SPECTRAL RADIUS AND HAMILTONICITY OF GRAPHS

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Abstract

In this paper, we study the Hamiltonicity of graphs with large minimum degree. Firstly, we present some conditions for a simple graph to be Hamilton-connected and traceable from every vertex in terms of the spectral radius of the graph or its complement, respectively. Secondly, we give the conditions for a nearly balanced bipartite graph to be traceable in terms of spectral radius, signless Laplacian spectral radius of the graph or its quasi-complement, respectively.

Keywords: spectral radius, singless Laplacian spectral radius, traceable, Hamiltonian-connected, traceable from every vertex, minimum degree.

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1. INTRODUCTION

Let $G = (V(G), E(G))$ be a simple graph of order $n$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Denote by $e(G) = |E(G)|$ the number of
edges of the graph $G$. Let $N_G(v)$ be the set of vertices which are adjacent to $v$ in $G$. The degree of $v$ is denoted by $d_G(v) = |N_G(v)|$ (or simply $d(v)$), the minimum degree of $G$ is denoted by $\delta(G)$. Let $X \subseteq V(G)$, $G - X$ is the graph obtained from $G$ by deleting all vertices in $X$. $G$ is called $k$-connected (for $k \in \mathbb{N}$) if $|V(G)| > k$ and $G - X$ is connected for every set $X \subseteq V(G)$ with $|X| < k$. We note that if $G$ is $k$-connected, then $\delta(G) \geq k$. A regular graph is one graph whose all vertices have the same degrees, and a bipartite semi-regular graph is a bipartite graph for which the vertices in the same part have the same degrees. The complement of $G$ is denoted by $\overline{G} = (V(\overline{G})E(\overline{G}))$, where $V(\overline{G}) = V(G)$, $E(\overline{G}) = \{xy : x, y \in V(G), xy \notin E(G)\}$. Let $G = (X, Y; E)$ be a bipartite graph with two partite sets $X, Y$. If $|X| = |Y|$, $G = (X, Y; E)$ is called a balanced bipartite graph. If $|X| = |Y| - 1$, $G = (X, Y; E)$ is called a nearly balanced bipartite graph. The quasi-complement of $G = (X, Y; E)$ is denoted by $\hat{G} := (X, Y; E')$, where $E' = \{xy : x \in X, y \in Y, xy \notin E\}$. For two disjoint graphs $G_1$ and $G_2$, the union of $G_1$ and $G_2$, denoted by $G_1 + G_2$, is defined as $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2)$, and the join of $G_1$ and $G_2$, denoted by $G_1 \vee G_2$, is defined as $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$, and $E(G_1 \vee G_2) = E(G_1 + G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$. Denote $K_n$ the complete graph on $n$ vertices, $O_n = \overline{K_n}$ the empty graph on $n$ vertices (without edges), $K_{n,m} = O_n \vee O_m$ the complete bipartite graph with two parts having $n, m$ vertices, $G - v \ (v \in V(G))$ the graph obtained from $G$ by deleting $v$, respectively.

The adjacency matrix of $G$ is defined to be a matrix $A(G) = [a_{ij}]$ of order $n$, where $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$, and $a_{ij} = 0$ otherwise. The degree matrix of $G$ is denoted by $D(G) = \text{diag} \ (d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$. The matrix $Q(G) = D(G) + A(G)$ is the signless Laplacian matrix (or $Q$-matrix) of $G$. Obviously, $A(G)$ and $Q(G)$ are real symmetric matrix. So their eigenvalues are real number and can be ordered. The largest eigenvalue of $A(G)$, denoted by $\mu(G)$, and the corresponding eigenvectors (whose all components are positive number) are called the spectral radius and the Perron vector of $G$, respectively. The largest eigenvalue of $Q(G)$, denoted by $q(G)$, is called the signless Laplacian spectral radius of $G$.

A Hamiltonian cycle of the graph $G$ is a cycle of order $n$ contained in $G$, and a Hamiltonian path of $G$ is a path of order $n$ contained in $G$, where $|V(G)| = n$. The graph $G$ is said to be Hamiltonian if it contains a Hamiltonian cycle, and is said to be traceable if it contains a Hamiltonian path. If every two vertices of $G$ are connected by a Hamiltonian path, it is said to be Hamilton-connected. A graph $G$ is traceable from a vertex $x$ if it has a Hamiltonian $x$-path. The problem of deciding whether a graph is Hamiltonian is one of the most difficult classical problems in graph theory. Indeed, determining whether a graph is Hamiltonian is NP-complete.

Recently, the spectral theory of graphs has been applied to this problem. Up
to now, there are some references on the spectral conditions for a graph to be traceable, Hamiltonian, Hamilton-connected or traceable from every vertex. We refer readers to see [4, 5, 7, 10, 12–17, 19, 21–25]. Particularly, Li and Ning [4, 5] and Nikiforov [19] study spectral sufficient conditions of graphs with large minimum degree. Li and Ning [4] present some (signless Laplacian) spectral radius conditions for a simple graph and a balanced bipartite graph to be traceable and Hamiltonian, respectively. Li and Ning [5] present some spectral radius conditions for a balanced bipartite graph and a nearly balanced bipartite graph to be traceable, respectively. Nikiforov [19] gives some spectral radius conditions for a simple graph to be traceable and Hamiltonian, respectively. Motivated by those papers, in this paper, we also study the graphs with large minimum degree. We will respectively present some conditions for a simple graph to be Hamilton-connected and traceable from every vertex in terms of the spectral radius of the graph or its complement in Section 2, and respectively give the conditions for a nearly balanced bipartite graph to be traceable in terms of spectral radius, signless Laplacian spectral radius of the graph or its quasi-complement in Section 3.

2. Spectral Radius Conditions for a Graph to Be Hamilton-Connected, and Traceable from Every Vertex

For an integer \( k \geq 0 \), the \( k \)-closure of a graph \( G \), denoted by \( C_k(G) \), is the graph obtained from \( G \) by successively joining pairs of nonadjacent vertices whose degree sum is at least \( k \) until no such pair remains, see [2]. The \( k \)-closure of the graph \( G \) is unique, independent of the order in which edges are added. Note that \( d_{C_k(G)}(u) + d_{C_k(G)}(v) \leq k - 1 \) for any pair of nonadjacent vertices \( u \) and \( v \) of \( C_k(G) \).

**Lemma 1** (Ore [20], Bondy and Chvátal [1]).

(i) If \( G \) is a 2-connected graph of order \( n \) and \( d_G(u) + d_G(v) \geq n + 1 \) for any two distant nonadjacent vertices \( u \) and \( v \), then \( G \) is Hamilton-connected.

(ii) A 2-connected graph \( G \) is Hamilton-connected if and only if \( C_{n+1}(G) \) is so.

**Lemma 2** (Yu, Ye, Cai and Cao [22]). Let \( G \) be a simple graph, with degree sequence \((d_G(v_1), d_G(v_2), \ldots, d_G(v_n))\), where \( d_G(v_1) \leq d_G(v_2) \leq \cdots \leq d_G(v_n) \) and \( n \geq 3 \). Suppose that there is no integer \( 2 \leq k \leq \frac{n}{2} \) such that \( d_G(v_{k-1}) \leq k \), and \( d_G(v_{n-k}) \leq n - k \), then \( G \) is Hamilton-connected.

**Lemma 3** (Hong, Shu and Fang [11], Nikiforov [18]). If \( G \) is a graph of order \( n \), with \( m \) edges and minimum degree \( \delta \), then

\[
\mu(G) \leq \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta + 1)^2}{4}}.
\]
Lemma 4 (Hong, Shu and Fang [11], Nikiforov [18]). If \( 2m \leq n(n - 1) \), then the function

\[
f(x) = \frac{x - 1}{2} + \sqrt{2m - nx + \frac{(x + 1)^2}{4}}
\]

is decreasing in \( x \) for \( x \leq n - 1 \).

