end{cases}
\]
Case 2. When \( n \equiv 1, 2, 3 \pmod{6} \), then

\[
\alpha_2(x_ix_{i+1}) = \begin{cases} 
  i & \text{for } i = 1, 2, \\
  i + \lfloor \frac{i}{3} \rfloor & \text{for } 3 \leq i \leq t - 1, \\
  \lfloor \frac{2n+2}{3} \rfloor - 1 & \text{for } i = t \text{ and } n \equiv 2 \pmod{6}, \\
  n - i + 3 + \lfloor \frac{n-i-2}{3} \rfloor & \text{for } i = t \text{ and } n \equiv 1, 3 \pmod{6}, \\
  n - i + 2 & \text{for } 1 \leq i \leq \lceil \frac{n}{2} \rceil, \\
  n - i + 3 + \lfloor \frac{n-i-2}{3} \rfloor & \text{for } t + 1 \leq i \leq n - 2, \\
  n - i + 2 & \text{for } i = n - 1, n,
\end{cases}
\]

where \( t = \begin{cases} 
  \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod{6}, \\
  \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod{6}.
\end{cases} \)

\[
\alpha_2(y_iy_{i+1}) = \begin{cases} 
  i + 1 + \lfloor \frac{i-1}{3} \rfloor & \text{for } 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1, \\
  \lfloor \frac{2n+2}{3} \rfloor & \text{for } i = \lceil \frac{n}{2} \rceil, \\
  n - i + 2 + \lfloor \frac{n-i-1}{3} \rfloor & \text{for } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n - 1, \\
  1 & \text{for } i = n,
\end{cases}
\]

\[
\alpha_2(x_iy_i) = \begin{cases} 
  i + \lfloor \frac{i-1}{3} \rfloor & \text{for } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, \\
  \lfloor \frac{2n+2}{3} \rfloor - 1 & \text{for } i = \lfloor \frac{n}{2} \rfloor + 1 \text{ and } n \equiv 2 \pmod{6}, \\
  n - i + 3 + \lfloor \frac{n-i-1}{3} \rfloor & \text{for } \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n - 1, \\
  2 & \text{for } i = n,
\end{cases}
\]

\[
\alpha_2(y_ix_{i+1}) = \begin{cases} 
  1 & \text{for } i = 1, \\
  i + 1 + \lfloor \frac{i-2}{3} \rfloor & \text{for } 2 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1, \\
  \lfloor \frac{2n+2}{3} \rfloor - 1 & \text{for } i = \lfloor \frac{n}{2} \rfloor \text{ and } n \equiv 1, 2 \pmod{6}, \\
  \lfloor \frac{2n+2}{3} \rfloor & \text{for } i = \lfloor \frac{n}{2} \rfloor \text{ and } n \equiv 3 \pmod{6}, \\
  n - i + 2 + \lfloor \frac{n-i}{3} \rfloor & \text{for } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n.
\end{cases}
\]

Now we compute the \( C_3 \)-weights under the edge labeling \( \alpha_2 \) as follows. For the weights of 3-cycles of the first type we get

\[
wt_{\alpha_2}(C_3^i) = \sum_{e \in E(C_3^i)} \alpha_2(e) = \alpha_2(x_ix_{i+1}) + \alpha_2(x_iy_i) + \alpha_2(y_ix_{i+1})
\]

\[
= \begin{cases} 
  4i - 1 & \text{for } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, \\
  4n - 4i + 6 & \text{for } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n,
\end{cases}
\]
and for the weights of 3-cycles of the second type we have

\[ wt_{\alpha_2}(C^i_3) = \sum_{e \in E(C^i_3)} \alpha_2(e) = \alpha_2(y_iy_{i+1}) + \alpha_2(y_ix_{i+1}) = 2i + 3 \]

\[
\begin{align*}
&= \begin{cases} 
4i + 1 & \text{for } 1 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor - 1, \\
4n - 4i + 4 & \text{for } \left\lfloor \frac{n+1}{2} \right\rfloor \leq i \leq n.
\end{cases}
\end{align*}
\]

Combining these two cases one can see that the weights of the cycles \(C^i_3\) are different to the weights of the cycles \(C^i_2\). This shows that \(\alpha_2\) is an edge \(C_3\)-irregular labeling of \(A_n\). Therefore, \(\text{ehs}(A_n, C_4) \leq \left\lceil \frac{2n+2}{3} \right\rceil\) and we arrive at the desired result.

