FAIR TOTAL DOMINATION NUMBER
IN CACTUS GRAPHS

Majid Hajian
Department of Mathematics
Shahrood University of Technology
Shahrood, Iran

e-mail: majid_hajian2000@yahoo.com

AND

Nader Jafari Rad
Department of Mathematics
Shahed University
Tehran, Iran

e-mail: n.jafarirad@gmail.com

Abstract

For $k \geq 1$, a $k$-fair total dominating set (or just kFTD-set) in a graph $G$ is a total dominating set $S$ such that $|N(v) \cap S| = k$ for every vertex $v \in V \setminus S$. The $k$-fair total domination number of $G$, denoted by $ftd_k(G)$, is the minimum cardinality of a kFTD-set. A fair total dominating set, abbreviated FTD-set, is a kFTD-set for some integer $k \geq 1$. The fair total domination number of a nonempty graph $G$, denoted by $ftd(G)$, of $G$ is the minimum cardinality of an FTD-set in $G$. In this paper, we present upper bounds for the 1-fair total domination number of cactus graphs, and characterize cactus graphs achieving equality for the upper bounds.

Keywords: fair total domination, cactus graph.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

For notation and graph theory terminology not given here, we follow [12]. Specifically, let $G$ be a graph with vertex set $V(G) = V$ of order $|V| = n$ and let $v$ be a vertex in $V$. The open neighborhood of $v$ is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and
the closed neighborhood of \( v \) is \( N_G[v] = \{v\} \cup N_G(v) \). If the graph \( G \) is clear from the context, we simply write \( N(v) \) rather than \( N_G(v) \). The degree of a vertex \( v \), is \( \deg(v) = |N(v)| \). A vertex of degree one is called a leaf and its neighbor a support vertex. We denote the set of leaves and support vertices of a graph \( G \) by \( L(G) \) and \( S(G) \), respectively. A strong support vertex is a support vertex adjacent to at least two leaves, and a weak support vertex is a support vertex adjacent to precisely one leaf. For a set \( S \subseteq V \), its open neighborhood is the set \( N(S) = \bigcup_{v \in S} N(v) \), and its closed neighborhood is the set \( N[S] = N(S) \cup S \). The 2-corona \( 2\text{-cor}(G) \) of a graph \( G \) is a graph obtained from \( G \) by adding a path \( P_2 \) for every vertex \( v \) and joining \( v \) to a leaf of \( P_2 \). Note that \( 2\text{-cor}(G) \) has order \( 3|V(G)| \). The distance \( d(u, v) \) between two vertices \( u \) and \( v \) in a graph \( G \) is the minimum number of edges of a path from \( u \) to \( v \). For a subset \( S \) of vertices of a graph \( G \), we denote by \( G[S] \) the subgraph of \( G \) induced by \( S \). A cactus graph is a graph such that no pair of cycles have a common edge.

A subset \( S \subseteq V \) is a dominating set of \( G \) if every vertex not in \( S \) is adjacent to a vertex in \( S \). The domination number of \( G \), denoted by \( \gamma(G) \), is the minimum cardinality of a dominating set of \( G \). A dominating set \( S \) in a graph \( G \) with no isolated vertex, is a total dominating set of \( G \) if every vertex in \( S \) is adjacent to a vertex in \( S \).

Caro et al. [1] studied the concept of fair domination in graphs. For \( k \geq 1 \), a \( k \)-fair dominating set, abbreviated \( kFD \)-set, in \( G \) is a dominating set \( S \) such that \( |N(v) \cap D| = k \) for every vertex \( v \in V \setminus D \). The \( k \)-fair domination number of \( G \), denoted by \( fd_k(G) \), is the minimum cardinality of a \( kFD \)-set. A \( kFD \)-set of \( G \) of cardinality \( fd_k(G) \) is called a \( fd_k(G) \)-set. A fair dominating set, abbreviated \( FD \)-set, in \( G \) is a \( kFD \)-set for some integer \( k \geq 1 \). The fair domination number, denoted by \( fd(G) \), of a graph \( G \) that is not the empty graph is the minimum cardinality of an \( FD \)-set in \( G \). An \( FD \)-set of \( G \) of cardinality \( fd(G) \) is called a \( fd(G) \)-set. A perfect dominating set in a graph \( G \) is a dominating set \( S \) such that every vertex in \( V(G) \setminus S \) is adjacent to exactly one vertex in \( S \). Hence a \( 1FD \)-set is precisely a perfect dominating set. The concept of perfect domination was introduced by Cockayne et al. in [4], and Fellows et al. [7] with a different terminology which they called semiperfect domination. This concept was further studied in, for example, [2, 3, 5, 6, 8, 9, 11].