Lemma 5 (Bondy and Murty [2]). Let \( G \) be a graph. Then \( G \) is traceable from every vertex if and only if \( G \lor K_1 \) is Hamilton-connected.

Given a graph \( G \) of order \( n \), a vector \( x \in \mathbb{R}^n \) is called to be defined on \( G \), if there is a 1-1 map \( \varphi \) from \( V(G) \) to the entries of \( x \); simply written \( x_u = \varphi(u) \) for each \( u \in V(G) \). If \( x \) is an eigenvector of \( A(G) \), then \( x \) is defined on \( G \) naturally, \( x_u \) is the entry of \( x \) corresponding to the vertex \( u \). One can find that

\[
x^T A(G)x = 2 \sum_{uv \in E(G)} x_u x_v,
\]

when \( \mu \) is an eigenvalue of \( G \) corresponding to the eigenvector \( x \) if and only if \( x \neq 0 \),

\[
\mu x_v = \sum_{u \in N_G(v)} x_u,
\]

for each vertex \( v \in V(G) \). Equation (2) is called the eigenvalue-equation for the graph \( G \). In addition, for an arbitrary unit vector \( x \in \mathbb{R}^n \),

\[
\mu(G) \geq x^T A(G)x,
\]

with equality holds if and only if \( x \) is an eigenvector of \( A(G) \) according to \( \mu(G) \).

Lemma 6 (Li and Ning [4]). Let \( G \) be a graph with non-empty edge set. Then

\[
\mu(G) \geq \min \left\{ \sqrt{d(u)d(v)} : uv \in E(G) \right\}.
\]

Moreover, if \( G \) is connected, then equality holds if and only if \( G \) is regular or bipartite semi-regular graph.

Lemma 7. Let \( G \) be a graph of order \( n \). Then

\[
\mu(G \lor K_1) > \frac{n - 1}{n} \mu(G) + 2 \sqrt{\frac{n - 1}{n}}.
\]

Proof. Let \( x \in \mathbb{R}^n \) be a unit Perron vector of \( G \). Then by (1) and (3),

\[
\mu(G) = x^T A(G)x = 2 \sum_{uv \in E(G)} x_u x_v.
\]
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Let \( w \in V(K_1), H = G \lor K_1, \) and let \( x' \in \mathbb{R}^{n+1}, x'_u = \sqrt{\frac{n-1}{n}} x_u, \) for every \( u \in V(G), \) \( x'_u = \frac{1}{\sqrt{n}}. \) Since \( \sum_{u \in V(G)} x_u^2 = 1, \) \( x_u > 0, \) we have

\[
\sum_{u \in V(H)} x_u^2 = \sum_{u \in V(G)} x'_u^2 + x'_w^2 = \frac{n-1}{n} \sum_{u \in V(G)} x_u^2 + \frac{1}{n} = 1,
\]

and \( \sum_{u \in V(G)} x_u > \sum_{u \in V(G)} x_u^2 = 1. \) Then by (1) and (3)

\[
\mu(G \lor K_1) = \mu(H) \geq x^T A(H) x' = 2 \sum_{uv \in E(G)} x'_u x'_v + 2 x'_w \sum_{u \in V(G)} x'_u
\]

\[
= 2 \frac{n-1}{n} \sum_{uv \in E(G)} x_u x_v + 2 \frac{1}{\sqrt{n}} \sqrt{\frac{n-1}{n}} \sum_{u \in V(G)} x_u
\]

\[
> \frac{n-1}{n} \mu(G) + 2 \sqrt{\frac{n-1}{n}}.
\]

So the result follows.

\[\Box\]

Lemma 8 (Tomescu [3]). Every \( t \)-regular graph on \( 2t \) (\( t \geq 3 \)) vertices not isomorphic to \( K_t, t, \) or of order \( 2t + 1 \) for even \( t \geq 4, \) is Hamilton-connected.

Lemma 9. Let \( k \geq 2, n \geq 2k^2 + 1, \) and \( G \) be a graph of order \( n. \) If \( G \) is a subgraph of \( K_2 \lor (K_{n-k-1} + K_{k-1}), \) with minimum degree \( \delta(G) \geq k, \) then \( \mu(G) < n-k, \) unless \( G = K_2 \lor (K_{n-k-1} + K_{k-1}). \)

**Proof.** Set for short \( \mu := \mu(G), \) and let \( x = (x_{v_1}, \ldots, x_{v_n})^T \) be a unit Perron vector of \( G. \) By (3), we have that

\[
\mu = x^T A(G) x.
\]

Assume that \( G \) is a proper subgraph of \( K_2 \lor (K_{n-k-1} + K_{k-1}). \) By Perron-Frobenius theorem, we can assume that \( G \) is obtained by omitting just one edge \( uv \) of \( K_2 \lor (K_{n-k-1} + K_{k-1}). \)

Write \( X \) for the set of vertices of \( K_2 \lor (K_{n-k-1} + K_{k-1}) \) of degree \( k, \) let \( Y \) be the set of their neighbors not in the set \( X, \) and let \( Z \) be the set of the remaining \( n - k - 1 \) vertices of \( K_2 \lor (K_{n-k-1} + K_{k-1}). \)

Since \( \delta(G) \geq k, \) we can see that \( G \) must contain all the edges between \( X \) and \( Y. \) Therefore, \( \{u, v\} \subset Y \cup Z, \) with three possible cases: (a) \( \{u, v\} \subset Y; \) (b) \( u \in Y, v \in Z; \) (c) \( \{u, v\} \subset Z. \) We shall show that case (c) yields a graph of no smaller spectral radius than case (b), and that case (b) yields a graph of no smaller spectral radius than case (a).
Indeed, by (2), we have $x_i = x_j$ for any $i, j \in X$; likewise, $x_i = x_j$ for any $i, j \in Y \{u, v\}$ and for any $i, j \in Z \{u, v\}$. Thus, let

$$x := x_i, i \in X,$$
$$y := x_i, i \in Y \{u, v\},$$
$$z := x_i, i \in Z \{u, v\}.$$ 

Suppose that case (a) holds, that is, $\{u, v\} \subset Y$. Choose a vertex $w \in Z$, remove the edge $vw$ and add the edge $uv$. Then the obtained graph $G'$ is covered by case (b).

If $x_w \leq x_u$, we have

$$x^T A(G') x - x^T A(G) x = 2x_v(x_u - x_w) \geq 0.$$ 

If $x_w > x_u$, swap the entries $x_u$ and $x_w$, write $x'$ for the resulting vector. We note that $x'$ is also a unit vector, and have that

$$x'^T A(G') x' - x^T A(G) x = 2(x_w - x_u) \sum_{i \in X} x_i \geq 0.$$ 

Then by (3), $\mu(G') \geq \mu(G)$, as claimed.

