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KALEIDOSCOPIC EDGE-COLORING OF COMPLETE GRAPHS AND r-REGULAR GRAPHS[1]

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Abstract

For an r-regular graph G, we define an edge-coloring c with colors from $\{1, 2, \ldots, k\}$, in such a way that any vertex of G is incident with at least one edge of each color. The multiset-color $c_m(v)$ of a vertex v is defined as the ordered tuple $(a_1, a_2, \ldots, a_k)$, where a_i $(1 \leq i \leq k)$ denotes the number of edges of color i which are incident with v in G. Then this edge-coloring c is called a *k-kaleidoscopic coloring* of G if every two distinct vertices in G have different multiset-colors and in this way the graph G is defined as a *k-kaleidoscope*. In this paper, we determine the integer k for a complete graph K_n to be a k-kaleidoscope, and hence solve a conjecture in [P. Zhang, A Kaleidoscopic View of Graph Colorings, (Springer Briefs in Math., New York, 2016)] that for any integers n and k with $n \geq k + 3 \geq 6$, the complete graph K_n is a k-kaleidoscope. Then, we construct an r-regular 3-kaleidoscope of order $\binom{r-1}{2} - 1$ for each integer $r \geq 7$, where $r \equiv 3 \pmod 4$, which solves another conjecture in [P. Zhang, A Kaleidoscopic View of Graph Colorings, (Springer Briefs in Math., New York, 2016)] on the maximum order of r-regular 3-kaleidoscopes.

Keywords: k-kaleidoscope, regular graph, edge-coloring.

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1. Introduction

In this paper, all graphs are simple, undirected and finite. For notation and terminology we follow the book [1]. An *edge-coloring* of a graph G is a mapping from the edges of G to a finite number of colors. In the early days, many classical colorings were put forward and studied such as proper edge-coloring,

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list edge-coloring, acyclic edge-coloring and so on. Recently, based on a variety of application instances in different fields, a number of new edge-colorings were put forward. For example, the rainbow edge-coloring has received wide attention due to its close connection with network security and many valuable results were derived in the papers such as [3, 4] and the book [5].

In this paper, we examine another kind of edge-coloring, the *kaleidoscopic coloring*, which was first introduced in [2]. Assume that a group of n computers, each having r ports, are needed to build a network. There are k types of connections and every pair of computers can have at most one connection between them. It is necessary to use every port so that the fail-safe connections would be maximized. Furthermore, distinct computers must have different numbers of types of connections so that the computer engineer is able to distinguish them. The above example is an application instance of a kaleidoscopic coloring. Actually it can model a lot of situations and be applied to many fields such as computer science and telecommunications. Next we will give a definition of kaleidoscopic coloring.

For an r-regular graph G, we define an edge-coloring with the colors $[k] = \{1, 2, 3, \ldots, k\}$ ($k \geq 3$) assigned to the edges of G such that any vertex in G is incident with at least one edge of each color. For a color set $S = \{i_1, i_2, \ldots, i_s\}$, the S-tuple of a vertex v is defined as $(a_{i_1}, a_{i_2}, \ldots, a_{i_s})$, where a_{i_j} ($1 \leq j \leq s$) denotes the number of edges of color i_j that are incident with v in G. In particular, the *multiset-color* $c_m(v)$ of the vertex v is an S-tuple for $S = [k]$. Then this edge-coloring c is called a *k-kaleidoscopic coloring* of G if every two distinct vertices in G have different multiset-colors and in this way the graph G is a *k-kaleidoscope*.

A proper edge-coloring of a graph G is a factorization of the edge set of G into $F_1, F_2, \ldots, F_k$ such that F_i ($1 \leq i \leq k$) is an independent edge set; what we need for the kaleidoscopic coloring is a factorization that enables distinct vertices to have distinct $[k]$-tuples, where a $[k]$-tuple for a vertex v in G is defined as $(\deg_{F_1}(v), \deg_{F_2}(v), \ldots, \deg_{F_k}(v))$.