Maravilla et al. [13] introduced the concept of fair total domination in graphs. For an integer \( k \geq 1 \) and a graph \( G \) with no isolated vertex, a \( k \)-fair total dominating set, abbreviated \( kFTD \)-set, is a total dominating set \( S \subseteq V(G) \) such that \( |N(u) \cap S| = k \) for every \( u \in V(G) \setminus S \). The \( k \)-fair total domination number of \( G \), denoted by \( ftd_k(G) \), is the minimum cardinality of a \( kFTD \)-set. A \( kFTD \)-set of \( G \) of cardinality \( ftd_k(G) \) is called an \( ftd_k(G) \)-set. A fair total dominating set, abbreviated \( FTD \)-set, in \( G \) is a \( kFTD \)-set for some integer \( k \geq 1 \). Thus, a fair total dominating set \( S \) of a graph \( G \) is a total dominating set \( S \) of \( G \) such that
for every two distinct vertices \( u \) and \( v \) of \( V(G) \setminus S \), \(|N(u) \in S| = |N(v) \cap S|\); that is, \( S \) is both a fair dominating set and a total dominating set of \( G \). The fair total domination number of \( G \), denoted by \( \text{ftd}(G) \), is the minimum cardinality of an FTD-set. A fair total dominating set of cardinality \( \text{ftd}(G) \) is called a minimum fair total dominating set or an FTD-set of \( G \).

In [10], Volkmann and we studied fair total domination in trees and unicyclic graphs. In this paper, we study 1-fair total domination in cactus graphs. We present upper bounds for the 1-fair total domination number of cactus graphs, and characterize cactus graphs achieving equality for the upper bounds. The techniques used in this paper are similar to those presented in [9]. The following observations are easily verified.

**Observation 1.** Any support vertex in a graph \( G \) with no isolated vertex belongs to every \( k \)FTD-set for each integer \( k \).

**Observation 2.** Let \( S \) be a 1FTD-set in a graph \( G \), and \( v \) be a vertex of degree at least two such that \( v \) is adjacent to a weak support vertex \( v' \). If \( S \) contains a vertex \( u \in N_G(v) \setminus \{v'\} \), then \( v \in S \).

2. **Unicyclic Graphs**

A vertex \( v \) of a graph is a special vertex if \( \deg_G(v) = 2 \) and \( v \) belongs to a cycle of \( G \). Let \( \mathcal{H}_1 \) be the class of all graphs \( G \) that can be obtained from the 2-corona \( 2\text{-cor}(C) \) of a cycle \( C \) by removing precisely one support vertex \( v \) and the leaf adjacent to \( v \). Let \( \mathcal{G}_1 \) be the class of all graphs \( G \) that can be obtained from a sequence \( G_1, G_2, \ldots, G_s = G \), where \( G_1 \in \mathcal{H}_1 \), and if \( s \geq 2 \), then \( G_{j+1} \) is obtained from \( G_j \) by one of the following Operations \( \mathcal{O}_1 \) or \( \mathcal{O}_2 \), for \( j = 1, 2, \ldots, s - 1 \).

**Operation \( \mathcal{O}_1 \).** Let \( v \) be a vertex of \( G_j \) with \( \deg(v) \geq 2 \) such that \( v \) is not a special vertex. Then \( G_{j+1} \) is obtained from \( G_j \) by adding a path \( P_3 \) and joining \( v \) to a leaf of \( P_3 \) by means of an edge.

**Operation \( \mathcal{O}_2 \).** Let \( v \) be a support vertex of \( G_j \) and let \( u \) be a leaf adjacent to \( v \). Then \( G_{j+1} \) is obtained from \( G_j \) by adding a vertex \( u' \) and a path \( P_2 \), and joining \( u \) to \( u' \) and \( v \) to a leaf of \( P_2 \).

**Observation 3.** If \( H \in \mathcal{H}_1 \), then \( H \) has precisely one special vertex.

**Observation 4** [10]. If \( G \in \mathcal{G}_1 \) has order \( n \), and \( C \) is the cycle of \( G \), then we have the following.

1. \( G \) has precisely one special vertex.
2. \( G \) has \( (n - 1)/3 \) leaves.
3. No vertex of \( C \) is a support vertex.
Lemma 5 [10]. If $G \in \mathcal{G}_1$, then every 1FTD-set in $G$ contains every vertex of $G$ of degree at least two.

Theorem 6 [10]. If $G$ is a unicyclic graph of order $n$, then $\text{ftd}_1(G) \leq \frac{2n+1}{3}$, with equality if and only if $G = C_7$ or $G \in \mathcal{G}_1$.

3. Main Result

Our aim in this paper is to give an upper bound for the fair total domination number of a cactus graph $G$ in terms of the number of cycles of $G$, and then characterize all cactus graphs achieving equality for the proposed bound. For this purpose we first introduce some families of graphs. Let $\mathcal{H}_1$ and $\mathcal{G}_1$ be the families of unicyclic graphs described in Section 2. For $i = 2, \ldots, k$, we construct a family $\mathcal{H}_i$ from $\mathcal{G}_i^{-1}$, and a family $\mathcal{G}_i$ from $\mathcal{H}_i$ as follows.

- Family $\mathcal{H}_i$. Let $\mathcal{H}_i$ be the family of all graphs $H_i$ such that $H_i$ can be obtained from a graph $H_1 \in \mathcal{H}_1$ and a graph $G \in \mathcal{G}_{i-1}$, by the following procedure.