Essentially the same argument proves that case (c) yields a graph of no smaller spectral radius than case (b). Therefore, we may assume that $\{u, v\} \subset Z$. Since the vertices $u$ and $v$ are symmetric, so $x_u = x_v$. Set $t := x_u$ and note that the $n$ eigenvalue-equations of $G$ are reduced to four equations involving just the unknowns $x, y, z$, and $t$.

$$\mu x = (k - 2)x + 2y,$$
$$\mu y = (k - 1)x + y + (n - k - 3)z + 2t,$$
$$\mu z = 2y + (n - k - 4)z + 2t,$$
$$\mu t = 2y + (n - k - 3)z.$$ 

We find that

$$x = \frac{2y}{\mu - k + 2},$$
$$z = \left(1 - \frac{2(k - 1)}{(\mu + 1)(\mu - k + 2)}\right) y,$$
$$t = \frac{\mu + 1}{\mu + 2} \left(1 - \frac{2(k - 1)}{(\mu + 1)(\mu - k + 2)}\right) y.$$ 

Furtherly, note that if we delete all edges incident to vertices in $X$, and add the edge $uv$ to $G$, we obtain the graph $K_{n-k+1} + \overline{K}_{k-1}$. Letting $x''$ be the
restriction of $x$ to $K_{n-k+1}$, we find that
\[ x''^T A(K_{n-k+1})x'' = x^T A(G)x + 2t^2 - 4(k-1)xy - (k-1)(k-2)x^2 \]
\[ = \mu + 2t^2 - 4(k-1)xy - (k-1)(k-2)x^2. \]

But since $\|x''\|^2 = 1 - (k-1)x^2$, we see that
\[ \mu + 2t^2 - 4(k-1)xy - (k-1)(k-2)x^2 \]
\[ = x''^T A(K_{n-k+1})x'' \leq \mu \|K_{n-k+1}\| \|x''\|^2 \]
\[ = (n-k)(1 - (k-1)x^2). \]

Assume for a contradiction that $\mu \geq n-k$. This assumption, together with above inequality, yields
\[ \mu + 2t^2 - 4(k-1)xy - (k-1)(k-2)x^2 \leq \mu(1 - (k-1)x^2), \]
and therefore
\[ 2(k-1)xy - \frac{(\mu - k + 2)(k-1)x^2}{2} \geq t^2. \]

Now, first combining above equality about $x$, then combining equality about $t$, we have
\[ \frac{2(k-1)y^2}{\mu - k + 2} \geq \left( \frac{\mu + 1}{\mu + 2} \right)^2 \left( 1 - \frac{2(k-1)}{(\mu + 1)(\mu - k + 2)} \right)^2 y^2. \]

Cancelling $y^2$ and applying Bernoulli’s inequality to the right side, we get
\[ 2(k-1) \geq (\mu - k + 2) \left( 1 - \frac{1}{\mu + 2} \right)^2 \left( 1 - \frac{2(k-1)}{(\mu + 1)(\mu - k + 2)} \right)^2 \]
\[ > (\mu - k + 2) \left( 1 - \frac{2}{\mu + 2} - \frac{4(k-1)}{(\mu + 1)(\mu - k + 2)} \right) \]
\[ = \mu - k + 2 - \frac{2\mu - 2k + 4}{\mu + 2} - \frac{4(k-1)}{\mu + 1} > \mu - k + 2 - \frac{2\mu + 2k}{\mu + 1}. \]

Using the inequalities $\mu \geq n-k \geq 2k^2 - k + 1$, we easily find that
\[ 2 < \frac{2\mu + 2k}{\mu + 1} < 3, \]
and so,
\[ 2(k-1) > 2k^2 - k + 1 - k + 2 - 3 = 2k^2 - 2k, \]
a contradiction, completing the proof.
Theorem 10. Let \( k \geq 2, n \geq 2k^2 + 1 \) and let \( G \) be a graph of order \( n \) with minimum degree \( \delta(G) \geq k \). If

\[
\mu(G) \geq n - k,
\]

then \( G \) is Hamilton-connected, unless \( G = K_2 \lor (K_{n-k-1} + K_{k-1}) \).

Proof. Assume that \( \mu(G) \geq n - k \), but \( G \) is not Hamilton-connected. Let \( H = C_{n+1}(G) \). Then \( H \) is not Hamilton-connected by Lemma 1, \( \delta(H) \geq \delta(G) \geq k \), and \( \mu(H) \geq \mu(G) \geq n - k \) by Perron-Frobenius theorem. Note that \( H \) is \( (n+1) \)-closure of \( G \), thus every two nonadjacent vertices \( u, v \) have degree sum at most \( n \), i.e.,

\[
d_H(u) + d_H(v) \leq n. \tag{5}
\]

Since \( H \) is not Hamilton-connected, by Lemma 2, there is an integer \( 2 \leq s \leq \frac{n}{2} \) such that \( d_H(v_{s-1}) \leq s \) and \( d_H(v_{n-s}) \leq n - s \), obviously, \( s \geq \delta(H) \geq k \). Write \( m \) for the number of edges of \( H \), set \( \delta(H) := \delta \), then we can get

\[
2m = \sum_{i=1}^{s-1} d_H(v_i) + \sum_{i=s}^{n-s} d_H(v_i) + \sum_{i=n-s+1}^{n} d_H(v_i) \tag{6}
\]

\[
\leq s(s-1) + (n - 2s + 1)(n-s) + s(n-1) \tag{7}
\]

\[
= 3s^2 + n^2 - 2ns + n - 3s. \tag{8}
\]

On the other hand, combining Lemmas 3 and 4, we have

\[
n - k \leq \mu(H) \leq \frac{k-1}{2} + \sqrt{2m - nk + \frac{(k+1)^2}{4}},
\]

which, after some algebra operations, gives

\[
2m \geq n^2 - 2kn + 2k^2 + n - 2k. \tag{9}
\]

Next, we will prove that \( s = k \). Suppose \( k+1 \leq s \leq \frac{n}{2} \). Let \( f(x) = 3x^2 + n^2 - 2nx + n - 3x \). We note \( f(x) \) is convex in \( x \), then \( f(s) \leq f(k+1) \) or \( f(s) \leq f \left( \frac{n}{2} \right) \).

Combining (8) and (9), we get

\[
n^2 - 2kn + 2k^2 + n - 2k \leq 2m \leq f(s) \leq f(k+1)
\]

\[
= 3(k+1)^2 + n^2 - 2n(k+1) + n - 3(k+1)
\]

or

\[
n^2 - 2kn + 2k^2 + n - 2k \leq 2m \leq f \left( \frac{n}{2} \right) \leq f \left( \frac{n}{2} \right) = \frac{3}{4}n^2 - \frac{n}{2}.
\]
Then $n \leq \frac{k^2 + 5k}{2}$ or $n^2 + (6 - 8k)n + 8k(k - 1) \leq 0$, each of these inequalities leads to a contradiction. So we have $s = k$, and thus $\delta(H) = k$, then,

$$d_H(v_1) = d_H(v_2) = \cdots = d_H(v_{k-1}) = k.$$ 

Our next goal is to show that $d_H(v_k) \geq n - k^2$. Indeed, suppose that

$$d_H(v_k) < n - k^2.$$

Also using Lemma 2, we get

$$2m = \sum_{i=1}^{k-1} d_H(v_i) + d_H(v_k) + \sum_{i=k+1}^{n-k} d_H(v_i) + \sum_{i=n-k+1}^{n} d_H(v_i) < (k - 1)k + n - k^2 + (n - 2k)(n - k) + k(n - 1) = n^2 - 2kn + 2k^2 + n - 2k,$$

contradicting (9). Hence $d_H(v_i) \geq n - k^2$ for every $i \in \{k, k + 1, \ldots, n\}$.

Next, we shall show that the vertices $v_k, v_{k+1}, \ldots, v_n$ induce a complete graph in $H$. Indeed, let $v_i, v_j \in \{v_k, v_{k+1}, \ldots, v_n\}$ be two distinct vertices of $H$. If they are nonadjacent, then

$$d_H(v_i) + d_H(v_j) \geq 2n - 2k^2 \geq n + 2k^2 + 1 - 2k^2 = n + 1,$$

contradicting (5).

Write $X$ for the vertex set $\{v_1, v_2, \ldots, v_{k-1}\}$. Write $Y$ for the set of vertices in $\{v_k, v_{k+1}, \ldots, v_n\}$ having neighbors in $X$. Let $Z$ be the set of remaining vertices of $V(G)$.