It is well known that every connected graph has at least two vertices with the same degree. Actually there is exactly one connected graph G of order n containing only two vertices of the same degree. We can describe the graph as follows. Label the vertices of G as $v_1, v_2, \ldots, v_n$, and add an edge v_iv_j if and only if $i + j \geq n + 1$. It is obvious that $v_{\lfloor \frac{n}{2} \rfloor}$ and $v_{\lfloor \frac{n}{2} \rfloor+1}$ have the same degree $\lfloor \frac{n}{2} \rfloor$. Therefore for any decomposition of any r-regular graph G into two graphs D_1 and D_2, at least two vertices u and v have the property: $\deg_{D_1}(u) = \deg_{D_1}(v)$ and $\deg_{D_2}(u) = \deg_{D_2}(v)$. So any r-regular graph G is definitely not a 2-kaleidoscope, thus we will consider r-regular graphs with $r \geq 3$. It can be easily seen that $r > k$ is required for an r-regular graph to be a k-kaleidoscope according to the definition. But in fact $r > k+1$ holds since $r = k+1$ would imply that at most k different $[k]$-tuples satisfy the demand of at least $r+1$ distinct $[k]$-tuples for the vertices in G, a contradiction. Combining with the discussion above, we come to the conclusion that for an r-regular graph G of order n, the possible values of k for G to be a k-kaleidoscope can only be integers between 3 and $n-3$.

In [6], the author solved the cases for k to be 3 or $n-3$ when G is complete. The following conjecture was also posed in the same book.

Conjecture 1.1 [6]**.** *For integers n and k with $n \geq k+3 \geq 6$, the complete graph K_n is a k-kaleidoscope.*

Another open problem from [6] is to determine the maximum order of an r-regular 3-kaleidoscope. A simple calculation shows that there are $\binom{r-1}{2}$ different [3]-tuples altogether for an r-regular graph. Therefore the number of vertices in an r-regular 3-kaleidoscopic graph cannot exceed $\binom{r-1}{2}$. Since for the integer $r \equiv 3 \pmod 4$, $\binom{r-1}{2}$ is odd, so there is no r-regular graph of order $\binom{r-1}{2}$. Thus the largest possible order for an r-regular 3-kaleidoscope is $\binom{r-1}{2} - 1$. Zhang [6] proved that for any r ($r \geq 5$) such that $r \not\equiv 3 \pmod 4$, there exists an r-regular 3-kaleidoscope of order $\binom{r-1}{2}$. Furthermore, the following conjecture was posed in the same book.

Conjecture 1.2 [6]**.** *For every integer r, $r \geq 7$ and $r \equiv 3 \pmod 4$, there is an r-regular 3-kaleidoscope of order $\binom{r-1}{2} - 1$.*

In this paper, we verify these two conjectures and give their proofs in Sections 2 and 3, respectively.

2. Proof of Conjecture 1.1

In the proof of Conjecture 1.1, we need two auxiliary lemmas from [6].

Lemma 2.1 [6]**.** *For each integer $n \geq 6$, the complete graph K_n is a 3-kaleidoscope.*

Lemma 2.2 [6]**.** *For each integer $k \geq 3$, the complete graph K_{k+3} is a k-kaleidoscope.*

***Proof of Conjecture* 1.1.** The case when $k = 3$ is verified in Lemma 2.1. For the complete graph K_n, we give our proof by induction on n. The base case when $n = 6$ follows from Lemma 2.1. We assume that K_m ($m \geq 6$) is a k-kaleidoscope for any k ($3 \leq k \leq m-3$), where m is any integer smaller than n. We distinguish two cases according to the range of k.