  Procedure A. Let $w_0 \in V(H_1)$ be a vertex of degree at least two of $H_1$ such that $w_0$ is adjacent to a weak support vertex $w_0'$, and $w \in V(G_{i-1})$ be a vertex of degree at least two of $G_{i-1}$ such that $w$ is adjacent to a weak support vertex $w'$ of degree two. We remove $w_0'$, the leaf adjacent to $w_0'$, $w'$ and the leaf adjacent to $w'$, and then identify the vertices $w_0$ and $w$.

- Family $\mathcal{G}_i$. Let $\mathcal{G}_i$ be the family of all graphs $G$ that can be obtained from a sequence $G_1, G_2, \ldots, G_s = G$, where $G_1 \in \mathcal{H}_i$, and if $s \geq 2$, then $G_{j+1}$ is obtained from $G_j$ by one of the Operations $O_1$ or $O_2$, described in Section 2, for $j = 1, 2, \ldots, s-1$.

  Note that $\mathcal{H}_i \subseteq \mathcal{G}_i$, for $i = 1, 2, \ldots, k$. Figure 1 demonstrates the construction of the family $\mathcal{G}_k$.

We will prove the following.
Theorem 7. If $G$ is a cactus graph of order $n \geq 5$ with $k \geq 1$ cycles, then $\text{ftd}_1(G) \leq (2(n+k) - 1)/3$, with equality if and only if $G = C_7$ or $G \in \mathcal{G}_k$.

4. Preliminary Results and Observations

4.1. Notation

We call a vertex $w$ in a cycle $C$ of a cactus graph $G$ a special cut-vertex if $w$ belongs to a shortest path from $C$ to a cycle $C' \neq C$. We call a cycle $C$ in $G$, a leaf-cycle if $C$ contains exactly one special cut-vertex. In the cactus graph presented in Figure 2, $v_i$ is a special cut-vertex, for $i = 1, 2, \ldots, 8$. Moreover, $C_j$ is a leaf-cycle for $j = 1, 2, 3$.

![Cactus Graph](image)

Figure 2. $C_i$ is a leaf-cycle for $i = 1, 2, 3$ and $v_j$ is a special cut-vertex for $j = 1, 2, \ldots, 8$.

Observation 8. Every cactus graph with at least two cycles contains at least two leaf-cycles.

4.2. Properties of the family $\mathcal{G}_k$

The following observation can be proved by a simple induction on $k$.

Observation 9. If $G \in \mathcal{G}_k$ is a cactus graph of order $n$, then we have the following.

1. No cycle of $G$ contains a support vertex. Furthermore, any cycle of $G$ contains precisely one special vertex.

2. If a vertex $v$ of $G$ belongs to a cycle of $G$, then $v$ is adjacent to at most one weak support vertex of degree two.

3. $|L(G)| = (n + 1)/3 - 2k/3$.

4. If a vertex $v$ of $G$ belongs to at least two cycles of $G$, then $v$ is not adjacent to a weak support vertex, and $v$ belongs to precisely two cycles of $G$. 
Proof. Let $G \in G_k$ be a cactus graph of order $n$. To show (1), (2) or (3), we prove by an induction on $k$, that we call first-induction. For the base step, if $k = 1$, then $G \in G_1$, and the result follows by Observation 4. Assume the result holds for all graphs $G' \in G_{k'}$ with $k' < k$. Now consider the graph $G \in G_k$, where $k > 1$. Clearly, $G$ is obtained from a sequence $G_1, G_2, \ldots, G_l = G$, of cactus graphs such that $G_1 \in H_k$, and if $l \geq 2$, then $G_{i+1}$ is obtained from $G_i$ by one of the Operations $O_1$ or $O_2$ for $i = 1, 2, \ldots, l - 1$. We prove by an induction on $l$, that we call second-induction. For the base step of the second-induction, let $l = 1$. Thus $G \in H_k$. By the construction of graphs in the family $H_k$, there are graphs $H \in H_1$ and $G' \in G_{k-1}$ such that $G$ is obtained from $H$ and $G'$ by Procedure $A$. It is easy to see that the base step of the second-induction holds. Assume that the result (for the second-induction) holds for $2 \leq l' < l$. Now let $G = G_l$. Clearly, $G$ is obtained from $G_{l-1}$ by applying one of the Operations $O_1$ or $O_2$. It is easy to see that the result holds.

The proof for (4) is similarly verified. 

Observation 10. Let $G \in G_k$ be obtained from a sequence $G_1, G_2, \ldots, G_s = G$ $(s \geq 2)$ such that $G_1 \in H_1$ and $G_{j+1}$ is obtained from $G_j$ by one of the Operations $O_1$ or $O_2$ or Procedure $A$, for $j = 1, 2, \ldots, s - 1$. If $v$ is a vertex of $G$ belonging to two cycles of $G$, then there is an integer $i \in \{2, 3, \ldots, s\}$ such that $G_i$ is obtained from $G_{i-1}$ by applying Procedure $A$ on the vertex $v$ using a graph $H \in H_1$, such that $v$ belongs to a cycle of $G_{i-1}$.

