ON INDEPENDENT DOMINATION IN PLANAR CUBIC GRAPHS

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Abstract

A set \(S\) of vertices in a graph \(G\) is an independent dominating set of \(G\) if \(S\) is an independent set and every vertex not in \(S\) is adjacent to a vertex in \(S\). The independent domination number, \(i(G)\), of \(G\) is the minimum cardinality of an independent dominating set. Goddard and Henning [Discrete Math. 313 (2013) 839–854] posed the conjecture that if \(G \notin \{K_{3,3}, C_5 \square K_2\}\) is a connected, cubic graph on \(n\) vertices, then \(i(G) \leq \frac{3}{2}n\), where \(C_5 \square K_2\) is the 5-prism. As an application of known result, we observe that this conjecture is true when \(G\) is 2-connected and planar, and we provide an infinite family of such graphs that achieve the bound. We conjecture that if \(G\) is a bipartite, planar, cubic graph of order \(n\), then \(i(G) \leq \frac{3}{2}n\), and we provide an infinite family of such graphs that achieve this bound.

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1. Introduction

In this note, we continue the study of independent domination in cubic graphs. A set is independent in a graph if no two vertices in the set are adjacent. An independent dominating set, abbreviated ID-set, in a graph is a set that is both dominating and independent. Equivalently, an independent dominating set is a maximal independent set. The independent domination number of a graph $G$, denoted by $i(G)$, is the minimum cardinality of an independent dominating set, and an independent dominating set of cardinality $i(G)$ in $G$ is called an $i(G)$-set. Independent dominating sets have been studied extensively in the literature (see, for example, [1, 2, 4, 5, 7, 8, 9, 10, 12] and the so-called domination book [6]). A recent survey on independent domination in graphs can be found in [3].

Recall that $K_{3,3}$ denotes the bipartite complete graph with both partite sets on three vertices. The 5-prism, $C_5 \square K_2$, is the Cartesian product of a 5-cycle with a copy of $K_2$. The graphs $K_{3,3}$ and $C_5 \square K_2$ are shown in Figure 1(a) and 1(b), respectively.

As remarked in [4], the question of best possible bounds on the independent domination number of a connected, cubic graph remains unresolved. Lam, Shiu and Sun [9] established the following upper bound on the independent domination number of a connected, cubic graph. Equality in Theorem 1 holds for the prism $C_5 \square K_2$ (see Figure 1).

**Theorem 1** [9], For a connected, cubic graph $G$ on $n$ vertices, $i(G) \leq \frac{2}{5}n$ except for $K_{3,3}$.

Goddard and Henning [3] conjectured that the graphs $K_{3,3}$ and $C_5 \square K_2$ are the only exceptions for an upper bound of $\frac{3}{8}n$. We state their conjecture formally as follows.

**Conjecture 2** [3]. If $G \notin \{K_{3,3}, C_5 \square K_2\}$ is a connected, cubic graph on $n$ vertices, then $i(G) \leq \frac{3}{8}n$.

Dorbec et al. [2] proved Conjecture 2 when $G$ does not have a subgraph isomorphic to $K_{2,3}$. 
Theorem 3 [2]. If \( G \not\cong C_5 \square K_2 \) is a connected, cubic graph on \( n \) vertices that does not have a subgraph isomorphic to \( K_{2,3} \), then \( i(G) \leq \frac{3}{8} n \).

A graph \( G \) is \( k \)-vertex connected, which we shall simply write as \( k \)-connected, if there does not exist a set of \( k-1 \) vertices whose removal disconnects the graph, i.e., the vertex connectivity of \( G \) is at least \( k \). In particular, if a connected graph does not have a cut-vertex, then it is 2-connected. As a simple application of Theorem 3, we observe that Conjecture 2 is true for 2-connected, planar, cubic graphs.

Theorem 4. If \( G \not\cong C_5 \square K_2 \) is a 2-connected, planar, cubic graph on \( n \) vertices, then \( i(G) \leq \frac{3}{8} n \).