\[ \square \]

3. Triangular Ladder and Diagonal Ladder

Let \(L_n \cong P_n \sqcup P_2\), \(n \geq 2\), be a ladder with the vertex set \(V(L_n) = \{x_i, y_i : i = 1, 2, \ldots, n\}\) and the edge set \(E(L_n) = \{x_ix_{i+1}, y_iy_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{x_iy_i : i = 1, 2, \ldots, n\}\). The triangular ladder \(TL_n, n \geq 2\), is obtained from a ladder \(L_n\) by adding the edges \(y_ix_{i+1}\) for \(i = 1, 2, \ldots, n-1\).

**Theorem 4.** Let \(TL_n, n \geq 2\), be a triangular ladder. Then

\[ \text{ehs}(TL_n, C_3) = \left\lceil \frac{2n}{3} \right\rceil. \]

**Proof.** The triangular ladder \(TL_n, n \geq 2\), admits a \(C_3\)-covering with exactly \(2(n-1)\) cycles \(C_3\). There are two types of cycles \(C_3\) that cover \(TL_n\). The first type of cycles \(C_3\) has the vertex set \(V(C^i_3) = \{x_i, x_{i+1}, y_i : 1 \leq i \leq n-1\}\) and the edge set \(E(C^i_3) = \{x_ix_{i+1}, y_ix_{i+1}, y_iy_{i+1} : 1 \leq i \leq n-1\}\). The second type of cycles \(C_3\) has the vertex set \(V(C^i_3) = \{y_i, y_{i+1}, x_{i+1} : 1 \leq i \leq n-1\}\) and the edge set \(E(C^i_3) = \{y_iy_{i+1}, y_ix_{i+1}, y_{i+1}y_{i+2} : 1 \leq i \leq n-1\}\).

According to Theorem 1 it follows that \(\text{ehs}(TL_n, C_3) \geq \left\lceil \frac{2n}{3} \right\rceil\). To show that \(\left\lceil \frac{2n}{3} \right\rceil\) is an upper bound for the edge \(C_3\)-irregularity strength of \(TL_n\) we define a \(C_3\)-irregular edge labeling \(\alpha_3 : E(TL_n) \to \{1, 2, \ldots, \left\lceil \frac{2n}{3} \right\rceil\}\) as follows. Let us consider three cases.

**Case 1.** When \(i \equiv 0 \pmod{3}\), then

\[
\begin{align*}
\alpha_3(x_ix_{i+1}) &= \alpha_3(y_iz_{i+1}) = \frac{2i}{3} & \text{for } i = 3, 6, \ldots, n-1, \\
\alpha_3(y_ix_{i+1}) &= \alpha_3(y_iz_{i+1}) = \frac{2i+3}{3} & \text{for } i = 3, 6, \ldots, n-1, \\
\alpha_3(x_iz_{i+1}) &= \frac{2i}{3} & \text{for } i = 3, 6, \ldots, n.
\end{align*}
\]
Case 2. When \( i \equiv 1 \pmod{3} \), then
\[
\alpha_3(x_i x_{i+1}) = \alpha_3(y_i y_{i+1}) = \alpha_3(y_i x_{i+1}) = \frac{2i+1}{3} \quad \text{for } i = 1, 4, \ldots, n - 1,
\]
\[
\alpha_3(x_i y_i) = \frac{2i+1}{3} \quad \text{for } i = 1, 4, \ldots, n.
\]

Case 3. When \( i \equiv 2 \pmod{3} \), then
\[
\alpha_3(x_i x_{i+1}) = \frac{2i-1}{3} \quad \text{for } i = 2, 5, \ldots, n - 1,
\]
\[
\alpha_3(y_i y_{i+1}) = \alpha_3(y_i x_{i+1}) = \frac{2i+2}{3} \quad \text{for } i = 2, 5, \ldots, n - 1,
\]
\[
\alpha_3(x_i y_i) = \frac{2i+2}{3} \quad \text{for } i = 2, 5, \ldots, n.
\]