Observation 11. Assume that $G \in G_k$ and $v \in V(G)$ is a vertex of degree four belonging to two cycles. Let $D_1$ and $D_2$ be the components of $G - v$, $G_1^*$ be the graph obtained from $G[D_1 \cup \{v\}]$ by joining $v$ to a leaf of a path $P_2$, and $G_2^*$ be the graph obtained from $G[D_2 \cup \{v\}]$ by joining $v$ to a leaf of a path $P_2$. Then there exists an integer $k' < k$ such that $G_1^* \in G_{k'}$ or $G_2^* \in G_{k'}$.

Proof. Let $G \in G_k$. Then $G$ is obtained from a sequence $G_1, G_2, \ldots, G_s = G$ $(s \geq 2)$ such that $G_1 \in H_1$ and $G_{j+1}$ is obtained from $G_j$ by one of the Operations $O_1$ or $O_2$ or Procedure $A$, for $j = 1, 2, \ldots, s - 1$. Note that $s \geq k$. We define the $j$-th Procedure-Operation or just $PO_j$ as one of the Operation $O_1$, Operation $O_2$, or Procedure $A$ that can be applied to obtain $G_{j+1}$ from $G_j$. Thus $G$ is obtained from $G_1$ by Procedure-Operations $PO_1, PO_2, \ldots, PO_{s-1}$.

Let $v$ be a vertex of $G$ of degree four belonging to two cycles of $G$, and $D_1$ and $D_2$ be the components of $G - v$. By Observation 10, there is an integer $i \in \{2, 3, \ldots, s\}$ such that $G_i$ is obtained from $G_{i-1}$ by applying Procedure $A$ on the vertex $v$ using a graph $H \in H_1$. Note that $v$ is adjacent to a weak support vertex $v'$ of $G_{i-1}$. Let $v''$ be the leaf of $v'$ in $G_{i-1}$ that is removed in Procedure $A$. Clearly, either $V(G_{i-1}) \cap D_1 \neq \emptyset$ or $V(G_{i-1}) \cap D_2 \neq \emptyset$. Without loss of generality, assume that $V(G_{i-1}) \cap D_1 \neq \emptyset$. Among $PO_1, PO_{i+1}, \ldots, PO_{s-1}$, let $PO_{r_1}, PO_{r_2}, \ldots, PO_{r_t}$, be those procedure-operations applied on a vertex of $D_1$. 


Note that $i \leq t \leq s - 1$. Let $G_{r_0} = G_{i-1}$ and $G_{r_{t+1}}$ be obtained from $G_{r_t}$ by $PO_{l+1}$, for $l = 0, 1, 2, \ldots, t - 1$. Clearly, by an induction on $t$, we can deduce that there is an integer $k^* < k$ such that $G_{r_t} \in \mathcal{G}_{k^*}$. Note that $G_{r_t} = G^*_t$. 

**Lemma 12.** If $G \in \mathcal{G}_k$, then every 1FTD-set in $G$ contains each vertex of $G$ of degree at least two.

**Proof.** Let $G \in \mathcal{G}_k$, and $S$ be a 1FTD-set in $G$. We prove by an induction on $k$, that we call first-induction, that $S$ contains every vertex of $G$ of degree at least two. For the base step, if $k = 1$, then $G \in \mathcal{G}_1$, and the result follows by Lemma 5. Assume the result holds for all graphs $H \in \mathcal{G}_k$ with $k' < k$. Now consider the graph $G \in \mathcal{G}_k$, where $k > 1$. Clearly, $G$ is obtained from a sequence $G_1, G_2, \ldots, G_l = G$, of cactus graphs such that $G_1 \in \mathcal{H}_k$, and if $l \geq 2$, then $G_{l+1}$ is obtained from $G_l$ by one of the Operations $O_1$ or $O_2$ for $i = 1, 2, \ldots, l - 1$.

We prove by an induction on $l$, that we call second-induction, that $S$ contains every vertex of $G$ of degree at least two.

For the base step of the second-induction, let $l = 1$. Thus $G \in \mathcal{H}_k$. By the construction of graphs in the family $\mathcal{H}_k$, there are graphs $H \in \mathcal{H}_1$ and $G' \in \mathcal{G}_{k-1}$ such that $G$ is obtained from $H$ and $G'$ by Procedure A. Clearly, $H$ is obtained from the 2-corona $2\text{-cor}(C)$ of a cycle $C$, by removing precisely one support vertex $v$ and the leaf adjacent to $v$ of $2\text{-cor}(C)$.

Let $C = c_0c_1\cdots c_r c_{0}$ be the cycle of $H$, where $c_0$ is a vertex of degree at least two of $H$ that is adjacent to a weak support vertex $c'_0$, and let $c'_0$ and its leaf (that we call $c''_{0}$) be removed according to Procedure A. By Observation 3, $H$ has precisely one special vertex. Let $c_t$ be the special vertex of $H$. Let $w \in V(G')$ be a vertex of degree at least two of $G'$ that is adjacent to a weak support vertex $w'$, and let $w'$ and its leaf (that we call $w''$) be removed according to Procedure A.