Since $|X| = k - 1$, and $d_H(v_1) = d_H(v_2) = \cdots = d_H(v_{k-1}) = k$, we get $Y \neq \emptyset$, and any vertex in $X$ must have at least two neighbors in $\{v_k, v_{k+1}, \ldots, v_n\}$.

In fact, every vertex from $Y$ is adjacent to every vertex in $X$. Indeed, suppose that this is not the case, and let $w \in \{v_k, v_{k+1}, \ldots, v_n\}, u \in X, v \in X$, such that $w$ is adjacent to $u$, but not to $v$. We see that

$$d_H(w) + d_H(v) \geq n - k + 1 + k = n + 1,$$

contradicting (5).

Next, let $l = |Y|$ and note that $2 \leq l \leq k$.

If $l = 2$, then $H = K_2 \lor (K_{n-k-1} + K_{k-1})$. Since $G \subseteq H$, by Lemma 9, if $G$ is a proper subgraph of $H$, $\mu(G) < n - k$, then $G = K_2 \lor (K_{n-k-1} + K_{k-1})$, a contradiction.

If $3 \leq l \leq k - 1$, we can get $H$ is Hamilton-connected, which contradicts the assumptions of $H$. 


Indeed, let $I$ be the graph induced by $X \cup Y \{u\}$, where $u \in Y$. Since $K_{l-1} \lor K_{k-1} \subset I$, and $l \geq 3$, we see that $I$ is 2-connected. Furtherly, if $x$ and $y$ are distinct nonadjacent vertices of $I$,

$$d_I(x) + d_I(y) \geq 2k - 2 \geq k + l - 1,$$

then $I$ is Hamilton-connected by Lemma 1.

Then for any two distinct vertices $x, y$ of $H$, we can get a Hamilton path of $H$ with $x, y$ as endpoint. So, $H$ is Hamilton-connected. For example, for any $x, y \in X$. Let $xP_4vP_2y$ be a Hamilton path of $I$, where $v \in Y$. Let $M$ be a subgraph of $H$, which is induced by $V(H) \setminus V(I)$. We note that $M$ is a complete graph, then $M$ is Hamiltonian. So, there is a Hamilton cycle $C : uP_3v_1u$ of $M$.

Now we delete the edges $u_1v, uv_1$, and add the edges $u_1u, vv_1$, then we get a path $xP_4vP_2y$ being a Hamilton path of $H$. Similar methods prove the other cases.

If $l = k$, we also can find that $H$ is Hamilton-connected, which contradicts the assumptions of $H$. For example, for any $x \in X, y \in Z$. Because every vertex in $Y$ is adjacent to every vertex in $X$, there is a path $xP_4v$, which contains all vertices of $X \cup Y$, where $v \in Y$. Let $N$ be a subgraph of $H$, which is induced by $Z \cup \{v\}$. We note that $N$ is a complete graph, then $N$ is Hamilton-connected. So, there is a Hamilton path $vP_3w$ of $N$. Now, we get a path $xP_4vP_3w$ being a Hamilton path of $H$. Similar methods prove the other cases.

So, the result follows.

**Theorem 11.** Let $k \geq 1, n \geq 2(k + 1)^2$, and let $G$ be a graph of order $n$ with minimum degree $\delta(G) \geq k$. If

$$\mu(G) \geq \frac{n^2}{n-1} - \frac{nk}{n-1} - \frac{2}{\sqrt{n-2}},$$

then $G$ is traceable from every vertex, unless $G = K_1 \lor (K_{n-k-1} + K_k)$.

**Proof.** Let $H = G \lor K_1$. Then $H$ be a graph of order $n + 1$, with minimum degree $\delta(H) \geq k + 1$. By Lemma 7 and the assumption, we have

$$\mu(H) > \frac{n-1}{n} - \frac{\mu(G)}{n} + \frac{2\sqrt{n-1}}{n}$$

$$\geq \frac{n-1}{n} \left( \frac{n^2}{n-1} - \frac{nk}{n-1} - \frac{2}{\sqrt{n-2}} \right) + \frac{2\sqrt{n-1}}{n}$$

$$= (n+1) - (k + 1).$$

Then by Theorem 10, we get $H$ is Hamilton-connected, unless $H = K_2 \lor (K_{n-k-1} + K_k)$.

So, according to Lemma 5, $G$ is traceable from every vertex, unless $G = K_1 \lor (K_{n-k-1} + K_k)$.
Let $E_{S_n}$ be the set of following graphs of even order $n$:

(i) $K_{\frac{n}{2}, \frac{n}{2}}$;
(ii) $G_1 \lor G_2$, where $G_1$ is a regular graph of order $n - r$ with degree $\frac{n}{2} - r$, $G_2$ has $r$ vertices, $1 \leq r \leq \frac{n}{2}$.

Let $E_{W_n}$ be the set of following graphs of odd order $n$:

$G_1 \lor G_2$, where $G_1$ is a regular graph of order $n + 1 - r$ with degree $\frac{n+1}{2} - r$, $G_2$ has $r - 1$ vertices, $1 \leq r \leq \frac{n+1}{2}$.

**Theorem 12.** Let $G$ be a graph of order $n \geq 2k$, where $k \geq 2$. If $\delta(G) \geq k$ and

$$\mu(\overline{G}) \leq \sqrt{(k - 1)(n - k - 1)},$$

then $G$ is Hamilton-connected, unless $G = K_{k-1, n-k-1} \lor K_2$ or $G = K_{k-1, n-k-1} \lor O_2$ or $G \in E_{S_n}$ and $n = 2k$.

**Proof.** Let $H = C_{n+1}(G)$. If $H$ is Hamilton-connected, then so is $G$ by Lemma 1. Now we assume that $H$ is not Hamilton-connected. Note that $H$ is $(n+1)$-closure of $G$, thus every two nonadjacent vertices $u, v$ of $H$ have degree sum at most $n$, i.e.,

$$d_{\overline{H}}(u) + d_{\overline{H}}(v) \geq n - 2, \text{ for any edge } uv \in E(\overline{H}).$$

Since $d_G(u) \geq k$ and $d_G(v) \geq k$, we have $d_{\overline{H}}(u) \leq n - k - 1$ and $d_{\overline{H}}(v) \leq n - k - 1$. Then combining (10), $k - 1 \leq d_{\overline{H}}(u) \leq n - k - 1, k - 1 \leq d_{\overline{H}}(v) \leq n - k - 1$, this implies that

$$d_{\overline{H}}(u)d_{\overline{H}}(v) \geq d_{\overline{H}}(u)(n - 2 - d_{\overline{H}}(u)) \geq (k - 1)(n - k - 1),$$

with equality if and only if (up to symmetry), $d_{\overline{H}}(u) = k - 1$ and $d_{\overline{H}}(v) = n - k - 1$.

By Lemma 6, Perron-Frobenius theorem, and the assumption,

$$\sqrt{(k - 1)(n - k - 1)} \geq \mu(\overline{G}) \geq \mu(\overline{H}) \geq \min_{uv \in E(\overline{H})} \sqrt{d_{\overline{H}}(u)d_{\overline{H}}(v)}$$

$$\geq \sqrt{(k - 1)(n - k - 1)}.$$

Therefore, $\mu(\overline{G}) = \mu(\overline{H}) = \sqrt{(k - 1)(n - k - 1)}$, and $d_{\overline{H}}(u) + d_{\overline{H}}(v) = n - 2$ for any edge $uv \in E(\overline{H})$, and $d_{\overline{H}}(u) = k - 1, d_{\overline{H}}(v) = n - k - 1$. Note that every nontrivial component of $\overline{H}$ has a vertex of degree at least $\frac{n}{2} - 1$ and hence of order at least $\frac{n}{2}$. This implies that $\overline{H} = K_{\frac{n}{2}} + K_{\frac{n}{2}}$ for $n = 2k$, or $\overline{H}$ contains exactly one nontrivial component $F$ which is either regular or semi-regular, and $\frac{n}{2} \leq |V(F)| \leq n$. 
Noting that $\mu(\overline{G}) = \mu(\overline{H})$, $G \supseteq H$, if $H = K_{\frac{n}{2}} + K_{\frac{n}{2}}$ and $n = 2k$, then $G = \overline{H}$ by the Perron-Frobenius theorem. So $G = K_{\frac{n}{2}, \frac{n}{2}} \in ES_n$ and $n = 2k$, a contradiction. Therefore we assume that $\overline{H}$ contains exactly one nontrivial component $F$.