Proof. We show firstly that \( G \) has no subgraph isomorphic to \( K_{2,3} \). Suppose, to the contrary, that \( G \) has a subgraph \( F \), isomorphic to \( K_{2,3} \), with partite sets \( \{a, f\} \) and \( \{b, c, d\} \). Consider an embedding of \( G \) in the plane. For every embedding of \( K_{2,3} \) in the plane there is a cycle which has a vertex in its interior. Without loss of generality, suppose that \( c \) is a vertex in the interior of the cycle \( C \), where \( C: abfda \). Let \( x \) be the neighbor of \( c \) different from \( a \) and \( f \). Either the vertex \( x \) is in the interior of the cycle \( C \) or the vertex \( x \) belongs to \( C \), in which case \( x = b \) or \( x = d \). If \( x = b \), then the vertex \( d \) is a cut-vertex in \( G \), contradicting the 2-connectivity of \( G \). Hence, \( x \neq b \). Analogously, \( x \neq d \). Therefore, the vertex \( x \) is in the interior of \( C \). Renaming vertices, if necessary, we may assume that \( x \) is in the interior of cycle \( abfca \). Let \( X \) be the subgraph of \( G \) that lies in the interior of the cycle \( abfca \). By assumption, \( x \in X \). If the vertex \( b \) is adjacent to a vertex of \( X \), then the vertex \( d \) is a cut-vertex of \( G \), a contradiction. Therefore, the vertex \( b \) is not adjacent to a vertex of \( X \). However, then, the vertex \( c \) is a cut-vertex of \( G \), a contradiction. Hence, \( G \) has no subgraph isomorphic to \( K_{2,3} \). Thus, by Theorem 3, \( i(G) \leq 3n/8 \). \( \blacksquare \)

We pose the following conjecture.

Conjecture 5. If \( G \not\cong C_5 \square K_2 \) is a connected, planar, cubic graph on \( n \) vertices, then \( i(G) \leq \frac{3}{8} n \).

The following conjecture was posed by Zhu and Wu [13].

Conjecture 6 [13]. If \( G \) is a 2-connected, planar, cubic graph of order \( n \), then \( \gamma(G) \leq \frac{1}{3} n \).

We pose the following two conjectures.

Conjecture 7. If \( G \) is a bipartite, planar, cubic graph of order \( n \), then \( i(G) \leq \frac{1}{3} n \).

Conjecture 8. If \( G \) is a bipartite, planar, cubic graph of order \( n \), then \( \gamma(G) \leq \frac{1}{3} n \).
We remark that every bipartite, cubic graph has no cut-vertex, and therefore each of its components is a 2-connected, cubic (bipartite) graph. Hence, Conjecture 6 implies Conjecture 8, and so Conjecture 8 is a weaker conjecture than Conjecture 6. We also remark that Conjecture 7 implies Conjecture 8, and so Conjecture 8 is a weaker conjecture than Conjecture 7. A computer search confirms that Conjecture 7 is true when $n \leq 24$.

We have three immediate aims in this paper.

Our first aim is to provide an infinite family, $G_{\text{cubic}}$, of 2-connected, planar, cubic graphs that achieve the upper bound of Theorem 4. The family $G_{\text{cubic}}$ is constructed in Section 2.

Our second aim is to provide an infinite family, $F_{\text{cubic}}$, of connected, planar, cubic graphs that are not 2-connected that achieve the upper bound of Conjecture 5. The family $F_{\text{cubic}}$ is constructed in Section 3.

Our third aim is to provide an infinite family, $H_{\text{cubic}}$, of bipartite, planar, cubic graphs that achieve the upper bound of Conjecture 7 and Conjecture 8. The family $H_{\text{cubic}}$ is constructed in Section 4.

For $k \geq 1$, we use the notation $[k] = \{1, \ldots, k\}$.

2. THE GRAPH FAMILY $G_{\text{cubic}}$

We denote the graph obtained from a 5-prism by deleting an edge that does not belong to a 5-cycle by $(C_5 \Box K_2)^-$. The graph $(C_5 \Box K_2)^-$ is illustrated in Figure 2.

![Figure 2](image-url)

Figure 2. The graph $(C_5 \Box K_2)^-$.