It is a routine matter to verify that under the labeling \( \alpha_3 \) all edge labels are at most \( \left\lceil \frac{2n}{3} \right\rceil \). It is not difficult to see that under the edge labeling \( \alpha_3 \) the weights of the cycles \( C_3^i \), \( 1 \leq i \leq n - 1 \), are of the form
\[
wt_{\alpha_3}(C_3^i) = \alpha_3(x_i x_{i+1}) + \alpha_3(y_i y_{i+1}) + \alpha_3(y_i x_{i+1}) = 2i + 1.
\]

The weights of the cycles \( C_3^i \), \( 1 \leq i \leq n - 1 \), are of the form
\[
wt_{\alpha_3}(C_3^i) = \alpha_3(y_i y_{i+1}) + \alpha_3(x_i x_{i+1}) + \alpha_3(y_i x_{i+1}) = 2(i + 1).
\]

Combining these two cases we obtained that the weights are different for any two distinct cycles \( C_3 \). Thus \( \ehs(DL_n, C_3) \leq \left\lceil \frac{2n}{3} \right\rceil \). This completes the proof. 

The diagonal ladder \( DL_n \) is obtained from a ladder \( L_n \) by adding the edges \( \{x_i y_{i+1}, x_{i+1} y_i : 1 \leq i \leq n - 1\} \). So the diagonal ladder \( DL_n \) contains \( 2n \) vertices and \( 5n - 4 \) edges.

**Theorem 5.** Let \( DL_n \), \( n \geq 2 \), be a diagonal ladder. Then
\[
\ehs(DL_n, K_4) = \left\lceil \frac{n + 4}{6} \right\rceil.
\]

**Proof.** The diagonal ladder \( DL_n \), \( n \geq 2 \), admits a \( K_4 \)-covering with exactly \( (n - 1) \) complete graphs \( K_4 \). The \( K_4^i \) has the vertex set \( V(K_4^i) = \{x_i, y_i, x_{i+1}, y_{i+1} : 1 \leq i \leq n - 1\} \) and the edge set \( E(K_4^i) = \{x_i x_{i+1}, x_i y_i, x_i y_{i+1}, y_i y_{i+1}, y_i x_{i+1}, x_{i+1} y_{i+1} : 1 \leq i \leq n - 1\} \).

With respect to Theorem 1 it follows that \( \ehs(DL_n, K_4) \geq \left\lceil \frac{n + 4}{6} \right\rceil \). To show that \( \ehs(DL_n, K_4) \leq \left\lceil \frac{n + 4}{6} \right\rceil \) we define a \( K_4 \)-irregular edge labeling \( \alpha_4 : \)
\[ E(DL_n) \rightarrow \{1, 2, \ldots, \left\lceil \frac{n+4}{6} \right\rceil \}, \] in the following way.

\[
\begin{align*}
\alpha_4(y_iy_{i+1}) &= \left\lfloor \frac{i}{6} \right\rfloor \quad \text{for } 1 \leq i \leq n-1, \\
\alpha_4(x_ix_{i+1}) &= \begin{cases} 
\left\lfloor \frac{i}{6} \right\rfloor & \text{for } 1 \leq i \leq 5, \\
\left\lceil \frac{i+1}{6} \right\rceil & \text{for } 6 \leq i \leq n-1, 
\end{cases} \\
\alpha_4(x_iy_i) &= \begin{cases} 
\left\lfloor \frac{i}{6} \right\rfloor & \text{for } 1 \leq i \leq 4, \\
\left\lceil \frac{i+2}{6} \right\rceil & \text{for } 5 \leq i \leq n, 
\end{cases} \\
\alpha_4(x_iy_{i+1}) &= \begin{cases} 
1 & \text{for } i = 1, \\
\left\lceil \frac{i+5}{6} \right\rceil & \text{for } 2 \leq i \leq n-1, 
\end{cases} \\
\alpha_4(x_{i+1}y_i) &= \begin{cases} 
1 & \text{for } i = 1, 2, \\
\left\lceil \frac{i+4}{6} \right\rceil & \text{for } 3 \leq i \leq n-1. 
\end{cases}
\end{align*}
\]

One can verify that under the labeling \( \alpha_4 \) all edge labels are at least 1 and at most \( \left\lceil \frac{n+4}{6} \right\rceil \). To show that \( \alpha_4 \) is edge \( K_4 \)-irregular labeling it will be enough to show that \( wt_{\alpha_4}(K^i_4) < wt_{\alpha_4}(K^{i+1}_4) \). It is a simple mathematical exercise that the weights of the subgraphs \( K^i_4, i = 1, 2, 3, 4, 5 \) are \( wt_{\alpha_4}(K^i_4) = 5 + i \).

