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ABOUT (k,l)-KERNELS, SEMIKERNELS AND GRUNDY FUNCTIONS IN PARTIAL LINE DIGRAPHS[1]

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Abstract

Let D be a digraph of minimum in-degree at least 1. We prove that for any two natural numbers k, l such that $1 \leq l \leq k$, the number of (k,l)-kernels of D is less than or equal to the number of (k,l)-kernels of any partial line digraph $\mathcal{L}D$. Moreover, if $l < k$ and the girth of D is at least $l+1$, then these two numbers are equal. We also prove that the number of semikernels of D is equal to the number of semikernels of $\mathcal{L}D$. Furthermore, we introduce the concept of (k,l)-Grundy function as a generalization of the concept of

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Grundy function and we prove that the number of (k, l)-Grundy functions of D is equal to the number of (k, l)-Grundy functions of any partial line digraph $\mathcal{L}D$.

Keywords: digraphs, in-domination, kernel, Grundy function.

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1. Introduction

Throughout the paper, $D = (V, A)$ denotes a loopless digraph with vertex set V and arc set A. Let $\omega^-(x)$ stand for the set of arcs having vertex x as their terminal vertex, and $\omega^+(x)$ stand for the set of arcs having vertex x as their initial vertex. Thus, the *in-degree* of x is $d^-(x) = |\omega^-(x)|$ and the *out-degree* of x is $d^+(x) = |\omega^+(x)|$. The *minimum in-degree* (*minimum out-degree*) of D is $\delta^-(D) = \min\{d^-_D(x) : x \in V\}$ ($\delta^+(D) = \min\{d^+_D(x) : x \in V\}$ respectively). Moreover, given a set $U \subseteq V$, $\omega^-(U) = \{(x, y) \in A : y \in U \text{ and } x \notin U\}$. Given a set of arcs $\Omega \subseteq A$, the *heads* of Ω are the vertices in the set $H(\Omega) = \{y : (x, y) \in \Omega\}$. For any pair of vertices $x, y \in V$, a directed path $(x, x_1, \ldots, x_{n-1}, y)$ from x to y is called an $x \to y$ path. The *distance* from x to y is denoted by $d_D(x, y)$ and it is defined to be the length of a shortest $x \to y$ path.

A set $K \subset V(D)$ is said to be a *kernel* if it is both independent (for every two vertices $x, y \in K$, $d_D(x, y) \geq 2$) and absorbing (a vertex not in K has a successor in K). This concept was first introduced in [13] by Von Neumann and Morgensten in the context of Game Theory as a solution for cooperative n-player games. The concept of a kernel is important to the theory of digraphs because it arises naturally in applications such as Nim-type games, logic, and facility location, to name a few. Several authors have been investigating sufficient conditions for the existence of kernels in digraphs, for a comprehensive survey see for example [4] and [7]. Also see Chapter 15 of [12] for a summary.

Let l, k be two integers such that $l \geq 1$ and $k \geq 2$. A (k, l)-*kernel of a digraph* D is a subset of vertices K which is both k-*independent* ($d_D(u, v) \geq k$ for all $u, v \in K$) and l-*absorbing* ($d_D(x, K) \leq l$ for all $x \in V \setminus K$). Observe that any kernel is a $(2, 1)$-kernel and a quasikernel, introduced in [10], is a $(2, 2)$-kernel. The concept of (k, l)-kernel is a nice and wide generalization of the concept of kernel; (k, l)-kernels have been deeply studied by several authors, see for example [8, 9, 18–20].

Grundy functions are very useful in the context of game theory and they are nearly related to kernels since a digraph with Grundy function has also a kernel. Also the concept of semikernel is very close to that of kernel, because a digraph such that every induced subdigraph has a nomempty semikernel has a kernel.