Let $F \cong (C_5 \Box K_2)^-$, where $V(F) = \{r_1, r_2, \ldots, r_5, s_1, s_2, \ldots, s_5\}$, where $r_1r_2 \cdots r_5r_1$ and $s_1s_2 \cdots s_5s_1$ are the two 5-cycles in $F$ and $r_is_i \in E(F)$ for $i \in \{2, 3, 4, 5\}$. Let $H \cong (C_5 \Box K_2)^-$, where $V(H) = \{p_1, p_2, \ldots, p_5, q_1, q_2, \ldots, q_5\}$, where $p_1p_2 \cdots p_5p_1$ and $q_1q_2 \cdots q_5q_1$ are the two 5-cycles in $H$ and $p_iq_i \in E(H)$ for $i \in \{2, 3, 4, 5\}$. An infinite family, $G_{\text{cubic}}$, of 2-connected, planar, cubic graphs can be constructed as follows. For $k \geq 1$, define the graph $G_k$ as described below. Consider two copies of the path $P_{4k+2}$ with respective vertex sequences...
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c_0d_0a_1b_1c_1d_1 \cdots a_kb_kc_kd_k \text{ and } y_0z_0w_1x_1y_1z_1 \cdots w_kx_ky_kz_k. \text{ Join } c_0 \text{ to } z_0, \text{ and join } d_0 \text{ to } y_0, \text{ and for each } i \in [k], \text{ join } a_i \text{ to } w_i, \text{ b_i to } x_i, \text{ c_i to } z_i, \text{ and } d_i \text{ to } y_i.

To complete } G_k \text{ add a disjoint copy of } F \text{ and } H, \text{ and join } c_0 \text{ to } r_1, \text{ y_0 to } s_1, \text{ d_k to } p_1, \text{ and } z_k \text{ to } q_1. \text{ We note that the graph } G_k \text{ has order } 8k + 24. \text{ Let } G_{\text{cubic}} = \{ G_k : k \geq 1 \}. \text{ An embedding of the graph } G_2 \in G_{\text{cubic}} \text{ (of order 40) in the plane is illustrated in Figure 3.}

Figure 3. A planar drawing of the graph } G_2.

For simplicity, the graph } G_2 \text{ is redrawn in Figure 4.

Figure 4. The graph } G_2.

We are now in a position to prove the following result.

**Proposition 9.** If } G \in G_{\text{cubic}} \text{ has order } n, \text{ then } i(G) = \frac{3}{8}n.

**Proof.** Let } G \in G_{\text{cubic}} \text{ have order } n. \text{ Then, } G = G_k \text{ for some } k \geq 1, \text{ and so } G \text{ has order } n = 8k + 24. \text{ We show that } i(G) = 3k + 9. \text{ Let } V_0 = \{ c_0, d_0, y_0, z_0 \}, \text{ and let } V_i = \{ a_i, b_i, c_i, d_i, w_i, x_i, y_i, z_i \} \text{ for } i \in [k]. \text{ The set } \{r_2, r_4, s_1, s_3\} \cup \{ p_2, p_4, q_1, q_3 \} \cup \{ z_0 \} \cup \left( \bigcup_{i=1}^{k} \{ a_i, c_i, y_i \} \right)

is an ID-set of } G \text{ of cardinality } 3k + 9, \text{ implying that } i(G) \leq 3k + 9. \text{ We show next that } i(G) \geq 3k + 9. \text{ We adopt the following notation. If } X \text{ is a subset of vertices of } G, \text{ we let } X_F = X \cap V(F) \text{ and let } X_H = X \cap V(H). \text{ Further, we let } X_0 = V_0 \cap X, \text{ and for } i \in [k], \text{ we let } X_i = V_i \cap X.
Let $X$ be an $i(G)$-set. In order to dominate $\{d_0, z_0\}$, we note that $|X_0| \geq 1$ since at most one of $q_1$ and $w_1$ belong to $X$. In order to dominate $\{b_1, c_1, x_1, y_1\}$, we note that $|X_1| \geq 2$. Let $I_X = \{ i \in [k] : |X_i| = 2 \}$. Among all $i(G)$-sets, let $X$ be chosen so that

1. $|X_F| + |X_H|$ is maximum.
2. Subject to (1), $|X_0|$ is minimum.
3. Subject to (2), $|I_X|$ is minimum.

We proceed further with the following series of claims. The statement and proof of our first claim is analogous to the statement and proof of a similar claim in [4]. For completeness, we include the proof of this claim.