First suppose $F$ is a bipartite semi-regular graph. By the condition of the degree sum of two adjacent vertices, we have $G$ contains at least $n - 2$ vertices. If $F$ contains $n - 2$ vertices, then $\overline{H} = K_{k-1,n-k-1} + O_2$. Noting that $\mu(\overline{G}) = \mu(\overline{H})$, $G \supseteq \overline{H}$, then $G = \overline{H}$ or $K_{k-1,n-k-1} + K_2$ by the Perron-Frobenius theorem. So $G = (K_{k-1} + K_{n-k-1}) \lor K_2$ or $(K_{k-1} + K_{n-k-1}) \lor O_2$, a contradiction. If $F$ contains $n - 1$ vertices, then the graph $F$ with two partite sets $X, Y$ has $|X| = k - 1$, $|Y| = n - k$ or $|X| = k + 1, |Y| = n - k - 1$ or $|X| = k - 1, |Y| = n - k + 1$. If $|X| = k, |Y| = n - k$, then according to the edge number of $F$, we have $(n - k)(k - 1) = (k - 1)(n - k - 1)$, a contradiction. If $F$ contains $n$ vertices, then the graph $F$ with two partite sets $X, Y$ has $|X| = k$, $|Y| = n - k$ or $|X| = k + 1, |Y| = n - k - 1$ or $|X| = k - 1, |Y| = n - k + 1$. If $|X| = k$, $|Y| = n - k$, then according to the edge number of $F$, we have $(n - k - 1)(k - 1) = (k - 1)(n - k - 1)$ or $(n - k - 1)(k - 1) = (k - 1)(n - k + 1)$, a contradiction.

Next we assume $F$ is a regular graph. Then for every $v \in V(F)$, $d_F(v) = \frac{n}{2} - 1$, and $n = 2k$. If $F = \overline{H}$, by a similar discussion as the above, $G = \overline{H}$, and hence $G = H$ is regular of degree $\frac{n}{2}$. By Lemma 8, $G = K_{\frac{n}{2}, \frac{n}{2}} \in ES_n$, or $G$ is Hamilton-connected, a contradiction. Otherwise, $\overline{H} = F \lor O_r$, where $r = n - |V(F)|$ and $1 \leq r \leq \frac{n}{2}$. Noting that $\mu(\overline{G}) = \mu(\overline{H})$, we have $G = F \lor F_1$, where $F_1$ is obtained from $O_r$ possibly adding some edges. Hence $G = F \lor F_1 \in ES_n$, a contradiction.

**Theorem 13.** Let $G$ be a graph of order $n \geq 2k + 1$, where $k \geq 2$. If $\delta(G) \geq k$ and

$$\mu(G) \leq \sqrt{k(n - k - 1)},$$

then $G$ is traceable from every vertex, unless $G = K_{k,n-k-1} \lor K_1$ or $G \in EW_n$ and $n = 2k + 1$.

**Proof.** Let $G' = G \lor K_1$. We note that $|V(G')| = n + 1$, $\mu(G') = \mu(\overline{G}) \leq \sqrt{k(n - k - 1)}$, $\delta(G') \geq k + 1$. By Theorem 12, we get $G'$ is Hamilton-connected, unless $G' = K_{k,n-k-1} \lor K_2$ or $G' = K_{k,n-k-1} \lor O_2$ or $G' \in ES_{n+1}$ and $n = 2k + 1$.

By Lemma 5 and the construction of $G'$, we have $G$ is traceable from every vertex, unless $G = K_{k,n-k-1} \lor K_1$ or $G \in EW_n$ and $n = 2k + 1$. ■
3. **(Signless Laplacian) Spectral Radius Conditions for a Nearly Balanced Bipartite Graph to be Traceable**

We note that if a bipartite graph $G = (X, Y; E)$ is traceable, $G$ is a balanced bipartite graph or a nearly balanced bipartite graph. Li and Ning [4] have presented some (signless Laplacian) spectral radius conditions for a balanced bipartite graph to be Hamiltonian. If $G = (X, Y; E)$ is a nearly balanced bipartite graph with $|X| = |Y| - 1$, we can obtained $G'$ from $G$ by adding a vertex which is adjacent to every vertex in $Y$, then $G'$ is a balanced partite graph. Note that $G$ is traceable if and only if $G'$ is Hamiltonian. Inspired by this, in this section, we will study the conditions for a nearly balanced bipartite graph to be traceable in terms of spectral radius, signless Laplacian spectral radius of the graph or its quasi-complement.

Let $G$ be balanced bipartite graph of order $2n$. The bipartite closure of $G$, denoted by $cl_B(G)$, is the graph obtained from $G$ by recursively joining pairs of nonadjacent vertices in different partite sets whose degree sum is at least $n + 1$ until no such pair remains. Note that $d_{cl_B(G)}(u) + d_{cl_B(G)}(v) \leq n$ for any pair of nonadjacent vertices $u$ and $v$ in the distant partite sets of $cl_B(G)$.

**Lemma 14** (Bondy and Chvátal [1]). A balanced bipartite graph $G$ is Hamiltonian if and only if $cl_B(G)$ is Hamiltonian.

Before introducing our results, we need some notations. In order to facilitate understanding, in this paper, when we mention a bipartite graph, we always fix its partite sets, e.g., $O_{n,m}$ and $O_{m,n}$ are considered as different bipartite graphs, unless $m = n$.

Let $G_1, G_2$ be two bipartite graphs, with the bipartition $\{X_1, Y_1\}$ and $\{X_2, Y_2\}$, respectively. We use $G_1 \cup G_2$ to denote the graph obtained from $G_1 + G_2$ by adding all possible edges between $X_1$ and $Y_2$ and all possible edges between $Y_1$ and $X_2$. We define some classes of graphs as follows.

$$B_n^k = O_{k, n-k} \cup K_{n-k, k} \quad (1 \leq k \leq n/2),$$

$$C_n^k = O_{k, n-k} \cup K_{n-k-1, k} \quad (1 \leq k \leq n/2).$$

Note that $e(B_n^k) = n(n-k) + k^2$, $e(C_n^k) = n(n-k-1) + k^2$, $\mu(B_n^k) = \mu(C_n^k) = \mu(K_{k, n-k}) = \sqrt{k(n-k)}$, and $B_n^k$ is not Hamiltonian, $C_n^k$ is not traceable. By Perron-Frobenius theorem, $\mu(B_n^k) > \mu(K_{n, n-k}) = \sqrt{n(n-k)}$, $\mu(C_n^k) > \mu(K_{n, n-k-1}) = \sqrt{n(n-k-1)}$.

Let $G = (X, Y)$ be a bipartite graph with two partite sets $X, Y$. Denote by $B_n^k (1 \leq k \leq n/2)$ the family of graphs $\{O_{k, n-k} \cup G(X, Y)\}$, where $|X| = n-k$, $|Y| = k$. Denote by $C_n^k (1 \leq k \leq n/2)$ the family of graphs $\{O_{k, n-k} \cup G(X, Y)\}$, where $|X| = n-k-1$, $|Y| = k$. 