Case 1. $\lceil \frac{n}{2} \rceil \leq k \leq n-3$. It is well known that for even values of n, $n \geq 4$, K_n can be decomposed into $\frac{n}{2} - 1$ Hamiltonian cycles $H_1, H_2, \ldots, H_{\frac{n}{2}-1}$ and a perfect matching F. We then put the $(n-3)$-kaleidoscopic coloring of $G = K_n$ to be an $(n-3)$-kaleidoscope depicted in the proof of Lemma 2.2 here. When n is even, for each i $\left(1 \leq i \leq \frac{n}{2}-2\right)$, we give a proper coloring to H_i with the colors $2i-1$ and $2i$. Furthermore, we assign color $n-3$ to all edges in F. The vertices $v_1, v_2, \ldots, v_n$ of the Hamiltonian cycle $H_{\frac{n}{2}-1}$ are placed in clockwise order, we assign color i $\left(1 \leq i \leq \frac{n}{2}\right)$ to the two edges incident with v_{2i} in $H_{\frac{n}{2}-1}$. While when n $(n \geq 7)$ is odd, let $v \in V(G)$, then $G-v$ can be decomposed into $\frac{n-1}{2}-1$ Hamiltonian cycles $H_1, H_2, H_3, \ldots, H_{\frac{n-1}{2}-1}$ and a perfect matching F, then color the edges of $H_1, H_2, H_3, \ldots, H_{\frac{n-1}{2}-2}$ and F as above. For $H_{\frac{n-1}{2}-1}$ containing $v_1, v_2, \ldots, v_{n-1}$ in the clockwise order, assign color i $\left(1 \leq i \leq \frac{n-1}{2}\right)$ to the edge $v_{2i-1}v_{2i}$ and color $n-3$ to the remaining edges. At last give color i $\left(1 \leq i \leq \frac{n-1}{2}\right)$ to the edge vv_{2i-1}, assign color $\frac{n-1}{2}+i$ $\left(1 \leq i \leq \frac{n-1}{2}-3\right)$ to the edge vv_{2i} and give color $n-3$ to the remaining edges in G. We denote the coloring depicted above by c and we give a k-kaleidoscopic coloring c' on the foundation of c for $\left\lceil \frac{n}{2} \right\rceil \leq k \leq n-4$. That is,

$$c'(e) = \begin{cases} c(e) & if\ 1 \leq c(e) \leq k-1, \\ k & if\ k \leq c(e) \leq n-3. \end{cases}$$

The coloring c' is k-kaleidoscopic since the $[k]$-tuples for different vertices were different in their former $\left\lceil \frac{n}{2} \right\rceil - 1$ positions, so we can still distinguish them after we combine the latter colors into one.

Case 2. For $4 \leq k \leq \left\lceil \frac{n}{2} \right\rceil - 1$, we again distinguish two subcases according to n being even or odd.

Subcase 1. When n is even. We first consider the case when $5 \leq k \leq \left\lceil \frac{n}{2} \right\rceil - 1$. Partition the n vertices of K_n into two copies of $K_{\frac{n}{2}}$, denoted by G_1 and G_2 with all edges between them. By the inductive hypothesis, there exists a $(k-2)$-kaleidoscopic coloring of $K_{\frac{n}{2}}$. Then we give this coloring using the colors from $\{3, 4, \ldots, k\}$ to both G_1 with vertices $v_1, v_2, \ldots, v_{\frac{n}{2}}$ and G_2 with vertices $v'_1, v'_2, \ldots, v'_{\frac{n}{2}}$. Each pair of vertices v_i and v'_i $\left(1 \leq i \leq \frac{n}{2}\right)$ share the same $\{3, 4, \ldots, k\}$-tuple. Thus it is enough to provide a coloring of the graph $K_{\frac{n}{2}, \frac{n}{2}}$ using the colors from $\{1, 2\}$ such that each vertex is incident with at least one edge of each color from $\{1, 2\}$ and v_i and v'_i have different $\{1, 2\}$-tuples. Let $w_i = v_{i-1}$ $\left(2 \leq i \leq \frac{n}{2}\right)$, $w_1 = v_{\frac{n}{2}}$, $w'_i = v'_{i+1}$ $\left(1 \leq i \leq \frac{n}{2}-1\right)$ and $w'_{\frac{n}{2}} = v'_1$. Color 1 is assigned to the edge $w_i w'_j$ if and only if $\frac{n}{2}+1 \leq i+j \leq n-1$ and color 2 is assigned to all the remaining edges in $K_{\frac{n}{2}, \frac{n}{2}}$. Then the $\{1, 2\}$-tuples of vertices of $K_{\frac{n}{2}, \frac{n}{2}}$ are listed in the chart below. As a result, for any $1 \leq i \leq \frac{n}{2}$, v_i and v'_i have distinct $\{1, 2\}$-tuples and then this coloring is just what we want.