The line digraph technique is a good general method for obtaining large digraphs with fixed degree and diameter. In the line digraph $L(D)$ of a digraph

D, each vertex represents an arc of D. Thus, $V(L(D)) = \{uv : (u,v) \in A(D)\}$, and a vertex uv is adjacent to a vertex xz if and only if $v = x$, that is, when the arc (u,v) is adjacent to the arc (x,z) in D. For any $h > 1$, the *h-iterated line digraph*, $L^h(D)$, is defined recursively by $L^h(D) = L(L^{h-1}(D))$. For more information about line digraphs see, for instance, Aigner [1], Fiol, Yebra and Alegre [5] and Reddy, Kuhl, Hosseini and Lee [17].

A wider family of digraphs, called partial line digraphs, was introduced in [6] as a generalization of line digraphs. Let $D = (V, A)$ be a digraph and consider an arc subset $A' \subseteq A$ and a surjective mapping $\phi : A \to A'$ such that

(i) the set of heads of A' is $H(A') = V$;
(ii) the map ϕ fixes the elements of A', that is, $\phi|A' = id$, and for every vertex $j \in V$, $\phi(\omega^-(j)) \subset \omega^-(j) \cap A'$.

Hence, $|V| \leq |A'| \leq |A|$. Note that the existence of such a subset A' is guaranteed when $\delta^-(i) \geq 1$ for every $i \in V$. Then, *the partial line digraph* of D, denoted by $\mathcal{L}_{(A',\phi)}D$ (for short $\mathcal{L}D$ if the pair (A', ϕ) is clear from the context), is the digraph with vertex set $V(\mathcal{L}D) = A'$ and set of arcs

$$A(\mathcal{L}D) = \{(ij, \phi((j,k))) : (j,k) \in A\}.$$

Remark 1. If $A' = A$, then $\phi = id$ and the partial line digraph $\mathcal{L}D$ coincides with the line digraph $L(D)$.

In what follows we write $\phi(jk)$ instead $\phi((j,k))$ for clarity. Figure 1 shows an example of a digraph D with 12 arcs and its partial line digraph with $|A'| = 9$ vertices. The arcs not in A' are drawn with dotted lines and have images $\phi(12) = 42$, $\phi(34) = 54$, and $\phi(65) = 25$.

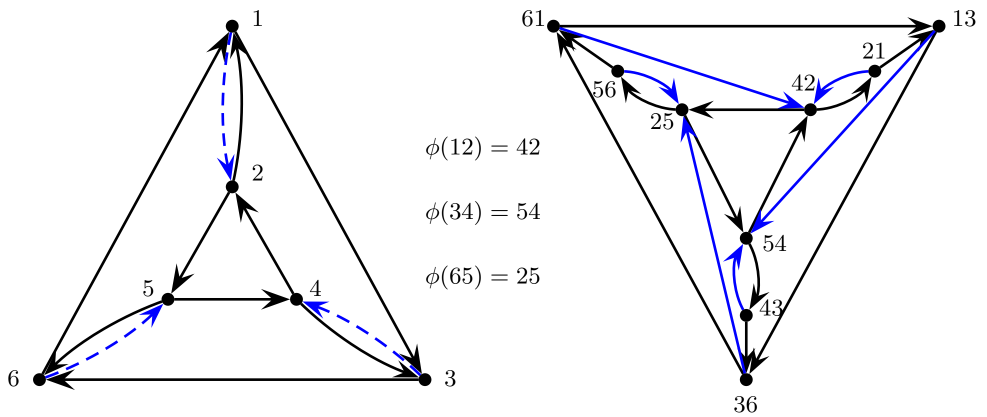

Figure 1. A digraph and its partial line digraph.

In this paper we study the relationship between the number of (k,l)-kernels (respectively, semikernels) of a digraph D and the corresponding number in any partial line digraph $\mathcal{L}D$. Also we introduce the concept of (k,l)-Grundy function as a generalization of the concept of Grundy function and we prove that the number of (k,l)-Grundy functions of D is equal to the number of (k,l)-Grundy functions of any partial line digraph $\mathcal{L}D$.

2. (k,l)-Kernels and Semikernels

In this section we will prove that the number of (k,l)-kernels of a digraph is less than or equal to the number of (k,l)-kernels of its partial line digraphs, and under certain conditions these two numbers are equal.