Lemma 15 (Li and Ning [4]). Let $G$ be a balanced bipartite graph of order $2n$. If $\delta(G) \geq k \geq 1$, $n \geq 2k + 1$ and

$$e(G) > n(n - k - 1) + (k + 1)^2,$$

then $G$ is Hamiltonian unless $G \subseteq B^k_n$.

Lemma 16. Let $G = (X, Y)$ be a nearly balanced bipartite graph of order $2n - 1$. If $\delta(G) \geq k \geq 1$, $n \geq 2k + 1$, and

$$e(G) > n(n - k - 2) + (k + 1)^2,$$

then $G$ is traceable unless $G \subseteq C^k_n$.

Proof. Let $|X| = n - 1$, $|Y| = n$, and let $G'$ be obtained from $G$ by adding a vertex which is adjacent to every vertex in $Y$. Then $G'$ be a balanced bipartite graph. Note that $G$ is traceable if and only if $G'$ is Hamiltonian. We have $|V(G')| = 2n$, $\delta(G') \geq \delta(G) \geq k \geq 1$, $n \geq 2k + 1$, and

$$e(G') = e(G) + n > n(n - k - 2) + (k + 1)^2 + n = n(n - k - 1) + (k + 1)^2.$$

By Lemma 15, $G'$ is Hamiltonian unless $G' \subseteq B^k_n$. Thus $G$ is traceable unless $G \subseteq C^k_n$. $\blacksquare$

Lemma 17 (Bhattacharya, Friedland and Peled [6]). Let $G$ be a bipartite graph. Then

$$\mu(G) \leq \sqrt{e(G)}.$$

Lemma 18 (Ferrara, Jacobson and Powell [9]). Let $G$ be a non-Hamiltonian balanced bipartite graph of order $2n$. If $d(u)+d(v) \geq n$ for every two nonadjacent vertices $u, v$ in distinct partite sets, then either $G \in \bigcup_{k=1}^{n/2} B^k_n$, or $G = \Gamma_1$ or $\Gamma_2$ for $n = 4$. 
Lemma 19 (Feng and Yu [8], Yu and Fan [21]). Let $G$ be a graph with non-empty edge set. Then

$$q(G) \leq \max \left\{ d(u) + \frac{\sum_{v \in N(u)} d(v)}{d(u)} : u \in V(G) \right\}. $$

Lemma 20. Let $G$ be a bipartite graph with two partite sets $X, Y$, and $\max\{|X|, |Y|\} = n$. Then

$$q(G) \leq \frac{e(G)}{n} + n. $$

Proof. If $G$ is an edgeless graph, then $q(G) = 0$, and the result is trivially true. Now assume $G$ contains at least one edge. Let $x \in V(G)$, and

$$d(x) + \frac{\sum_{v \in N(x)} d(v)}{d(x)} = \max \left\{ d(u) + \frac{\sum_{v \in N(u)} d(v)}{d(u)} : u \in V(G) \right\}. $$

By Lemma 19, for every $v \in V(G)$, $d_G(v) \leq \max\{|X|, |Y|\} = n$, we get

$$\frac{e(G)}{n} + n - q(G) \geq \left( \frac{\sum_{v \in N(x)} d(v)}{n} + n \right) - \left( d(x) + \frac{\sum_{v \in N(x)} d(v)}{d(x)} \right)
$$

$$= (n - d(x)) \left( 1 - \frac{\sum_{v \in N(x)} d(v)}{nd(x)} \right) \geq 0. $$

The result follows.

Lemma 21. Let $k \geq 1$, $n \geq \frac{k^3}{2} + k + 2$. If $G$ is a subgraph of $C_k^n$, $\delta(G) \geq k$, then

$$\mu(G) < \sqrt{n(n-k-1)}, $$

unless $G = C_k^n$. 

Figure 2. Graphs $\Gamma_1$ and $\Gamma_2$. 

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Proof. The proof is similar to the proof of Lemma 9. Set for short \( \mu := \mu(G) \), and let \( \mathbf{x} = (x_{v_1}, \ldots, x_{v_{2n-1}})^T \) be a unit Perron vector of \( G \). By (3), we have

\[
\mu = \mathbf{x}^T A(G) \mathbf{x}.
\]

Assume that \( G \) is a proper subgraph of \( C_n^k \). By Perron-Frobenius theorem, we may assume that \( G \) is obtained by omitting just one edge \( uv \) of \( C_k^n \).

Write \( X \) for the set of vertices of \( C_k^n \) of degree \( k \), let \( Y \) be the set of vertices of \( C_k^n \) of degree \( n \), write \( Z \) for the set of vertices of \( C_k^n \) of degree \( n - k - 1 \), let \( H \) be the set of the remaining \( k \) vertices of \( C_k^n \) of degree \( n - 1 \).

Since \( \delta(G) \geq k \), we can see that \( G \) must contain all the edges between \( X \) and \( H \). Therefore \( \{u, v\} \subset Y \cup H \) or \( \{u, v\} \subset Y \cup Z \), with two possible cases: (a) \( u \in Y, v \in H \); (b) \( u \in Y, v \in Z \). We shall show that case (b) yields a graph of no smaller spectral radius than case (a).

Indeed, by (2), we have \( x_i = x_j \) for any \( i, j \in X \); likewise \( x_i = x_j \) for any \( i, j \in Y \setminus \{u\} \), for any \( i, j \in Z \setminus \{v\} \) and for any \( i, j \in H \setminus \{v\} \). Thus, let

\[
x := x_i, i \in X,
y := y_i, i \in Y \setminus \{u\},
z := z_i, i \in Z \setminus \{v\},
h := h_i, i \in H \setminus \{v\}.
\]

Suppose that case (a) holds, that is, \( u \in Y, v \in H \). Choose a vertex \( w \in Z \), remove the edge \( uw \), and add the edge \( uv \). Then the obtained graph \( G' \) is covered by case (b).

If \( x_w \leq x_v \), we have

\[
\mathbf{x}^T A(G') \mathbf{x} - \mathbf{x}^T A(G) \mathbf{x} = 2x_w(x_v - x_w) \geq 0.
\]

If \( x_w > x_v \), swap the entries \( x_v \) and \( x_w \), write \( \mathbf{x}' \) for the resulting vector. We note that \( \mathbf{x}' \) is also a unit vector, and have that

\[
\mathbf{x}'^T A(G') \mathbf{x}' - \mathbf{x}^T A(G) \mathbf{x} = 2(x_w - x_v) \sum_{i \in X} x_i \geq 0.
\]

Then by (3), \( \mu(G') \geq \mu(G) \), as claimed.