First we show that $\{c_1, c_r\} \cap S \neq \emptyset$. Clearly, $S \cap \{c_{l-1}, c_t, c_{l+1}\} \neq \emptyset$, since $\deg_G(c_t) = 2$. Assume that $c_t \in S$. Since at least one of $c_{l-1}$ or $c_{l+1}$ is adjacent to a weak support vertex, by Observation 2, $\{c_{l-1}, c_t, c_{l+1}\} \cap S \neq \emptyset$. By applying Observation 2, we obtain that $\{c_1, c_r\} \cap S \neq \emptyset$, since any vertex of $\{c_1, \ldots, c_r\}\{c_t\}$ is adjacent to a weak support vertex of $G$. Thus assume that $c_t \notin S$. Then $\{c_{l-1}, c_t, c_{l+1}\} \cap S \neq \emptyset$, and so $\{c_1, c_r\} \cap S \neq \emptyset$, since any vertex of $\{c_1, \ldots, c_r\}\{c_t\}$ is adjacent to a weak support vertex of $G$. Hence, $\{c_1, c_r\} \cap S \neq \emptyset$. If $c_0 \notin S$, then $S \cup \{w', w''\}$ is a 1FTD-set for $G'$, and thus by the first-inductive hypothesis, $S'$ contains $w = c_0$, a contradiction. Thus $c_0 \in S$. By Observation 2, $V(C) \subseteq S$, since any vertex of $\{c_1, \ldots, c_r\} - \{c_t\}$ is adjacent to a weak support vertex of $G$. Thus $S \cap V(G')$ is a 1FTD-set for $G'$. By the first-inductive hypothesis, $(S \cap V(G')) \cup \{w', w''\}$ contains every vertex of $G'$ of degree at least two. Consequently, $S$ contains every vertex of $G$ of degree at least two. We conclude that the base step of the second-induction holds.
Assume that the result (for the second-induction) holds for $2 \leq l' < l$. Now let $G = G_l$. Clearly, $G$ is obtained from $G_{l-1}$ by applying one of the Operations $O_1$ or $O_2$.

Assume that $G$ is obtained from $G_{l-1}$ by applying Operation $O_2$. Let $x$ be a support vertex of $G_{l-1}$ and let $x'$ be a leaf adjacent to $x$. Let $G$ be obtained from $G_{l-1}$ by adding a vertex $u'$ and a path $P_2 = y_1y_2$, joining $x'$ to $u'$ and joining $x$ to $y_1$, according to Operation $O_2$. By Observation 1, $x', y_1 \in S$ and so $x \in S$. Thus $S \setminus \{y_1\}$ is a 1FTD-set for $G_{l-1}$. By the second-inductive hypothesis, $S$ contains all vertices of $G_{l-1}$ of degree at least two. Consequently, $S$ contains every vertex of $G_k$ of degree at least two.

Next assume that $G$ is obtained from $G_{l-1}$ by applying Operation $O_1$. Let $P_3 = x_1x_2x_3$ be a path and $x_1$ be joined to $y \in V(G_{l-1})$, where $\deg_{G_{l-1}}(y) \geq 2$ and $y$ is not a special vertex of $G_{l-1}$, according to Operation $O_2$. By Observation 1, $x_2 \in S$. Observe that $\{x_1, x_3\} \cap S \neq \emptyset$. If $x_1 \notin S$, then $x_3 \in S$ and $y \notin S$. Then $S \setminus \{x_2, x_3\}$ is a 1FTD-set for $G_{l-1}$ that does not contains $y$, a contradiction by the second-inductive hypothesis. Thus assume that $x_1 \in S$. Suppose that $y \notin S$. Clearly, $N_{G_{l-1}}(y) \cap S = \emptyset$. Assume that there exists a component $G'_1$ of $G_{l-1} - y$ such that $|V(G'_1) \cap N_{G_{l-1}}(y)| = 1$. Then clearly $S' = (S \cap V(G_{l-1})) \cup V(G'_1)$ is a 1FTD-set for $G_{l-1}$, and by the second-inductive hypothesis, $S'$ contains every vertex of $G_{l-1}$ of degree at least two. Thus $y \in S'$, and so $y \in S$, a contradiction. Next assume that every component of $G_{l-1} - y$ has at least two vertices in $N_{G_{l-1}}(y)$. Since $y$ is a non-special vertex of $G_{l-1}$, $y$ belongs to at least two cycles of $G_{l-1}$. By Observation 9(4), $y$ belongs to exactly two cycles of $G_{l-1}$. Thus $\deg_{G_{l-1}}(y) = 4$. By Observation 11, $G_{l-1} - y$ has exactly two components $D_1$ and $D_2$. Let $G^*$ be a graph obtained from $D_1 \cup \{y\}$ or $D_2 \cup \{y\}$ by adding a path $P_2 = y'y''$ to $y$. Then there exists $k' \leq k$ such that $G^* \in \mathcal{G}_{k'}$. Evidently, $S^* = (S \cap V(G^*)) \cup \{y', y''\}$ is a 1FTD-set for $G^*$, and so by the first-inductive hypothesis, $S^*$ contains every vertex of $G^*$ of degree at least two (since $G^* \in \mathcal{G}_{k'}$). Thus $y \in S^*$, and so $y \in S$, a contradiction. We conclude that $y \in S$. Observe that $S \cap V(G_{l-1})$ is a 1FTD-set for $G_{l-1}$, and so by the second-inductive hypothesis, $S \cap V(G_{l-1})$ contains every vertex of $G_{l-1}$ of degree at least two. Consequently, $S$ contains every vertex of $G$ of degree at least two. 