We start by proving a result concerning independent sets of a digraph and of those of their partial line digraphs.

Lemma 2. *Let D be a digraph with minimum in-degree at least 1. Let A' and ϕ satisfy the requirements of the definition of a partial line digraph, i.e., $\mathcal{L}_{(A',\phi)}D = \mathcal{L}D$. Let $k \geq 2$ be an integer number. Denote by $\mathcal{I}$ the set of all k-independent sets of D, and by $\mathcal{I}^*$ the set of all k-independent sets of $\mathcal{L}D$. Then the assignment $f : \mathcal{I} \to \mathcal{I}^*$ defined by $f(I) = \omega^-(I) \cap A'$ for all $I \in \mathcal{I}$ is an injective function. Therefore the number of k-independent sets of D is less than or equal to the number of k-independent sets of $\mathcal{L}D$.*

Proof. First of all let us see that f is a function. Let $ab, cd \in \omega^-(I) \cap A'$ be such that $d_{\mathcal{L}D}(ab, cd) = t$, and observe that $d_D(b,d) \geq k$ because $b, d \in I$. By definition of ϕ any shortest path from ab to cd in $\mathcal{L}D$ is $ab, \phi(bb_1), \phi(b_1b_2), \ldots, \phi(b_{t-1}b_t) = cd$, where $b_i \in V(D)$ and $(b,b_1), (b_i, b_{i+1}) \in A(D)$, $i = 1, \ldots, t-1$. Since $\phi(b_{t-1}b_t) = \alpha b_t$ for some $\alpha \in V(D)$, then $b_t = d$ yielding that a walk $b, b_1, \ldots, b_t = d$ from b to d of length t exists in D. This means that $t \geq d_D(b,d) \geq k$ and hence every two vertices of $\omega^-(I) \cap A'$ are mutually at distance at least k.

Let us prove that f is an injective function. Let $I_1, I_2 \in \mathcal{I}$ be such that $f(I_1) = f(I_2)$, that is $\omega^-(I_1) \cap A' = \omega^-(I_2) \cap A'$. Let us show that $I_1 = I_2$. Let $u \in I_1$. Note that by item (i) of definition of $\mathcal{L}D$ there is $y \in V(D)$ such that $yu \in A'$. Clearly, $yu \in \omega^-(I_1) \cap A'$ which implies that $yu \in \omega^-(I_2) \cap A'$, then $u \in I_2$, that is, $I_1 \subseteq I_2$. Reasoning analogously, $I_2 \subseteq I_1$ yielding that $I_1 = I_2$. Therefore f is an injective function and the lemma holds. ■

The concept of *Fibonacci number* for a graph G was introduced in [15] and it is defined as the number of independent subsets of G including the empty set. We extend this concept for digraphs, and we give an upper bound on the Fibonacci number of a digraph in terms of the Fibonacci number of its partial line digraph.

Corollary 3. *Let D be a digraph with minimum in-degree at least* 1*. Let A' and ϕ satisfy the requirements of the definition of a partial line digraph, i.e., $\mathcal{L}_{(A',\phi)}D = \mathcal{L}D$. Then the Fibonacci number of D is less than or equal to the Fibonacci number of $\mathcal{L}D$.*

2.1. (k,l)-kernels

Some known results about the existence of kernels and (k,l)-kernels in line digraphs can be seen in [11, 16]. The following theorem is proved in [2].

Theorem 4 [2]**.** *Let k, l be two natural numbers such that $1 \leq l < k$, and let D be a digraph with minimum in-degree at least* 1 *and girth at least $l+1$. Then D has a (k,l)-kernel if and only if any partial line digraph $\mathcal{L}D$ has a (k,l)-kernel.*

Note that, since a kernel is a $(2,1)$-kernel, it follows that D has a kernel if and only if any partial line digraph $\mathcal{L}D$ has a kernel. Next, we establish a relationship between the number of (k,l)-kernels of D and the number of (k,l)-kernels of $\mathcal{L}D$.