Therefore, we may assume that \( u \in Y, v \in Z \), and set \( t := x_u, s := x_v \). Note that the \( 2n - 1 \) eigenvalue-equations of \( G \) are reduced to six equations involving just the unknowns \( x, y, z, h, t, \) and \( s \).

\[
\begin{align*}
\mu x &= kh, \\
\mu y &= (n - k - 1)z + kh + s, \\
\mu z &= (n - k - 2)y + t,
\end{align*}
\]
\begin{align*}
\mu_h &= kx + (n - k - 2)y + t, \\
\mu_t &= (n - k - 1)z + kh, \\
\mu_s &= (n - k - 2)y.
\end{align*}

We find that
\begin{align*}
x &= \frac{k}{\mu}h, \\
t &= \frac{(n - k - 1)\left(\mu^2 - k^2\right) + k\mu^2}{\mu^3}h, \\
s &= \frac{(\mu^2 - (n - k - 1))\left(\mu^2 - k^2\right) - k\mu^2}{\mu^4}h.
\end{align*}

Furthermore, note that if we remove all edges between $X$ and $H$, and add the edge $uv$ to $G$, we obtain the graph $K_{n,n-k-1} + K_k$. Letting $x''$ be the restriction of $x$ to $K_{n,n-k-1}$, we find that
\begin{align*}
x''^T A(K_{n,n-k-1})x'' = x^T A(G)x + 2st - 2k^2xh = \mu + 2st - 2k^2xh.
\end{align*}

But since $||x''||^2 = 1 - kx^2$, we see that
\begin{align*}
\mu + 2st - 2k^2xh &= x''^T A(K_{n,n-k-1})x'' \leq \mu(K_{n,n-k-1})||x''||^2 \\
&= \sqrt{n(n - k - 1)(1 - kx^2)}.
\end{align*}

Assume for a contradiction that $\mu \geq \sqrt{n(n - k - 1)}$. This assumption together with the above inequality yields
\begin{align*}
\mu + 2st - 2k^2xh &\leq \mu \left(1 - kx^2\right),
\end{align*}

and therefore
\begin{align*}
2st - 2k^2xh &\leq -kx^2\mu.
\end{align*}

Now, first combining above equality about $x$, then combining above equalities about $t$ and $s$, we have
\begin{align*}
k^3 &\geq \frac{2\mu}{h^2}st \geq \frac{2(n - k - 1)\left(\mu^2 - (n - k - 1)\right)\left(\mu^2 - k^2\right)^2}{\mu^6} \\
&\quad - \frac{2k(n - k - 1)\left(\mu^2 - k^2\right)}{\mu^4}.
\end{align*}
Applying Bernoulli’s inequality to the right side, we get
\[ k^3 \geq 2(n - k - 1) \left( \frac{\mu^2 - (n - k - 1)}{\mu^2} \right) \left( \frac{\mu + k}{\mu} \right)^2 \left( \frac{\mu - k}{\mu} \right)^2 \]
\[ - \frac{2k(n - k - 1)}{\mu^2} \left( 1 - \frac{k^2}{\mu^2} \right) \]
\[ = 2(n - k - 1) \left( 1 - \frac{n - k - 1}{\mu^2} \right) \left( 1 + \frac{k}{\mu} \right)^2 \left( 1 - \frac{k}{\mu} \right)^2 \]
\[ - \frac{2k(n - k - 1)}{\mu^2} + \frac{2k^3(n - k - 1)}{\mu^4} \]
\[ > 2(n - k - 1) \left( 1 - \frac{n - k - 1}{\mu^2} \right) - \frac{2k(n - k - 1)}{\mu^2}. \]

Using the inequality \( \mu \geq \sqrt{n(n - k - 1)} \), we easily find that
\[ k^3 > 2(n - k - 1) - \frac{2(n - 1)}{n} > 2(n - k - 2), \]
and then \( n < \frac{k^2}{2} + k + 2 \), a contradiction.

**Theorem 22.** Let \( G \) be a nearly balanced bipartite graph of order \( 2n - 1 \) \( (n \geq \max \{ \frac{k^2}{2} + k + 2, (k + 1)^2 \}) \), where \( k \geq 1 \). If \( \delta(G) \geq k \) and
\[ \mu(G) > \sqrt{n(n - k - 1)}, \]
then \( G \) is traceable, unless \( G = C_n^k \).

**Proof.** By the assumption and Lemma 17,
\[ \sqrt{n(n - k - 1)} < \mu(G) \leq \sqrt{e(G)}. \]
Thus, we obtain
\[ e(G) > n(n - k - 1) \geq n(n - k - 2) + (k + 1)^2, \]
when \( n \geq \max \{ \frac{k^2}{2} + k + 2, (k + 1)^2 \} > 2k + 1 \), by Lemma 16, \( G \) is traceable or \( G \subseteq C_n^k \). But if \( G \subseteq C_n^k \), then \( \mu(G) < \sqrt{n(n - k - 1)} \), unless \( G = C_n^k \) by Lemma 21, a contradiction.

**Theorem 23.** Let \( G = (X, Y) \) be a nearly balanced bipartite graph of order \( 2n - 1 \) \( (n \geq 2k) \), where \( k \geq 1 \). If \( \delta(G) \geq k \), and
\[ \mu(\hat{G}) \leq \sqrt{k(n - k)}, \]
then \( G \) is traceable, unless \( G \in \bigcup_{k=1}^{n/2} C_n^k \) or \( \Gamma_2 - v \), where \( d_{\Gamma_2}(v) = 4 \).
Proof. Let \(|X| = n - 1, |Y| = n, G'\) be obtained from \(G\) by adding a vertex which is adjacent to every vertex in \(Y\). Then \(G'\) is a balanced partite graph. Note that \(G\) is traceable if and only if \(G'\) is Hamiltonian. Let \(H = \text{cl}_B(G')\). If \(H\) is Hamiltonian, then so is \(G'\) by Lemma 14. Now we assume that \(H\) is not Hamiltonian. Note that \(H\) is bipartite closure of \(G\), thus every two nonadjacent vertices \(u, v\) in distant partite sets of \(H\) have degree sum at most \(n\), i.e.,

\[
(11) \quad d_H(u) + d_H(v) = n - d_H(u) + n - d_H(v) \geq n,
\]

for any edge \(uv \in E(\hat{H})\).

This implies that \(\hat{H}\) contains only one component or \(\hat{H} = K_{s,n−s} + K_{t,n−t}\), \(s, t \geq 1\). If \(\hat{H} = K_{s,n−s} + K_{t,n−t}\), \(s, t \geq 1\), it contradicts the structure of \(H\) (it must contain an isolated vertex). So, \(\hat{H}\) contains only one component.

Since \(\delta(H) \geq \delta(G') \geq \delta(G) \geq k\), we can see that \(d_H(u) \leq n - k\) and \(d_H(v) \leq n - k\). Thus by (11), we have \(k \leq d_H(u) \leq n - k, k \leq d_H(v) \leq n - k\), which implies that

\[
d_H(u)d_H(v) \geq d_H(u)(n - d_H(u)) \geq k(n - k),
\]

with equality if and only if (up to symmetry) \(d_H(u) = k, d_H(v) = n - k\). By Lemma 6,

\[
\sqrt{k(n - k)} \geq \mu(\hat{G}) = \mu(G') \geq \mu(\hat{H}) \geq \min_{uv \in E(\hat{H})} \sqrt{d_H(u)d_H(v)} \geq \sqrt{k(n - k)},
\]

which implies that \(\mu(\hat{H}) = \sqrt{k(n - k)}\) and there is an edge \(uv \in E(\hat{H})\) such that \(d_H(u) = k, d_H(v) = n - k\). Let \(F\) be the component of \(\hat{H}\) which contains \(uv\). By Lemma 6, \(F\) is a bipartite semi-regular graph with partite sets \(X' \subseteq X\), and \(Y' \subseteq Y\), and for any vertex \(x \in X'\), \(d_F(x) = k\), and any vertex \(y \in Y'\), \(d_F(y) = n - k\). Then \(d_H(u) + d_H(v) = n\) for every two nonadjacent vertices \(u, v\) in distinct partite sets of \(H\). By Lemma 18, \(H \in \bigcup_{k=1}^{n/2} B_n^k\) or \(H = \Gamma_1\) or \(\Gamma_2\) for \(n = 4\) and \(k = 2\). Then \(G' \subseteq B_n^k\) (\(1 \leq k \leq n/2\)) or \(G' \subseteq \Gamma_1\) or \(\Gamma_2\) for \(n = 4\) and \(k = 2\). By Perron-Frobenius theorem, every (spanning) subgraph of \(\Gamma_1, \Gamma_2\) or \(B_n^k, 1 \leq k \leq n/2\), if it is not \(\Gamma_1\) or \(\Gamma_2\) or a graph in \(B_n^k\), \(1 \leq k \leq n/2\), then it has the quasi-complement with spectral radius greater than \(\sqrt{k(n - k)}\). Thus \(G' \in \bigcup_{k=1}^{n/2} B_n^k\) or \(\Gamma_1\) or \(\Gamma_2\) for \(n = 4\) and \(k = 2\). By the construction of \(G'\), we get \(G \in \bigcup_{k=1}^{n/2} C_n^k\) or \(\Gamma_2 - v\), where \(d_{\Gamma_2}(v) = 4\), a contradiction. \(\blacksquare\)