If $k = 4$, then G_i $(i = 1, 2)$ contains the unique connected spanning subgraph F_i with only two vertices sharing the same degree as we say in the introduction. List the vertices of F_i according to their degrees in the nondecreasing order and

vertex	v_1	$v_i\,(2 \leq i \leq \frac{n}{2}-2)$	$v_{\frac{n}{2}-1}$	$v_{\frac{n}{2}}$
$\{1,2\}$-tuple	$(2, \frac{n}{2}-2)$	$(i+1, \frac{n}{2}-i-1)$	$(\frac{n}{2}-1, 1)$	$(1, \frac{n}{2}-1)$
vertex	v'_1	$v'_i\,(2 \leq i \leq \frac{n}{2}-2)$	$v'_{\frac{n}{2}-1}$	$v'_{\frac{n}{2}}$
$\{1,2\}$-tuple	$(\frac{n}{2}-1, 1)$	$(i-1, \frac{n}{2}-i+1)$	$(\frac{n}{2}-2, 2)$	$(\frac{n}{2}-1, 1)$

Table 1. The corresponding $\{1,2\}$-tuples of the vertices of $K_{\frac{n}{2},\frac{n}{2}}$.

label them as $v_1, v_2, \ldots, v_{\frac{n}{2}}$ in F_1 and $v'_1, v'_2, \ldots, v'_{\frac{n}{2}}$ in F_2. Let $H_1 = F_1 - v_{\frac{n}{2}} v_{\lfloor \frac{n}{4} \rfloor}$ and $H_2 = F_2 - v'_{\frac{n}{2}} v'_{\lfloor \frac{n}{4} \rfloor}$. Assign color 3 to the edges of H_i $(i = 1, 2)$ and color 4 to the remaining edges in G_is. The remaining edges are colored as the above description except for a little change that the color of $w'_{\frac{n}{2}-2} w_{\frac{n}{2}}$ is 2 instead of 1. The checkout is similar.

Subcase 2. When n is odd. The simplest case is $k = 4$ and $n \geq 13$. In this case, split the graph K_n into $K_{\lceil \frac{n}{2} \rceil}$ and $K_{\lfloor \frac{n}{2} \rfloor}$ with all edges between them. Give a 3-kaleidoscopic coloring respectively to $K_{\lceil \frac{n}{2} \rceil}$ and $K_{\lfloor \frac{n}{2} \rfloor}$ using the colors 2, 3 and 4. And assign color 1 to all edges between them. For the case that $5 \leq k \leq \lceil \frac{n}{2} \rceil - 1$ and $n \geq 13$, label the vertices of $K_{\lceil \frac{n}{2} \rceil}$ and $K_{\lfloor \frac{n}{2} \rfloor}$ respectively as $v_1, v_2, \ldots, v_{\lceil \frac{n}{2} \rceil}$ and $v'_1, v'_2, \ldots, v'_{\lfloor \frac{n}{2} \rfloor}$, furthermore we each give them a $(k-3)$-kaleidoscopic coloring using the colors from $\{4, 5, \ldots, k\}$ if $k \neq 5$ and a 3-kaleidoscopic coloring using colors from $\{3, 4, 5\}$ if $k = 5$. For the former case, $v_i v'_j$ is assigned to color a $(a \in \{1, 2, 3\})$ if $i + j \equiv a \pmod 3$ while in the latter case, $v_i v'_j$ has color b $(b \in \{1, 2\})$ if $i + j \equiv b \pmod 2$.