**Claim A.** If $\{d_i, z_i\} \subseteq X_i$ for some $i \in [k]$, then $|X_i| = 3$ or $|X_i| = 4$. Further, if $|X_i| = 3$, then either $a_i$ or $w_i$ is not dominated by $X_i$.

**Proof.** If $\{a_i, w_i\} \cap X_i \neq \emptyset$, then either $a_i \in X_i$, in which case $x_i \in X_i$ in order to dominate $x_i$, or $w_i \in X_i$, in which case $b_i \in X_i$ in order to dominate $b_i$. In both cases, $|X_i| = 4$. On the other hand, if $\{a_i, w_i\} \cap X_i = \emptyset$, then either $b_i \in X_i$, in which case $w_i$ is not dominated by $X_i$, or $x_i \in X_i$, in which case $a_i$ is not dominated by $X_i$.

**Claim B.** $3 \leq |X_H| \leq 4$. Further, if $|X_H| = 3$, then neither $p_1$ nor $q_1$ belongs to $X_H$, and exactly one of $p_1$ and $q_1$ is not dominated by $X_H$.

**Proof.** Suppose that $\{p_1, q_1\} \subseteq X_H$. In this case, $p_3 \in X_H$ or $q_3 \in X_H$. We may assume, by symmetry, that $p_3 \in X_H$, which forces $q_3$ to belong to $X_H$, and so $|X_H| = 4$. Suppose that exactly one of $p_1$ and $q_1$ belongs to $X_H$. We may assume, by symmetry, that $p_1 \in X_H$, and so $q_1 \notin X_H$. In order to dominate $q_2$, either $q_2 \in X_H$ or $q_3 \in X_H$. If $q_2 \in X_H$, then in order to dominate $p_3$ and $q_3$, we note that $X_H$ contains two vertices in addition to $p_1$ and $q_2$, and so $|X_H| = 4$. If $q_3 \in X_H$, then $X_H = \{p_1, p_3, q_3, q_5\}$, and once again $|X_H| = 4$. Suppose that neither $p_1$ nor $q_1$ belongs to $X_H$. In this case, either $p_2 \in X_H$ or $q_2 \in X_H$. We may assume, by symmetry, that $p_2 \in X_H$. Now, either $|X_H| = 4$ or $X_H = \{p_2, p_5, q_3\}$ or $X_H = \{p_2, p_5, q_4\}$. In particular, if $|X_H| = 3$, then $q_1$ is not dominated by $X_H$.

By symmetry, the proof of Claim C is analogous to that of Claim B, and is therefore omitted.

**Claim C.** $3 \leq |X_F| \leq 4$. Further, if $|X_F| = 3$, then neither $r_1$ nor $s_1$ belongs to $X_F$, and exactly one of $r_1$ and $s_1$ is not dominated by $X_F$.

**Claim D.** $|X_F| = |X_H| = 4$. 

**Proof.** Suppose, to the contrary, that \(|X_F| \neq 4\) or \(|X_H| \neq 4\). By symmetry, we may assume that \(|X_H| \neq 4\). Then, by Claim B, \(|X_H| = 3\), neither \(p_1\) nor \(q_1\) belongs to \(X_H\), and exactly one of \(p_1\) and \(q_1\) is not dominated by \(X_H\). We may assume, by symmetry, that \(p_1\) is not dominated by \(X_H\). In order to dominate the vertex \(p_1\), we have that \(d_k \in X_k\). But then \(z_k \in X_k\) in order to dominate \(z_k\), noting that \(q_1 \notin X_H\). Thus, \(\{d_k, z_k\} \subseteq X_k\). By Claim A, \(|X_k| = 3\) or \(|X_k| = 4\).

Suppose that \(|X_k| = 4\). In this case, either \(X_k = \{a_k, d_k, x_k, z_k\}\) or \(X_k = \{b_k, d_k, w_k, z_k\}\). We may assume, by symmetry, that \(X_k = \{a_k, d_k, x_k, z_k\}\). But then removing the five vertices in \(X_H \cup \{d_k, z_k\}\) from \(X\), and replacing them with the five vertices \(\{c_k, p_1, p_3, q_1, q_4\}\) produces a new \(i(G)\)-set \(X'\) satisfying \(|X'_F| = |X_F|\) and \(|X'_H| > |X_H|\), which is contrary to our choice of the set \(X\). Hence, \(|X_k| = 3\).