Theorem 5. *Let k, l be two natural numbers such that $l \geq 1$ and $k \geq 2$, and let D be a digraph with minimum in-degree at least* 1*. Let A' and ϕ satisfy the requirements of the definition of a partial line digraph, i.e., $\mathcal{L}_{(A',\phi)}D = \mathcal{L}D$. Then the number of (k,l)-kernels of D is less than or equal to the number of (k,l)-kernels of $\mathcal{L}D$. Moreover, if $l < k$ and the girth of D is at least $l+1$, then these numbers are equal.*

Proof. Denote by $\mathcal{K}$ the set of all (k,l)-kernels of D, and by $\mathcal{K}^*$ denote the set of all (k,l)-kernels of $\mathcal{L}D$.

Let $f : \mathcal{K} \to \mathcal{K}^*$ be defined by $f(K) = \omega^-(K) \cap A'$ for all $K \in \mathcal{K}$. In the proof of Theorem 2.1 of [2] it was proved that f is well defined. And from Lemma 2 it follows that f is injective. Therefore $|\mathcal{K}| \leq |\mathcal{K}^*|$.

Let $h : \mathcal{K}^* \to \mathcal{K}$ be defined by $h(\hat{K}) = H(\hat{K})$ for all $\hat{K} \in \mathcal{K}^*$. In the proof of Theorem 2.1 of [2] it was proved that h is well defined if $l < k$ and the girth of D is at least $l+1$. Moreover, we can check that $h = f^{-1}$ because $h(f(K)) = H(\omega^-(K) \cap A') = K$. Therefore the theorem holds. ∎

Corollary 6. *Let D be a digraph with minimum in-degree at least* 1*. Let A' and ϕ satisfy the requirements of the definition of a partial line digraph, i.e., $\mathcal{L}_{(A',\phi)}D = \mathcal{L}D$. Then the following assertions hold.*

(i) *The number of kernels of D is equal to the number of kernels of $\mathcal{L}D$.*
(ii) *The number of quasikernels of D is less than or equal to the number of quasikernels of $\mathcal{L}D$.*

Let us observe that the number of quasikernels of D can be strictly less than the number of quasikernels of its line digraph $L(D)$. A quasikernel is a $(2,2)$-kernel as we mentioned before, i.e., $k = l = 2$. The digraphs shown in Figure 2 have different number of quasikernels. This proves that the hypothesis $l < k$ of Theorem 5 cannot be avoided to guarantee that the number of quasikernels of D and $L(D)$ is equal. In this example the digraph D on the left side has 3 quasikernels, namely, $\{x\}$, $\{z\}$ and $\{y,t\}$, while its line digraph on the right side has 5 quasikernels which are $\{zx\}$, $\{tz,yz\}$, $\{xy,xt\}$, $\{xt,yz\}$ and $\{xy,tz\}$.

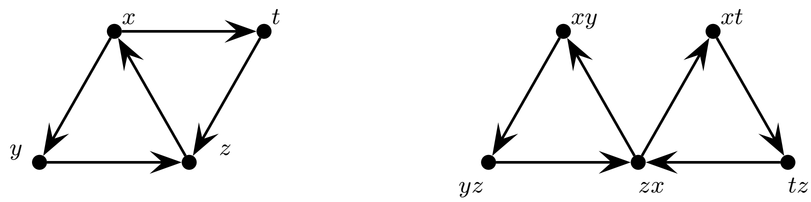

Figure 2. A digraph with 3 quasikernels and its line digraph with 5 quasikernels.

2.2. Semikernels

Let S be an independent set of D. We say that S is a *semikernel* of D if for all $sx \in \omega^+(S)$ there exists $xs' \in \omega^-(S)$. Thus, a vertex of out-degree zero forms a semikernel. Also a vertex only incident with symmetric arcs forms a semikernel. Figure 3 depicts a digraph having a semikernel but not a kernel. In [14] it was proved that if every induced subdigraph of D has a (nonempty) semikernel, then every induced subdigraph of D has a kernel, and so D.