Only three particular cases are left. That is, $k = 4$ when $n = 9$ and $k = 4$ or 5 when $n = 11$. Like our discussion in Subcase 1, we again find the three unique connected spanning subgraph F_4, F_5 and F_6 with only two vertices of the same degree contained in K_4, K_5 and K_6 appearing in the decomposition of K_9 and K_{11}. Similarly, the vertices of F_4, F_5 and F_6 are ordered in nondecreasing sequence according to their degrees as v_i $(1 \leq i \leq 4)$, v'_i $(1 \leq i \leq 5)$ and v''_i $(1 \leq i \leq 6)$. Let $H_4 = F_4 - v_2 v_4$, $H_5 = F_5 - v'_2 v'_5$ and $H_6 = F_6 - v''_3 v''_6$. Color the edges of H_i $(i = 4, 5, 6)$ with 1 and all the remaining edges in K_i $(i = 4, 5, 6)$ with 2. Thus v_1 and v_2, v_3 and v_4, v'_1 and v'_2, v'_4 and v'_5, v''_2 and v''_3, in addition with v''_5 and v''_6, each couple of these six has the same $\{1,2\}$-tuple. As a result, we only need to provide a coloring to the edges of $K_{4,5}$ or $K_{5,6}$ using the colors $3, 4$ and to the edges of $K_{5,6}$ using the colors $3, 4, 5$ such that the above six couples cannot have the same $\{3,4\}$-tuples or $\{3,4,5\}$-tuples. The vertices can be matched properly to v_i $(1 \leq i \leq 4)$, v'_k $(1 \leq k \leq 5)$ or v''_t $(1 \leq t \leq 6)$ as we show in Figure 1 so that the six couples would not have the same multiset-colors. Thus, our proof is done. ■

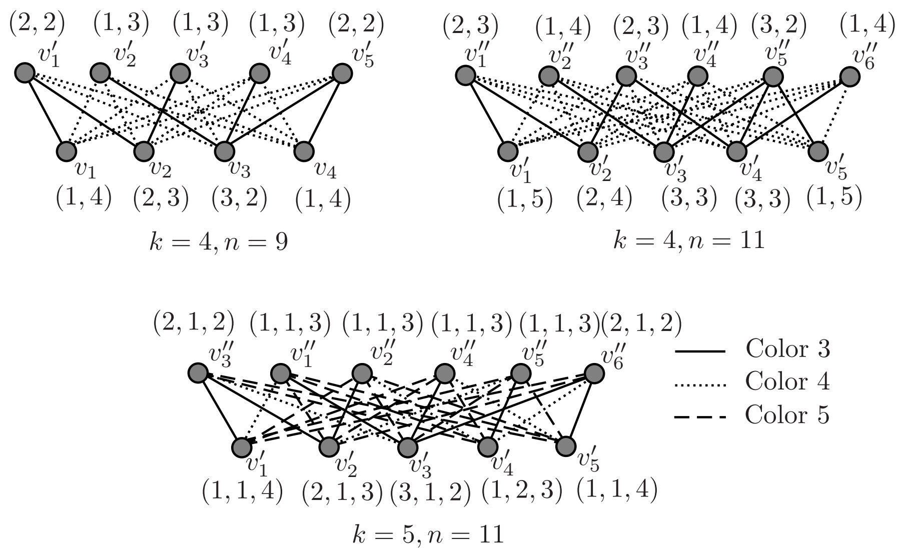

Figure 1. Three colorings of $K_{4,5}$ and $K_{5,6}$ in three different cases.