Since \(|X_k| = 3\), either \(X_k = \{b_k, d_k, z_k\}\) or \(X_k = \{x_k, d_k, z_k\}\). We may assume, by symmetry, that \(X_k = \{b_k, d_k, z_k\}\). But then removing the five vertices in \(X_H \cup \{d_k, z_k\}\) from \(X\), and replacing them with the five vertices \(\{y_k, p_1, p_3, q_1, q_4\}\) produces a new \(i(G)\)-set \(X'\) satisfying \(|X'_F| = |X_F|\) and \(|X'_H| > |X_H|\), which is contrary to our choice of the set \(X\).

**Claim E.** \(|X_0| = 1\).

**Proof.** As observed earlier, \(|X_0| \geq 1\). Suppose, to the contrary, that \(|X_0| \geq 2\). Then, either \(X_0 = \{c_0, y_0\}\) or \(X_0 = \{d_0, z_0\}\). If \(X_0 = \{c_0, y_0\}\), then removing the four vertices in \(X_F\) from \(X\), and replacing them with the three vertices \(\{r_2, r_5, s_3\}\) produces an ID-set of \(G\) of cardinality \(|X| - 1\), contradicting the fact that \(X\) is an \(i(G)\)-set. Hence, \(X_0 = \{d_0, z_0\}\). This implies that neither \(a_1\) nor \(w_1\) belongs to \(X\), and at most one of \(b_1\) and \(x_1\) belongs to \(X\). By symmetry, we may assume that \(b_1 \notin X\). The set \(X' = (X \setminus \{d_0\}) \cup \{a_1\}\) produces a new \(i(G)\)-set satisfying \(|X'_F| + |X'_H| = |X_F| + |X_H|\) and \(|X'_0| < |X_0|\), which is contrary to our choice of the set \(X\).

The proof of the following claim uses some of the arguments presented in [4].

**Claim F.** \(I_X = \emptyset\).

**Proof.** Suppose, to the contrary, that \(|I_X| \geq 1\). Let \(i\) be the largest integer such that \(|X_i| = 2\). In order to dominate \(\{b_i, c_i, x_i, y_i\}\), we may assume, by symmetry, that \(X_i = \{b_i, y_i\}\) or \(X_i = \{b_i, z_i\}\) or \(X_i = \{b_i, d_i\}\) or \(X_i = \{c_i, y_i\}\). In all four cases, the vertex \(w_i\) is not dominated by \(X_i\). If \(i = 1\), then this would imply that in order to dominate the vertex \(w_i\), we have that \(z_0 \in X_0\). But then \(d_0 \in X_0\), and so \(X_0 = \{d_0, z_0\}\), contradicting Claim E.

Thus, \(i \geq 2\). We now consider the set \(X_{i-1}\). In order to dominate the vertex \(w_i\), we have that \(z_{i-1} \in X_{i-1}\). But then \(d_{i-1} \in X_{i-1}\) in order to dominate \(d_{i-1}\). Thus, \(\{d_{i-1}, z_{i-1}\} \subseteq X_{i-1}\). By Claim A, either \(|X_{i-1}| = 3\) or \(|X_{i-1}| = 4\).
Suppose that \(|X_{i-1}| = 4\). We may assume, by symmetry, that \(a_{i-1} \in X_{i-1}\); that is, \(X_{i-1} = \{a_{i-1}, d_{i-1}, x_{i-1}, z_{i-1}\}\). But then the set \(X' = (X \setminus \{d_{i-1}, x_{i-1}, z_{i-1}\}) \cup \{c_{i-1}, y_{i-1}, w_{i}\}\) is an \(i(G)\)-set such that \(|X'_F| + |X'_H| = |X_F| + |X_H|\), \(|X'_0| = |X_0|\), and \(|I_{X'}| < |I_X|\), contradicting our choice of the set \(X\). Hence, \(|X_{i-1}| = 3\).