As a consequence of Observation 9(3) and Lemma 12, we obtain the following.

**Corollary 13.** If $G \in \mathcal{G}_k$ is a cactus graph of order $n$, then $V(G) \setminus L(G)$ is the unique ftd$_1(G)$-set.

In what follows, we present an upper bound for the 1-fair domination number of a cactus graph in terms of the order and the number of cycles.

**Theorem 14.** If $G$ is a cactus graph of order $n \geq 4$ with $k \geq 1$ cycles, then ftd$_1(G) \leq (2(n + k) - 1)/3$. 
Proof. The result follows by Theorem 6 if $k = 1$. Thus assume that $k \geq 2$. Suppose to the contrary that $ftd_1(G) > (2(n(G) + k) - 1)/3$. Assume that $G$ has the minimum order, and among all such graphs, we may assume that the size of $G$ is minimum. Let $C_1, C_2, \ldots, C_k$ be the $k$ cycles of $G$. Let $C_i$ be a leaf-cycle of $G$, where $i \in \{1, 2, \ldots, k\}$. Let $C_i = c_0c_1 \cdots c_{ri}$, where $c_0$ is the special cut-vertex of $G$. Suppose that $G$ has a strong support vertex $u$, and $u_1, u_2$ are leaves adjacent to $u$. Let $G_0 = G - u_1$. By the choice of $G$, $ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2/3$. Let $S'$ be an $ftd_1(G')$-set. By Observation 1, $u \in S'$. Clearly, $S'$ is a 1FTD-set in $G$ and so $ftd_1(G) \leq (2(n + k) - 1)/3 - 2/3$, a contradiction. We deduce that every support vertex of $G$ is adjacent to precisely one leaf.

Assume that $deg_G(v_j) = 2$ for each $j = 1, 2, \ldots, r$. Let $G' = G - c_2$. Then by the choice of $G$, $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 2$. Let $S'$ be an $ftd_1(G')$-set. By Observation 1, $c_0 \in S'$. If $|S' \cap \{c_1, c_3\}| = 1$, then $S'$ is a 1FTD-set for $G$ cardinality at most $(2(n + k) - 1)/3 - 2/3$, a contradiction. Thus assume that $|S' \cap \{c_1, c_3\}| = 2$. Then $\{c_2\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 1/3$, a contradiction. Thus assume that $|S' \cap \{c_1, c_3\}| = 0$. Now $\{c_1\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 1/3$, a contradiction. We deduce that $deg_G(c_i) \geq 3$ for some $i \in \{1, 2, \ldots, r\}$.

Let $v_d$ be a leaf of $G$ such that $d(v_d, C_i - c_0)$ is as maximum as possible, the shortest path from $v_d$ to $C_i$ does not contain $c_0$ and $deg(v_{d-1})$ is as maximum as possible, where $v_{d-1}$ is the neighbor of $v_d$ on the shortest path from $v_d$ to a vertex $v_0 \in C_i$.

Assume that $d \geq 3$. Observe that $deg_G(v_{d-1}) = 2$, since $G$ has no strong support vertex. Assume that $deg_G(v_{d-2}) = 2$. Let $G' = G - \{v_d, v_{d-1}, v_{d-2}\}$. By the choice of $G$, $ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$. Let $S'$ be an $ftd_1(G')$-set. If $v_{d-3} \in S'$, then $\{v_{d-1}, v_{d-2}\} \cup S'$ is a 1FTD-set in $G$ and so $ftd_1(G) \leq (2(n + k) - 1)/3$, a contradiction. If $v_{d-3} \notin S'$, then $\{v_{d-1}, v_d\} \cup S'$ is a 1FTD-set in $G$ and so $ftd_1(G) \leq (2(n + k) - 1)/3$, a contradiction. Thus assume that $deg_G(v_{d-2}) \geq 3$. Assume that $v_{d-2}$ is a support vertex. Let $G'' = G - \{v_{d-1}, v_d\}$. By the choice of $G$, $ftd_1(G'') \leq (2(n(G'') + k) - 1)/3 = (2(n + k) - 1)/3 - 4/3$. Let $S''$ be an $ftd_1(G'')$-set. By Observation 1, $v_{d-2} \in S''$. Then $\{v_{d-1}, v_d\} \cup S''$ is a 1FTD-set in $G$ and so $ftd_1(G) \leq (2(n + k) - 1)/3$, a contradiction. Thus assume that $v_{d-2}$ is not a support vertex of $G$. Let $x \neq v_{d-1}, v_{d-2}$ be a support vertex of $G$ such that $x \in N(v_{d-2})$. By the choice of the path $v_0v_1 \cdots v_d$, (the part “$deg(v_{d-1})$ is as maximum as possible”), $deg(x) = 2$. Let $y$ be the leaf adjacent to $x$ and $G' = G - \{v_d, v_{d-1}, y\}$. By the choice of $G$, $ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$. Let $S'$ be an $ftd_1(G')$-set. By Observation 1, $v_{d-2} \in S'$, since $v_{d-2}$ is a support vertex of $G'$. Thus $\{v_{d-1}, x\} \cup S'$ is a 1FTD-set in $G$ and so $ftd_1(G) \leq (2(n + k) - 1)/3$, a contradiction.
Next assume that $d = 2$. Assume that $\deg_G(c_i) = 2$ for some $i \in \{1, 2, \ldots, r\}$. Let $\deg_G(c_j) = 2$. Assume that $\deg_G(c_{j+1}) = 2$. Let $G' = G - c_j$. Then by the choice of $G$, $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$. Let $S'$ be an $ftd_1(G')$-set. By Observation 1, $c_{j+2} \in S'$. If $|S' \cap \{c_{j-1}, c_{j+1}\}| = 1$, then $S'$ is a 1FTD-set for $G$ of cardinality at most $(2(n + k) - 1)/3 - 4/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 2$. Then $\{c_j\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 4/3$, a contradiction.

Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 0$ and so $\{c_{j+1}\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 4/3$, a contradiction. Thus assume that $\deg_G(c_j) = 2$. Assume that $\deg_G(c_{j-1}) = 2$. By the choice of $G$, $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$. Let $S'$ be an $ftd_1(G')$-set. By Observation 1, $c_{j+1} \in S'$. If $c_{j-1} \notin S'$, then $S'$ is a 1FTD-set for $G$ of cardinality at most $(2(n + k) - 1)/3 - 4/3$, a contradiction. Thus assume that $c_{j-1} \notin S'$ and so $\{c_j\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 4/3$, a contradiction. Clearly, $c_{j+1}$ is not a support vertex of $G$. Let $c_{j+1}' \in N(c_{j+1}) \setminus V(c_1)$. Clearly, $c_{j+1}'$ is a support vertex, since $d = 2$. Observe that $\deg_G(c_{j+1}') = 2$, since $G$ has no strong support vertex. Let $c_{j+1}'$ be the leaf of $G_{j+1}$. Let $G' = G - c_j - c_{j+1}'$. By the choice of $G$, $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 2$. Let $S'$ be an $ftd_1(G')$-set. By Observation 1, $c_{j+1} \in S'$, since $c_{j+1}$ is a support vertex in $G'$. If $c_{j-1} \notin S'$, then $S' \cup \{c_{j+1}'\}$ is a 1FTD-set for $G$ of cardinality at most $(2(n + k) - 1)/3 - 1$, a contradiction. Thus assume that $c_{j-1} \in S'$. Then $\{c_j, c_{j+1}'\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3$, a contradiction. Thus $\deg(c_j) \geq 3$ for $1 \leq j \leq r$. Let $G^* = G - c_0c_1 \cdots c_r$. Let $G_1^*$ be the component of $G^*$ containing $c_r$, and $G_2^*$ be the component of $G^*$ containing $c_0$. Let $D = S(G_1^*) \setminus V(c_1)$. Clearly, $S' = D \cup \{c_1, c_2, \ldots, c_r\}$ is a 1FTD-set for $G_1^*$ of cardinality at most $2n(G_1^*)/3$. Let $G_3^* = G[V(G_2^*) \cup \{c_1\}]$. By the choice of $G$, $ftd_1(G_3^*) \leq (2(n(G_3^*) + k - 1) - 1)/3$. Let $S''$ be an $ftd_1(G_3^*)$-set. By Observation 1, $c_0 \in S''$. Clearly, $S'' \cup S''$ is a 1FTD-set for $G$ and so $ftd_1(G) \leq (2(n(G_3^*) + k - 1) - 1)/3 \leq 2n(G_1^*)/3 = (2(n + k) - 1)/3$, a contradiction.

Now assume that $d = 1$. Assume that $\deg_G(c_i) = 2$ for some $i \in \{1, 2, \ldots, r\}$. Let $\deg_G(c_j) = 2$. Assume that $\deg_G(c_{j+1}) = 2$. Let $G' = G - c_j$. By the choice of $G$, $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$. Let $S'$ be an $ftd_1(G')$-set. By Observation 1, $c_{j+2} \in S'$. If $|S' \cap \{c_{j-1}, c_{j+1}\}| = 1$, then $S'$ is a 1FTD-set for $G$ of cardinality at most $(2(n + k) - 1)/3 - 4/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 2$. Then $\{c_j\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 1/3$, a contradiction. Thus $\deg_G(c_{j+1}) \geq 3$. Similarly, $\deg_G(c_{j-1}) \geq 3$. Clearly, $c_{j+1} \neq c_0$ or $c_{j-1} \neq c_0$. Assume, without loss of general
ity, that $c_{j+1} \neq c_0$. Thus $c_{j+1}$ is a support vertex of $G$. Let $G' = G - c_j$. Then by the choice of $G$, $\text{ftd}_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$.