3. Proof of Conjecture 1.2

***Proof of Conjecture* 1.2.** For a fixed integer $r \geq 5$, note that $\binom{r-1}{2} = \sum_{i=1}^{i=r-2} i$. So for $1 \leq i \leq r-2$, let V_i be a set of $r-1-i$ vertices, and for each V_i, order the vertices as $v_{i,1}, v_{i,2}, \ldots, v_{i,r-1-i}$. Then we arrange these vertices in V_i $(1 \leq i \leq r-2)$ in the shape of an equilateral triangle. That is, the distance between any couple of nearest vertices is 1. We provide an example of the location of the vertices for $r = 7$ in Figure 2(a). We then rotate the triangle counterclockwise through an angle of $2\pi/3$ radians and denote like above as V'_i $(1 \leq i \leq r-2)$ and $v'_{i,j}$ $(1 \leq j \leq r-1-i)$. In the same way V''_i $(1 \leq i \leq r-2)$ and $v''_{i,j}$ $(1 \leq j \leq r-1-i)$ are obtained after a counterclockwise rotating by the angle $4\pi/3$. We denote by e_1 the $v_{1,1}v_{1,r-2}$-side, e_2 the $v_{1,r-2}v_{r-2,1}$-side, and e_3 the $v_{1,1}v_{r-2,1}$-side. For any point x in the triangle, we denote by $d_i(x)$ $(i = 1, 2, 3)$ the distance from x to e_{i+2} (mod 3) on a line segment parallel to e_i. More details are showed in Figure 2(b). It is obvious that for any vertex x, $d_1(x) + d_2(x) + d_3(x) = d(e_1) = r - 3$, that is, the length of a side of this equilateral triangle. However, any vertex x can be denoted as $v_{i,s_1(x)}$, $v'_{j,s_2(x)}$ or $v''_{k,s_3(x)}$, where $s_i(x) = d_i(x) + 1$. As a result $s_1(x) + s_2(x) + s_3(x) = r$ and every vertex is endowed with a unique coordinate $(s_1(x), s_2(x), s_3(x))$. We are going to give a coloring that enables any vertex x to have $s_i(x)$ $(i = 1, 2, 3)$ edges with color i.

We start by removing the vertex $v_{r-2,1}$. On the set V_i, $1 \leq i \leq r-3$, we construct the unique connected graph F_i with only two vertices of the same degree

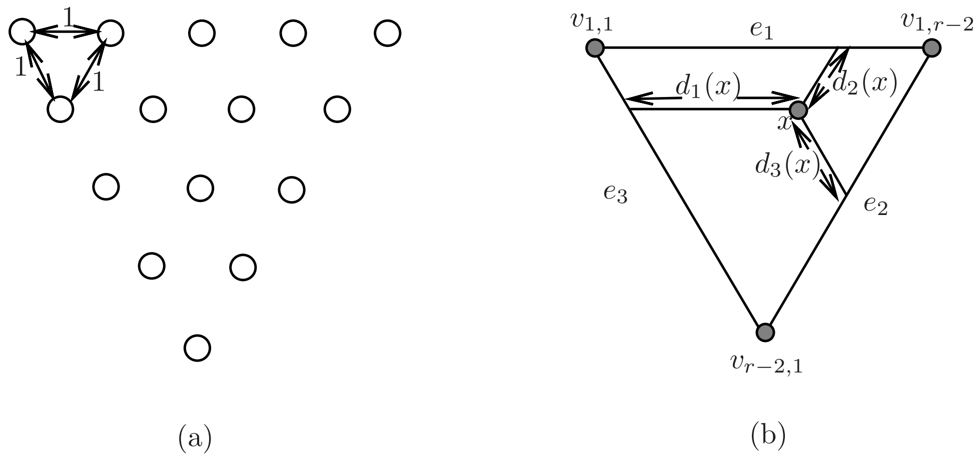

Figure 2. The location of the vertices for $r = 7$ and the diagram for $d_i(x)s$ $(i = 1, 2, 3)$.