Theorem 24. Let \(G\) be a nearly balanced bipartite graph of order \(2n - 1\) \((n \geq (k + 1)^2)\), where \(k \geq 1\). If \(\delta(G) \geq k\) and

\[
q(G) > \frac{n(2n - k - 2) + (k + 1)^2}{n},
\]

then \(G\) is traceable, unless \(G \subseteq C_n^k\). 

Proof. By the assumption and Lemma 20,
\[
\frac{n(2n - k - 2) + (k + 1)^2}{n} < q(G) \leq \frac{e(G)}{n} + n.
\]
Thus, we obtain
\[
e(G) > n(n - k - 1) \geq n(n - k - 2) + (k + 1)^2,
\]
when \( n \geq (k + 1)^2 \), by Lemma 16, \( G \) is traceable or \( G \subseteq C^k_n \).

Remark 25. In Theorem 24, we cannot change \( G \subseteq C^k_n \) to \( G = C^k_n \) like in Theorem 22. In fact, we can find a subgraph \( G \subset C^k_n \), which satisfies the conditions of Theorem 24, such that \( q(G) > 2n - k - 1 \geq \frac{n(2n - k - 2) + (k + 1)^2}{n} \).

Proof. Assume that \( G \) is a proper subgraph of \( C^k_n \), \( n \geq (k + 1)^2 \), \( k \geq 1 \), \( \delta(G) \geq k \), and has the maximum signless Laplacian spectral. By Perron-Frobenius theorem, \( G \) is obtained by omitting just one edge \( uv \) of \( C^k_n \). Set for short \( q := q(G) \), and let \( x = (x_{v_1}, \ldots, x_{v_{2n-1}})^T \) be a positive unit eigenvector to \( q \). We have
\[
q = x^T Q(G) x = 2 \sum_{uv \in E(G)} (x_u + x_v)^2,
\]
and
\[
(q - d_G(v)) x_v = \sum_{u \in N_G(v)} x_u,
\]
for each vertex \( v \in V(G) \). Equation (13) is called the signless Laplacian eigenvalue-equation for the graph \( G \). In addition, for an arbitrary unit vector \( x \in \mathbb{R}^n \),
\[
q \geq x^T Q(G) x,
\]
with equality holds if and only if \( x \) is an eigenvector of \( Q(G) \) according to \( q \).

Write \( X \) for the set of vertices of \( C^k_n \) of degree \( k \), let \( Y \) be the set of vertices of \( C^k_n \) of degree \( n \), write \( Z \) for the set of vertices of \( C^k_n \) of degree \( n - k - 1 \), let \( H \) be the set of the remaining \( k \) vertices of \( C^k_n \) of degree \( n - 1 \).

Since \( \delta(G) \geq k \), we see that \( G \) must contain all the edges between \( X \) and \( H \). Therefore \( \{u, v\} \subset Y \cup H \) or \( \{u, v\} \subset Y \cup Z \), with two possible cases: (a) \( u \in Y, v \in H \); (b) \( u \in Y, v \in Z \). We shall show that case (b) yields a graph of no smaller signless Laplacian spectral radius than case (a).

Indeed, by (13), we have \( x_i = x_j \) for any \( i, j \in X \); likewise \( x_i = x_j \) for any \( i, j \in Y \setminus \{u\} \), for any \( i, j \in Z \setminus \{v\} \) and for any \( i, j \in H \setminus \{v\} \). Thus, let
\[
x := x_i, i \in X,
y := y_i, i \in Y \setminus \{u\},
z := z_i, i \in Z \setminus \{v\},
h := h_i, i \in H \setminus \{v\}.
\]
Suppose that case (a) holds, that is, \( u \in Y, v \in H \). Choose a vertex \( w \in Z \), remove the edge \( uw \), and add the edge \( uv \). Then the obtained graph \( G' \) is covered by case (b).

If \( x_w \leq x_v \), we have

\[
x^T Q(G')x - x^T Q(G)x = 2(x_v + x_u)^2 - 2(x_w + x_u)^2 \geq 0.
\]

If \( x_w > x_v \), swap the entries \( x_v \) and \( x_w \), write \( x' \) for the resulting vector. We note that \( x' \) is a unit vector, and have that

\[
x^T Q(G')x' - x^T Q(G)x = 2 \left( x_w + \sum_{i \in X} x_i \right)^2 - 2 \left( x_v + \sum_{i \in X} x_i \right)^2 \geq 0.
\]

Then by (14), \( q(G') \geq q(G) \).

Therefore, \( G \) is obtained by omitting just one edge \( uv \) of \( C_n^k \), where \( u \in Y, v \in Z \). Now set \( t := x_u, s := x_v \). Note that the \( 2n - 1 \) signless Laplacian eigenvalue-equations of \( G \) are reduced to six equations involving just the unknowns \( x, y, z, h, t, \) and \( s \). By Equation (13), we have

\[
\begin{align*}
(q - k)x &= kh, \\
(q - n)y &= (n - k - 1)z + kh + s, \\
(q - (n - k - 1))z &= (n - k - 2)y + t, \\
(q - (n - 1))h &= kx + (n - k - 2)y + t, \\
(q - (n - 1))t &= (n - k - 1)z + kh, \\
(q - (n - k - 2))s &= (n - k - 2)y.
\end{align*}
\]

Transform the above equations into a matrix equation \((B - qI)x = 0\), where

\[
x = (x, y, z, h, t, s)^T,
\]

\[
B = \begin{pmatrix}
k & 0 & 0 & k & 0 & 0 \\
0 & n & n - k - 1 & k & 0 & 1 \\
0 & n - k - 2 & n - k - 1 & 0 & 1 & 0 \\
k & n - k - 2 & 0 & n - 1 & 1 & 0 \\
0 & 0 & n - k - 1 & k & n - 1 & 0 \\
0 & n - k - 2 & 0 & 0 & 0 & n - k - 2
\end{pmatrix}.
\]

Let

\[
f(x) = \det(B - xI) = -x(n - 1 - x)(x^4 + (-4n + k + 4)x^3 \\
+ (-nk + 6 + 5n^2 - 2k^2 - 11n + k)x^2 \\
+ (7n^2 + 5nk + 2 + 6nk^2 - 2n^2k - 7n - 2n^3 - 6k^2 - 2k^3 - 3k)x \\
+ 2nk^3 - k^3 - 2k - 3k^2 + 8nk^2 + 7nk + 2n^3k - 4n^2k^2 - 7n^2k).
\]
Thus, \( q \) is the largest root of \( f(x) = 0 \), and when \( x > q \), \( f(x) \) is monotonically increasing.

But when \( k \geq 1, n \geq (k+1)^2 \), we have

\[
f(2n-k-1) = (k+1-2n)(k-n)((4-k^2)n^2 + (-6k+k^3-4)n + 1 + 3k^2 + 3k + k^3) < 0,
\]

which implies that \( q > 2n-k-1 \), and the result follows.

\[\square\]

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