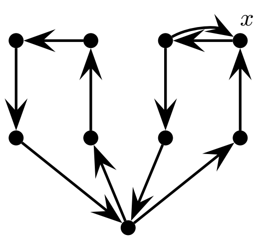

Figure 3. A digraph with semikernel $\{x\}$ but without kernels.

In Theorem 2.1 of [10] it was proved that the number of semikernels of a digraph with minimum in-degree at least one is less than or equal to the number of semikernels of the line digraph. Next we improve and generalize this result by stating the equality for every partial line digraph.

Theorem 7. *Let D be a digraph with minimum in-degree at least* 1. *Let A' and ϕ satisfy the requirements of the definition of a partial line digraph, i.e., $\mathcal{L}_{(A',\phi)}D = \mathcal{L}D$. Then the number of semikernels of D is less than or equal to the number of semikernels of $\mathcal{L}D$.*

Proof. Denote by $\mathcal{S}$ the set of all semikernels of D, and by $\mathcal{S}^*$ denote the set of all semikernels of $\mathcal{L}D$. Let $f : \mathcal{S} \to \mathcal{S}^*$ be defined by $f(K) = \omega^-(K) \cap A'$ for all $K \in \mathcal{S}$. Let us see that $f(K) \in \mathcal{S}^*$.

By Lemma 2, we know that f is an injective function and $f(K)$ is an independent set. It remains to prove that if $(e', e) \in \omega^+(f(K))$, then there exists $(e, e'') \in \omega^-(f(K))$. Let $(e', e) \in \omega^+(f(K))$. Then $e' = x'y' \in f(K) = \omega^-(K) \cap A'$, yielding that $y' \in K$. Moreover, $e = \phi(y'y)$ because $(e', e) \in A(\mathcal{L}D)$, which implies that $(y', y) \in \omega^+(K)$ since $y' \in K$. Since K is a semikernel, there exists $(y, y'') \in \omega^-(K)$, implying $e'' = \phi(yy'') \in \omega^-(K) \cap A' = f(K)$, then $(e, e'') \in \omega^-(f(K))$, implying that $f(K)$ is a semikernel. ∎

Figure 4 shows both a digraph and its line digraph with different number of semikernels.

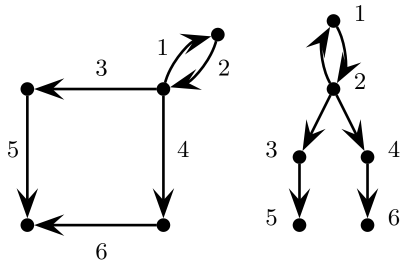

Figure 4. A digraph with 4 semikernels and its line digraph, with 8 semikernels.

Theorem 8. *Let D be a digraph with minimum in-degree at least* 1. *Let A' and ϕ satisfy the requirements of the definition of a partial line digraph, i.e., $\mathcal{L}_{(A',\phi)}D = \mathcal{L}D$. Then $\mathcal{L}D$ has a semikernel if and only if D has a semikernel.*

Proof. From Theorem 7 it follows that if D has a semikernel, then $\mathcal{L}D$ has a semikernel. To see the converse let us consider the function $h : \mathcal{S}^* \to \mathcal{S}$ defined by $h(K^*) = H(K^*)$. First, let us see that $H(K^*)$ is an independent set of D. Let $u, v \in H(K^*)$. Then $u'u, v'v \in K^*$ for some $u', v' \in V(D)$, yielding that $u'u, v'v$ are not adjacent in $\mathcal{L}D$. We reason by contradiction assuming that $uv \in A(D)$. Then $(u'u, \phi(uv)) \in \omega^+(K^*)$. Since K^* is a semikernel, there is $(\phi(uv), \phi(vw)) \in \omega^-(K^*)$. But $v'v, \phi(vw) \in K^*$ and they are adjacent which is a contradiction. Therefore u, v are not adjacent and so $H(K^*)$ is independent.