For \( i = 6, 7, \ldots, n-1 \) we get

\[
wt_{\alpha_4}(K^i_4) = \sum_{e \in E(K^i_4)} \alpha_4(e) = \alpha_4(x_ix_{i+1}) + \alpha_4(y_iy_{i+1}) + \alpha_4(x_iy_i) + \alpha_4(x_{i+1}y_{i+1}) \\
+ \alpha_4(x_iy_{i+1}) + \alpha_4(x_{i+1}y_i) = \left\lfloor \frac{i+1}{6} \right\rfloor + \left\lfloor \frac{i}{6} \right\rfloor + \left\lceil \frac{i+2}{6} \right\rceil + \left\lceil \frac{i+3}{6} \right\rceil + \left\lceil \frac{i+5}{6} \right\rceil \\
+ \left\lceil \frac{i+4}{6} \right\rceil = 5 + i.
\]

This proves that \( wt_{\alpha_4}(K^{i+1}_4) = wt_{\alpha_4}(K^i_4) + 1 \) for \( i = 1, 2, \ldots, n-1 \). Therefore, \( \alpha_4 \) is an edge \( K_4 \)-irregular labeling of \( DL_n \). Thus \( \text{ehs}(DL_n, K_4) \leq \left\lceil \frac{n+4}{6} \right\rceil \). This concludes the proof.

4. Wheel and Gear Graph

A **wheel** \( W_n \), \( n \geq 3 \), is a graph obtained by joining all vertices of cycle \( C_n \) to a further vertex \( c \), called the **center**. Thus \( W_n \) contains \( n + 1 \) vertices, say, \( c, x_1, x_2, \ldots, x_n \) and \( 2n \) edges, say, \( cx_i, x_ix_{i+1}, 1 \leq i \leq n \), where the indices are taken modulo \( n \).

**Theorem 6.** Let \( W_n \), \( n \geq 4 \), be a wheel. Then

\[
\text{ehs}(W_n, C_3) = \left\lfloor \frac{n+2}{3} \right\rfloor.
\]
Proof. The wheel $W_n$, $n \geq 4$, admits a $C_3$-covering with exactly $n$ cycles $C_3$. Every cycle $C_3$ in $W_n$ is of the form $C_3 = cx_ix_{i+1}$, where $i = 1, 2, \ldots, n$ with indices taken modulo $n$.

According to Theorem 1 we have that $ehs(W_n, C_3) \geq \lceil \frac{n+2}{3} \rceil$. To show that $\lceil \frac{n+2}{3} \rceil$ is an upper bound for the edge $C_3$-irregularity strength of $W_n$ we define a $C_3$-irregular edge labeling $\alpha_5 : E(W_n) \to \{1, 2, \ldots, \lceil \frac{n+2}{3} \rceil \}$ as follows.

$$\alpha_5(x_ix_{i+1}) = \begin{cases} 
    i & \text{for } i = 1, 2, \\
    i - \left\lfloor \frac{i}{3} \right\rfloor & \text{for } 3 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor - 1, \\
    \left\lceil \frac{n+2}{3} \right\rceil & \text{for } i = \left\lceil \frac{n+1}{2} \right\rceil, \\
    n - i + 1 - \left\lfloor \frac{n-i}{3} \right\rfloor & \text{for } \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \leq i \leq n,
\end{cases}$$

$$\alpha_5(cx_i) = \begin{cases} 
    1 & \text{for } i = 1, \\
    i - 1 - \left\lfloor \frac{i-2}{3} \right\rfloor & \text{for } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
    \left\lceil \frac{n+2}{3} \right\rceil & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1 \text{ and } n \equiv 0, 1 \pmod{3}, \\
    \left\lceil \frac{n+2}{3} \right\rceil - 1 & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \equiv 2 \pmod{3}, \\
    n - i + 1 - \left\lfloor \frac{n-i-1}{3} \right\rfloor & \text{for } \left\lceil \frac{n}{2} \right\rceil + 2 \leq i \leq n - 1, \\
2 & \text{for } i = n.
\end{cases}$$