Since \(|X_{i-1}| = 3\), either \(X_{i-1} = \{b_{i-1}, d_{i-1}, z_{i-1}\}\) or \(X_{i-1} = \{x_{i-1}, d_{i-1}, z_{i-1}\}\). We may assume, by symmetry, that \(X_{i-1} = \{b_{i-1}, d_{i-1}, z_{i-1}\}\). Thus, \(w_{i-1}\) is not dominated by \(X_{i-1}\). If \(i = 2\), then this would imply that in order to dominate the vertex \(w_{i-1}\), we have that \(z_0 \in X_0\). But then \(d_0 \in X_0\), and so \(X_0 = \{d_0, z_0\}\), contradicting Claim E. Thus, \(i \geq 3\). We now consider the set \(X_{i-2}\). In order to dominate the vertex \(w_{i-1}\), we have that \(\{d_{i-2}, z_{i-2}\} \subseteq X_{i-2}\).

Continuing this process, there is a smallest positive integer \(j < i\) such that \(\{d_{i-j}, z_{i-j}\} \subseteq X_{i-j}\) and \(|X_{i-j}| = 4\). We may assume, by symmetry, that \(a_{i-j} \in X_{i-j}\); that is, \(X_{i-j} = \{a_{i-j}, d_{i-j}, x_{i-j}, z_{i-j}\}\). We now define the set \(X'\) of vertices of \(G\) as follows. For \(\ell \in [k]\), let \(X'_\ell = V_i \cap X'\) be the set defined as follows. Let \(X'_i = X_i \cup \{w_i\}\) and let \(X'_{i-j} = \{a_{i-j}, c_{i-j}, y_{i-j}\}\). If \(j \geq 2\), then for \(i - j + 1 \leq \ell \leq i - 1\), let \(X'_\ell = \{a_{\ell}, c_{\ell}, y_{\ell}\}\). If \(j \leq i - 1\), then for \(0 \leq \ell \leq i - j - 1\), let \(X'_\ell = X_F\). If \(i < k\), then for \(i + 1 \leq \ell \leq k\), let \(X'_\ell = X_F\). Then, \(|X'_i| = |X_i| + 1 = 3\), \(|X'_{i-j}| = |X_{i-j}| - 1 = 3\), and \(|X'_{\ell}| = |X_{\ell}|\) for all \(\ell \notin \{i, i-j\}\), where \(\ell \in [k] \cup \{0\}\).

Further, let \(X'_F = X_F\) and \(X'_H = X_H\). Thus,

\[X' = X'_F \cup X'_H \cup \bigcup_{i=1}^{k} X'_i,\]

and \(|X'| = |X|\). Since the set \(X\) is an ID-set, by construction so too is the set \(X'\), implying that the set \(X'\) is an \(i(G)\)-set. However, \(|X'_F| = |X_F|\), \(|X'_H| = |X_H|\), \(|X'_0| = |X_0|\) and \(|I_{X'}| < |I_X|\), contradicting our choice of the set \(X\). Consequently, \(I_X = \emptyset\).

By Claim F, \(I_X = \emptyset\), implying that \(|X_i| \geq 3\) for all \(i \in [k]\). Thus, by Claim D and Claim E, we note that \(i(G) = |X| \geq 3k + 9\). As observed earlier, \(i(G) \leq 3k + 9\). Consequently, \(i(G) = 3k + 9 = 3n/8\). This completes the proof of Proposition 9.

3. The Graph Family \(\mathcal{F}_{\text{cubic}}\)

Following the notation introduced in Section 2, we construct an infinite family, \(\mathcal{F}_{\text{cubic}}\), of connected, planar, cubic graphs that are not 2-connected as follows. Let \(G_1^*\) be the graph obtained from the graph \(G_1 \in \mathcal{G}_{\text{cubic}}\) by deleting the vertices in \(V(F)\), and adding a new vertex \(v\) and adding the edges \(vc_0\) and \(vy_0\). The resulting graph, \(G_1^*\), is illustrated in Figure 5.
We note that $G^*_1$ has order 23. An analogous, but simpler, proof than that of Proposition 9 (or simple use a computer) shows that $i(G^*_1) = 9$. The set $\{p_1, p_3, q_1, q_4, c_1, y_1, d_0, z_0, v\}$ is an example of an $i(G^*_1)$-set.