Second, $vu \in \omega^+(H(K^*))$. As $v \in H(K^*)$, there is $v'v \in K^*$ and $(v'v, \phi(vu)) \in \omega^+(K^*)$. Since K^* is a semikernel, it follows that there exists $(\phi(vu), \phi(uw)) \in \omega^-(K^*)$. Then $\phi(uw) \in K^*$ and so $w \in H(K^*)$. Therefore $uw \in \omega^-(H(K^*))$ and the proof is finished. ∎

3. Grundy Function

Definition. Consider a simple digraph $D = (V, A)$. Following Berge [3], a non-negative integer function g on V is defined as a *Grundy function* if the following two requirements hold:

(1) $g(x) = k > 0$ implies that for each $j < k$, there is $y \in N^+(x)$ with $g(y) = j$;

(2) $g(x) = k$ implies that each $y \in N^+(x)$ satisfies $g(y) \neq k$.

This concept was first defined by Grundy in 1939 for acyclic digraphs as follows.

For every $x \in V$, $g(x) = \min(\mathbb{N} \setminus \{g(y) : y \in N^+(x)\})$.
Furthermore, Grundy proved that every acyclic digraph has a unique Grundy function. However, there are digraphs without Grundy function, for instance the odd directed cycles. One of the most relevant properties of a Grundy function is that if D has a Grundy function g, then D has a kernel $K = \{x \in V : g(x) = 0\}$.

Next, we propose a generalization of a Grundy function called (k, l)-Grundy function. To do that we need to introduce some notation. The out-neighborhood at distance r from a vertex $x \in V$ is $N_r^+(x) = \{y \in V : 1 \leq d(x, y) \leq r\}$.

Definition. Consider a simple digraph $D = (V, A)$ and let $l \geq 1$ and $k \geq 2$ be two integers. A non-negative integer function g on V is defined as a (k, l)-*Grundy function* if the following two requirements hold:

(1) $g(x) = t > 0$ implies that for each $j < t$, there is $y \in N_l^+(x)$ with $g(y) = j$;

(2) $g(x) = t$ implies that each $y \in N_{k-1}^+(x)$ satisfies $g(y) \neq t$.

Figure 5 depicts on the left side a digraph with a $(2, 2)$-Grundy function and on the right side a digraph with a $(3, 2)$-Grundy function.

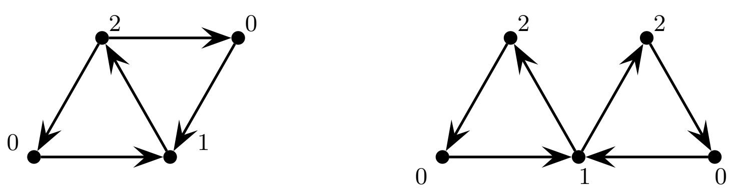

Figure 5. A digraph with a $(2, 2)$-Grundy function and a digraph with a $(3, 2)$-Grundy function.

Remark 9. If a digraph D has a (k, l)-Grundy function g, then D has a (k, l)-kernel $K = \{x \in V : g(x) = 0\}$.

The number of Grundy functions of D has been proved to be equal to the number of Grundy functions of its line digraph, see [10]. Next, we extend this result to (k, l)-Grundy functions and to partial line digraphs. First we prove that a digraph has a (k, l)-Grundy function if and only if any partial line digraph has a (k, l)-Grundy function.

Lemma 10. *Let $l \geq 1$ and $k \geq 2$ be two integers. Let D be a digraph with minimum in-degree at least 1 having a (k, l)-Grundy function g. Let A' and ϕ satisfy the requirements of the definition of a partial line digraph, i.e., $\mathcal{L}_{(A',\phi)}D = \mathcal{L}D$. Then $g_L : A' \to \mathbb{N}$ defined as $g_L(yx) = g(x)$ is a (k, l)-Grundy function on $\mathcal{L}D$.*