satisfying the inequalities $\deg_{F_i} v_{i,1} \le \deg_{F_i} v_{i,2} \le \cdots \le \deg_{F_i} v_{i,r-1-i}$. Besides, define $A = \left\{v_{1,\frac{r-1}{2}+2i} v_{2,\frac{r-1}{2}+2i-1} : 1 \le i \le \frac{r-3}{4}\right\}$. For $1 \le j \le \frac{r-3}{4}$, let $E_j = \left\{v_{i,r-i-2j} v_{i+2,r-i-2j-1} : 1 \le i \le r-4j-1\right\}$ and $B = \left\{v_{4i-3,\frac{r-4i+3}{2}} v_{4i-1,\frac{r-4i+1}{2}} : 1 \le i \le \frac{r-3}{4}\right\}$. Note that A, B and each E_j are all independent edge sets. Now we assign color 1 to each edge of A, B, E_j $\left(1 \le j \le \frac{r-3}{4}\right)$ and F_i $(1 \le i \le r-3)$. Then for any vertex x, it is incident with exactly $s_1(x)$ edges with color 1. The connected graphs F'_i on V'_i $(2 \le i \le r-2)$, F'_1 on $V'_1 - v'_{1,r-2}$ and the edge sets A', E'_j are defined similarly. However, let $B' = \left\{v'_{4i-1,\frac{r-4i+1}{2}} v'_{4i+1,\frac{r-4i-1}{2}} : 1 \le i \le \frac{r-3}{4}\right\}$. Again we assign color 2 to each edge of A', B', E'_j and F'_i $(1 \le i \le r-2)$. Then any vertex x is incident with $s_2(x)$ edges of color 2 except $v'_{2,r-3}$ and $v'_{1,\frac{r-1}{2}}$ with a difference of 1 respectively. So we add an edge $v'_{2,r-3} v'_{1,\frac{r-1}{2}}$ and give it color 2.

Since the vertex $v''_{1,1} (v_{r-2,1})$ has been removed, take the vertex $v''_{2,1}$ in place of $v''_{1,1}$ ($v''_{2,1}$ is used twice, both in F''_1 and F''_2). Then F''_i on V''_i $(1 \le i \le r-2)$, A'', E''_j and B'' are obtained after the same procedure as F_i, A, E_j and B' and we give color 3 to each edge of them. The vertices $v''_{1,\frac{r-1}{2}}$ and $v''_{2,1}$ are incident with $s_3\left(v''_{1,\frac{r-1}{2}}\right) - 1$ and $s_3\left(v''_{2,1}\right) + 1$ numbers of edges of color 3, respectively. So remove the edge $v''_{2,1} v''_{2,r-3}$ and add the edge $v''_{1,\frac{r-1}{2}} v''_{2,r-3}$ with color 3. Then any vertex x is incident with $s_3(x)$ edges of color 3.

The construction for the r-regular graph is complete and the coloring is given. It can be verified that each edge has been colored and any vertex x does

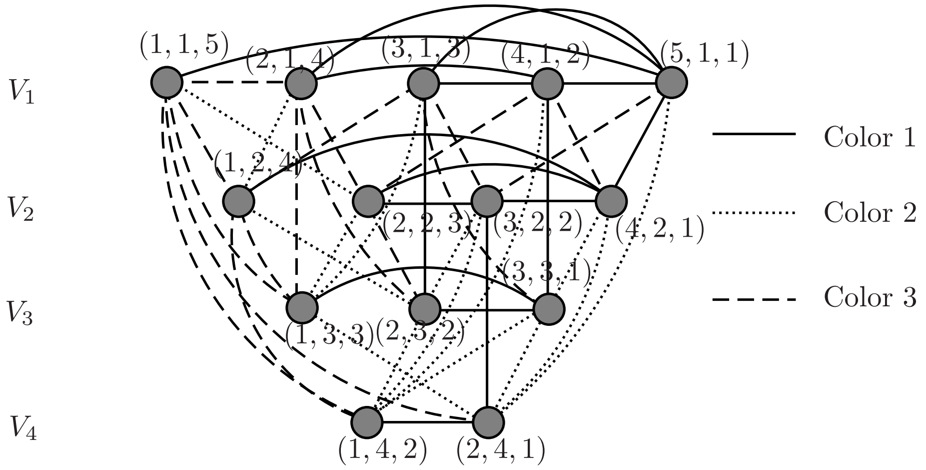

Figure 3. A 3-kaleidoscopic coloring of the r-regular graph when $r = 7$.

have a unique $\{1, 2, 3\}$-tuple $(s_1(x), s_2(x), s_3(x))$ under this coloring. We give an example of a 3-kaleidoscopic 7-regular graph in Figure 3.

The proof is thus complete. ■

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