For $k \geq 3$, let $F_1, F_2, \ldots, F_k$ be $k$ vertex-disjoint copies of the graph $G^*_1$, and let $v_i$ be the vertex of degree 2 in $F_i$ for $i \in [k]$. Let $C: u_1 u_2 \cdots u_k u_1$ be a $k$-cycle that has no vertex in common with these $k$ copies of the graph $G^*_1$. Let $F^*_k$ be the graph obtained from the disjoint union, $F_1 \cup F_2 \cup \cdots \cup F_k \cup C$, of these $k + 1$ graphs by adding the $k$ edges $u_i v_i$ for $i \in [k]$. Let $\mathcal{F}_{cubic} = \{ F^*_k : k \geq 3 \}$. The graph $F^*_4$ (of order 96) in the family $\mathcal{F}_{cubic}$ is illustrated in Figure 6.

For each $k \geq 3$, the graph $F^*_k$ has order $n = 24k$. Further, since $i(G^*_1) = 9$ and there exists an $i(G^*_1)$-set containing the vertex $v$ of degree 2 in $G^*_1$, we observe that $i(F^*_k) = 9k = 3n/8$. We state this formally as follows.

**Proposition 10.** If $G \in \mathcal{F}_{cubic}$ has order $n$, then $G$ is a connected, planar, cubic graph satisfying $i(G) = \frac{3}{8}n$.

4. **The Graph Family $\mathcal{H}_{cubic}$**

An infinite family, $\mathcal{H}_{cubic}$, of bipartite, planar, cubic graphs can be constructed as follows. For $k \geq 2$, define the graph $H_k$ as described below. Consider two copies of the cycle $C_{2k}$ with respective vertex sequences $a_1 b_1 a_2 b_2 \cdots a_k b_k a_1$ and $c_1 d_1 c_2 d_2 \cdots c_k d_k c_1$. To complete $H_k$, add $2k$ new vertices $e_1, e_2, \ldots, e_k$ and
Let $S$ be a set of vertices in a graph $G$ and let $v \in S$. The open neighborhood of $v$ in $G$ is $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood of $v$ is $N_G[v] = \{v\} \cup N_G(v)$. The $S$-private neighborhood of $v$ is defined by $pn[v,S] = \{w \in V(G) : N_G[w] \cap S = \{v\}\}$. A classical result of Ore [11] states that if $S$ is dominating set in a graph $G$, then $S$ is a minimal dominating set of $G$ if and only if for each $v \in S$, $pn[v,S] \neq \emptyset$.

We are now in a position to prove the following result.

**Proposition 11.** If $G \in \mathcal{H}_{\text{cubic}}$ has order $n$, then $\gamma(G) = i(G) = \frac{1}{2}n$.

**Proof.** Let $G \in \mathcal{H}_{\text{cubic}}$ have order $n$. Then, $G = H_k$ for some $k \geq 2$, and so $G$ has order $n = 6k$. We show that $\gamma(G) = i(G) = 2k$. Let $X_i = \{a_i, b_i, c_i, d_i, e_i, f_i\}$ for $i \in [k]$. The set $D_k = \bigcup_{i=1}^k \{a_i, b_i\}$ is an ID-set of $G$ of cardinality $2k$, implying that $i(G) \leq |D_k| = 2k$. We show next that $\gamma(G) \geq 2k$. Let $S$ be a $\gamma(G)$-set. By the minimality of $S$, and by construction of the graph $G$, we note that $1 \leq |S \cap X_i| \leq 4$ for all $i \in [k]$. For $j \in [4]$, let $S_j = \{i \in [k] : |S \cap X_i| = j\}$. Thus, $(S_1, S_2, S_3, S_4)$ is a (weak) partition of the set $[k]$, where some of the sets may be empty. We note that $|S| = \sum_{i=1}^k |S_i|$ and $k = \sum_{i=1}^4 |S_i|$. In what follows, we take addition modulo $k$. Among all $\gamma(G)$-sets, we choose $S$ so that $|S_4|$ is a minimum. We proceed further with the following two claims.