Proof. Let $g : V \to \mathbb{N}$ be a (k, l)-Grundy function on $D = (V, A)$. Next, we prove that $g_L : A' \to \mathbb{N}$ be defined as $g_L(yx) = g(x)$ is a (k, l)-Grundy function on $\mathcal{L}D$. Let $yx \in V(\mathcal{L}D)$. First, suppose that $g_L(yx) = t > 0$. Since $g_L(yx) = g(x) = t > 0$, by (1) of Definition 3, it follows that for each $j < t$, there is $w \in N_l^+(x)$ with $g(w) = j$. Hence, there is a path $(x, x_1, \ldots, x_r = w)$ in D with $r \leq l$, which produces a path $(yx, \phi(xx_1), \ldots, \phi(x_{r-1}x_r))$ in $\mathcal{L}D$ of length r, yielding that $\phi(x_{r-1}w) \in N_l^+(yx) \subset V(\mathcal{L}D)$. Therefore for each $j < t$, there is $\phi(x_{r-1}w) \in N_l^+(yx) \subset V(\mathcal{L}D)$ and $g_L(\phi(x_{r-1}w)) = g(w) = j$. Thus, g_L meets requirement (1) of Definition 3. Now suppose that $g_L(yx) = t$, so $g(x) = t$. Let $uv \in N_{k-1}^+(yx) \subset V(\mathcal{L}D)$, then there is a path $(yx, \phi(xx_1), \ldots, \phi(x_{r-1}x_r) = uv)$ of length $r \leq k-1$ in $\mathcal{L}D$. Hence there is a path $(x, x_1, \ldots, x_r = v)$ in D with $r \leq k-1$, yielding that $g(v) \neq t$ because $v \in N_{k-1}^+(x)$ applying (2) of Definition 3. Therefore for all $uv \in N_{k-1}^+(yx)$, we have $g_L(uv) = g_L(\phi(x_{r-1}x_r)) = g(v) \neq t$. Thus, g_L meets requirement (2) of Definition 3, concluding that g_L is a (k, l)-Grundy function on $\mathcal{L}D$. ■

Lemma 11. *Let $l \geq 1$ and $k \geq 2$ be two integers such that $l \leq k-1$. Let D be a digraph with minimum in-degree at least 1. Let A' and ϕ satisfy the requirements of the definition of a partial line digraph, i.e., $\mathcal{L}_{(A',\phi)}D = \mathcal{L}D$. Suppose that g is a (k, l)-Grundy function on $\mathcal{L}D$. Then $g_D : V \to \mathbb{N}$ defined as $g_D(x) = g(yx)$, $yx \in A'$, is a (k, l)-Grundy function on D.*

Proof. Let $g : A' \to \mathbb{N}$ be a (k, l)-Grundy function on $\mathcal{L}D$. First, let us prove that $g_D : V \to \mathbb{N}$ defined as $g_D(x) = g(yx)$ with $x \in V$ and $yx \in A'$ is a function. So assume that there are two arcs $yx, y'x \in A'$, such that $g(yx) \neq g(y'x)$. Suppose $0 \leq h = g(yx) < g(y'x)$, then there exists $uv \in N_l^+(y'x) \subset V(\mathcal{L}D)$ such that $g(uv) = h$ by condition (1) of Definition 3. Then there is a path $(y'x, \phi(xx_1), \ldots, \phi(x_{r-1}x_r) = uv)$ of length $r \leq l$ in $\mathcal{L}D$, and also a path $(yx, \phi(xx_1), \ldots, \phi(x_{r-1}x_r) = uv)$ of length $r \leq l$ in $\mathcal{L}D$ implying that $uv \in N_l^+(yx) \subseteq N_{k-1}^+(yx)$ because $l \leq k-1$, and $g(yx) = g(uv) = h$ which is a contradiction with (2) of Definition 3. Therefore $g(yx) = g(y'x)$. Furthermore, for every $x \in V$, there is an arc $yx \in A'$ by definition of $\mathcal{L}D$. Hence, $g_D(x)$ exists for all $x \in V$. Thus g_D is a function.