It is a matter of routine checking that under the labeling $\alpha_5$ all edge labels are at most $\lceil \frac{n+2}{3} \rceil$. For the $C_3$-weight of the cycle $C_3^i$ we get

$$wt_{\alpha_5}(C_3^i) = \alpha_5(cx_i) + \alpha_5(cx_{i+1}) + \alpha_5(x_ix_{i+1})$$

$$= \begin{cases} 
    2i + 1 & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
    2(n+2-i) & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n.
\end{cases}$$

Clearly, the weights of $C_3^i$ for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$ are odd and increasing. On the other hand the weights of $C_3^i$ for $\left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n$ are even and decreasing. So, it concludes that all the weights of $C_3^i$ are different. Thus $\alpha_5$ is an edge $C_3$-irregular labeling of $W_n$ and $ehs(W_n, C_3) \leq \lceil \frac{n+2}{3} \rceil$. This completes the proof. 

A gear graph $G_n$ is obtained from $W_n$ by inserting a vertex to each edge on the cycle $C_n$. Then the vertex set of $G_n$ is $V(G_n) = \{c, x_i, y_i : 1 \leq i \leq n\}$ and the edge set is $E(W_n) = \{x_iy_i, y_ix_{i+1}, cx_i : 1 \leq i \leq n\}$ with indices taken modulo $n$.

Theorem 7. Let $G_n$, $n \geq 3$, be a gear graph. Then

$$ehs(G_n, C_4) = \left\lceil \frac{n+3}{4} \right\rceil.$$

Proof. The gear $G_n$, $n \geq 3$, admits a $C_4$-covering with exactly $n$ cycles $C_4$. According to Theorem 1 we obtain that $ehs(G_n, C_4) \geq \left\lceil \frac{n+3}{4} \right\rceil$. To show that
\( \lceil \frac{n+3}{4} \rceil \) is an upper bound for the edge \( C_4 \)-irregularity strength of \( G_n \) we define a \( C_4 \)-irregular edge labeling \( \alpha_6 : E(G_n) \to \{1, 2, \ldots, \lceil \frac{n+3}{4} \rceil \} \), in the following way.

\[
\alpha_6(x_iy_i) = \begin{cases} 
\left\lceil \frac{i}{2} \right\rceil & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1, \\
\left\lceil \frac{n}{4} \right\rceil & \text{for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and } n \not\equiv 2 \pmod{4}, \\
\left\lceil \frac{n-i+2}{2} \right\rceil & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n,
\end{cases}
\]

\[
\alpha_6(y_ix_{i+1}) = \begin{cases} 
\left\lceil \frac{i+1}{2} \right\rceil & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\
\left\lceil \frac{n-i+1}{2} \right\rceil & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n,
\end{cases}
\]

\[
\alpha_6(cx_i) = \begin{cases} 
\left\lceil \frac{i}{2} \right\rceil & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\
\left\lceil \frac{n-i+3}{2} \right\rceil & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n.
\end{cases}
\]

It is easy to verify that under the labeling \( \alpha_6 \) all edge labels are at most \( \lceil \frac{n+3}{4} \rceil \). For the \( C_4 \)-weight of the cycle \( C_i^4 \), \( i = 1, 2, \ldots, n \), under the edge labeling \( \alpha_6 \), we get

\[
wt_{\alpha_6}(C_i^4) = \sum_{e \in E(C_i^4)} \alpha_6(e) = \alpha_6(cx_i) + \alpha_6(cx_{i+1}) + \alpha_6(x_iy_i) + \alpha_6(y_ix_{i+1})
\]

\[
= \begin{cases} 
2i + 2 & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\
2n + 5 - 2i & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n.
\end{cases}
\]

Clearly, the weights of \( C_i^4 \) for \( 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \) are even and increasing. On the other hand the weights of \( C_i^4 \) for \( \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n \) are odd and decreasing. So, it concluded that all the weights of \( C_i^4 \) are different. Thus \( \alpha_6 \) is an edge \( C_4 \)-irregular labeling of \( G_n \). Hence \( ehs(G_n, C_4) \leq \left\lceil \frac{n+2}{3} \right\rceil \). This completes the proof of theorem. \( \blacksquare \)

5. Conclusion

In this paper we have investigated the edge \( H \)-irregularity strength of some graphs. We have found the exact values of this parameter for several families of graphs namely, prisms, antiprisms, triangular ladders, diagonal ladders, wheels and gear graphs.

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