**Claim I.** $|S_4| = 0$.

**Proof.** Suppose, to the contrary, that $|S_4| \geq 1$. Thus, $|S \cap X_i| = 4$ for some $i \in [k]$. By the minimality of the set $S$, we note that $S \cap X_i = \{a_i, b_i, c_i, d_i\}$. By Ore’s Theorem [11] and the structure of the graph $G$, we note that $pn[a_i, S] = \{b_{i-1}\}$ and $pn[c_i, S] = \{d_{i-1}\}$. This implies that $S \cap X_{i-1} = \{e_{i-1}\}$. We now consider the set $S' = (S \setminus \{a_i, c_i\}) \cup \{e_i, f_{i-1}\}$. The resulting set $S'$ is a dominating set of $G$ satisfying $|S'| = |S|$, and is therefore a $\gamma(G)$-set. However, $|S'_4| = |S_4| - 1$, contradicting our choice of the set $S$. Therefore, $|S_4| = 0$. 

Figure 7. The bipartite, planar, cubic graph $H_5$. 

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Claim II. \(|S_3| \geq |S_1|\).

Proof. Suppose that \(i \in S_1\) for some \(i \in [k]\), and so \(|S \cap X_i| = 1\). In order to dominate \(e_i\) and \(f_i\), we note that either \(e_i \in S\) or \(f_i \in S\). Suppose that \(e_i \in S\).

In order to dominate \(b_i\), the vertex \(a_{i+1} \in S\), while in order to dominate \(d_i\), the vertex \(c_{i+1} \in S\). In order to dominate the vertex \(f_{i+1}\), the set \(S\) contains a vertex of \(X_{i+1}\) different from \(a_{i+1}\) and \(c_{i+1}\), implying that \(|S \cap X_{i+1}| \geq 3\). By Claim I, \(|S \cap X_{i+1}| \leq 3\). Consequently, \(|S \cap X_{i+1}| = 3\), and so \(i + 1 \in S_3\). Hence, if \(e_i \in S\), then \(i + 1 \in S_3\), \(\{a_{i+1}, c_{i+1}\} \subset S\) and \(|S \cap \{b_{i+1}, d_{i+1}\}| \leq 1\). Analogously, if \(f_i \in S\), then \(i - 1 \in S_3\), \(\{b_{i-1}, d_{i-1}\} \subset S\) and \(|S \cap \{a_{i-1}, c_{i-1}\}| \leq 1\). This implies that if \(i \in S_1\), then either \(e_i \in S\), in which case we can uniquely associate \(i + 1 \in S_3\) with \(i\), or \(f_i \in S\), in which case we can uniquely associate \(i - 1 \in S_3\) with \(i\). Therefore, \(|S_3| \geq |S_1|\).

We now return to the proof of Proposition 11. By Claim I, \(|S_4| = 0\), and so \(|S| = \sum_{i=1}^k i|S_i|\) and \(k = \sum_{i=1}^3 |S_i|\).

By Claim II, \(|S_3| \geq |S_1|\), and so \(k = |S_1| + |S_2| + |S_3| \geq 2|S_1| + |S_2|\), or, equivalently, \(k - 2|S_1| - |S_2| \geq 0\). Thus,

\[
|S| = |S_1| + 2|S_2| + 3|S_3| = |S_1| + 2|S_2| + 3(k - |S_1| - |S_2|) = 3k - 2|S_1| - |S_2| = 2k + (k - 2|S_1| - |S_2|) \geq 2k.
\]

Thus, \(2k \leq |S| = \gamma(G) \leq i(G) \leq 2k\). Consequently, we must have equality throughout this inequality chain. In particular, \(\gamma(G) = i(G) = 2k = \frac{1}{2}n\). ■

5. Summary of Results

In this paper, we consider five conjectures which we name as Conjectures 2, 5, 6, 7 and 8. We first consider Conjecture 2. We prove in Theorem 4 that Conjecture 2 is true for 2-connected graphs. Our first main result constructs an infinite family, \(G_{\text{cubic}}\), of 2-connected, planar, cubic graphs in Section 2 to show that in this case the bound is tight.

We next consider Conjecture 5. By our previous result, it suffices to prove Conjecture 5 for connected, planar, cubic graphs that contain cut-vertices. Our second result constructs an infinite family, \(F_{\text{cubic}}\), of connected, planar, cubic graphs that contain cut-vertices in Section 3 to show that if Conjecture 5 is true for graphs with cut-vertices, then the bound is tight.

We finally consider Conjectures 6, 7 and 8. Our third result constructs an infinite family, \(H_{\text{cubic}}\), of bipartite, planar, cubic graphs in Section 4 to show that if Conjectures 7 and 8 are true, then the bounds are tight.
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