Next, we prove that g_D is a (k,l)-Grundy function on D. Let $x \in V$. First, suppose that $g_D(x) = t > 0$. Since $g_D(x) = g(wx) = t > 0$ where $wx \in A'$, by (1) of Definition 3, it follows that for each $j < t$, there is $uv \in N_l^+(wx) \subset V(\mathcal{L}D)$ with $g(uv) = j$. Then there is a path $(wx, \phi(xx_1), \ldots, \phi(x_{r-1}x_r) = uv)$ of length r in $\mathcal{L}D$, implying that $v \in N_l^+(x) \subset V(D)$ and $g_D(v) = j$. Hence, g_D satisfies (1) of Definition 3. Finally, suppose that $g_D(x) = t$, let us see that for all $y \in N_{k-1}^+(x)$, $g_D(y) \neq t$. We have $t = g_D(x) = g(wx)$ for $wx \in A'$. Since for all $y \in N_{k-1}^+(x)$, there exists a path $(x, x_1, \ldots, x_r = y)$ of length $r \leq k-1$ in D, it follows that $\phi(x_{r-1}y) \in N_l^+(wx) \subset V(\mathcal{L}D)$, yielding that $g(\phi(x_{r-1}y)) \neq g(wx) = t$ because (2) of Definition 3. As $g_D(y) = g(\phi(x_{r-1}y))$ it turns out that $g_D(y) \neq t$. Thus, g_D meets requirement (2) of Definition 3, and we conclude that g_D is a (k,l)-Grundy function. ∎

As an immediate consequence of both Lemma 10 and Lemma 11, we can write the following theorem.

Theorem 12. *Let $l \geq 1$ and $k \geq 2$ be two integers with $l \leq k-1$. A digraph D with minimum in-degree at least 1 has a (k,l)-Grundy function if and only if any partial line digraph $\mathcal{L}D$ has a (k,l)-Grundy function.*

Theorem 13. *Let $l \geq 1$ and $k \geq 2$ be two integers with $l \leq k-1$. Let D be a digraph with minimum in-degree at least 1. Then the number of (k,l)-Grundy functions of D is equal to number of (k,l)-Grundy functions of any partial line digraph $\mathcal{L}D$.*

Proof. Let A' and ϕ satisfy the requirements of the definition of a partial line digraph, i.e., $\mathcal{L}_{(A',\phi)}D = \mathcal{L}D$. Denote by $\mathcal{F}$ the set of all (k,l)-Grundy functions on D, and by $\mathcal{F}^*$ the set of all (k,l)-Grundy functions on $\mathcal{L}D$. If $g \in \mathcal{F}$, then the function g_L given by Lemma 10, belongs to $\mathcal{F}^*$; and if $h \in \mathcal{F}^*$, then the function h_D given by Lemma 11, belongs to $\mathcal{F}$.

Let $f : \mathcal{F} \to \mathcal{F}^*$ be defined by $f(g) = g_L$. Let us prove that f is an injective function.

Let $g, g' \in \mathcal{F}$ be such that $f(g) = f(g')$, that is $g_L = g'_L$. Let us show that $g = g'$. Since for all $x \in V$ there exists $yx \in A'$, and $g_L(yx) = g'_L(yx)$, it follows that $g_L(yx) = g(x) = g'(x) = g'_L(yx)$. Hence $g = g'$. Thus, f is an injective function yielding that $|\mathcal{F}| \leq |\mathcal{F}^*|$.

Let $f^* : \mathcal{F}^* \to \mathcal{F}$ be defined by $f^*(h) = h_D$. Let us prove that f^* is an injective function.

Let $h, h' \in \mathcal{F}^*$ be such that $f^*(h) = f^*(h')$, that is $h_D = h'_D$. Let us show that $h = h'$. Since for all $yx \in A'$ we have $h(yx) = h_D(x) = h'_D(x) = h'(yx)$, it follows that $h = h'$. Thus, f^* is an injective function yielding that $|\mathcal{F}^*| \leq |\mathcal{F}|$.

Hence we conclude that $|\mathcal{F}| = |\mathcal{F}^*|$. ∎

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