Let $S'$ be an $\text{ftd}_1(G')$-set. By Observation 1, $c_{j+1} \in S'$. If $c_{j-1} \notin S'$, then $S'$ is a 1FTD-set for $G$, a contradiction. Thus assume that $c_{j-1} \in S'$. Then $\{c_j\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 4/3$, a contradiction. We thus obtain that $\deg(c_i) \geq 3$ for $1 \leq i \leq r$. Let $G^* = G - c_0c_1 - c_0c_r$. Let $G^*_1$ be the component of $G^*$ containing $c_r$, and $G^*_2$ be the component of $G^*$ containing $c_0$. Clearly, $S' = \{c_1, c_2, \ldots, c_r\}$ is a 1FTD-set for $G^*_1$ of cardinality at most $n(G^*_1)/2$.

Let $G^*_3 = G[V(G^*_2) \cup \{c_1\}]$. By the choice of $G$, $\text{ftd}_1(G^*_3) \leq (2(n(G^*_2) + k - 1) - 1)/3$. Let $S''$ be an $\text{ftd}_1(G^*_3)$-set. By Observation 1, $c_0 \in S''$. Clearly, $S' \cup S''$ is a 1FTD-set for $G$ and so $\text{ftd}_1(G) \leq (2(n(G^*_3) + k - 1) - 1)/3 + n(G^*_1)/2 < (2(n + k) - 1)/3$. 

It is evident that for the cycle $C_7$ the equality of the bound given in Theorem 14 holds.

**Theorem 15.** If $G \neq C_7$ is a cactus graph of order $n \geq 5$ with $k \geq 1$ cycles, then $\text{ftd}_1(G) = (2(n + k) - 1)/3$ if and only if $G \in \mathcal{G}_k$.

**Proof.** We prove by an induction on $k$ to show that any cactus graph $G \neq C_7$ of order $n \geq 5$ with $k \geq 1$ cycles and $\text{ftd}_1(G) = (2(n + k) - 1)/3$ belongs to $\mathcal{G}_k$. The base step of the induction follows by Theorem 6. Assume the result holds for all cactus graphs $G' \neq C_7$ with $k' < k$ cycles. Now let $G \neq C_7$ be a cactus graph of order $n$ with $k \geq 2$ cycles and $\text{ftd}_1(G) = (2(n + k) - 1)/3$. Suppose to the contrary that $G \notin \mathcal{G}_k$. Assume that $G$ has the minimum order, and among all such graphs, assume that the size of $G$ is minimum.

**Claim 1.** Every support vertex of $G$ is weak support vertex.

**Proof.** Suppose that $G$ has a strong support vertex $u$, and assume that $u_1$ and $u_2$ are two leaves adjacent to $u$. Let $G' = G - u_1$, and $S'$ be an $\text{ftd}_1(G')$-set. By Observation 1, $u \in S'$. By Theorem 14, $|S'| \leq (2(n(G') + 2) - 1)/3 = (2(n + k) - 1)/3 - 2/3$. Clearly, $S'$ is a 1FTD-set for $G$ of cardinality at most $(2(n + k) - 1)/3 - 2/3$, a contradiction.

By Observation 8, $G$ has at least two leaf-cycles. Let $C_1 = c_0c_1 \cdots c_rc_0$ be a leaf-cycle of $G$, where $c_0$ is a special cut-vertex of $G$. Let $G_1'$ be the component of $G - c_0c_1 - c_0c_r$ containing $c_1$.

**Claim 2.** $V(G_1') \neq \{c_1, \ldots, c_r\}$.

**Proof.** Suppose that $V(G_1') = \{c_1, \ldots, c_r\}$. Then $\deg_G(c_i) = 2$, for each $i = 1, 2, \ldots, r$. Let $G' = G - c_2$. By Theorem 14, $\text{ftd}_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$. Let $S'$ be an $\text{ftd}_1(G')$-set. By Observation 1,
$c_0 \in S'$. If $|S' \cap \{c_1, c_3\}| = 1$, then $S'$ is a 1FTD-set for $G$ of cardinality at most $(2(n+k) - 1)/3 - 4/3$, a contradiction. Thus assume that $|S' \cap \{c_1, c_3\}| = 2$. Then $\{c_2\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n+k) - 1)/3 - 1/3$, a contradiction. Thus assume that $|S' \cap \{c_1, c_3\}| = 0$. Then $\{c_1\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n+k) - 1)/3 - 1/3$, a contradiction. 

Let $v_d \in V(G'_1) \setminus \{c_1, \ldots, c_r\}$ be a leaf of $G'_1$ at maximum distance from $\{c_1, \ldots, c_r\}$, and assume that $\deg(v_{d-1})$ is as maximum as possible, $\deg_G(v_0)$ is as maximum as possible, and $\deg_G(v_1)$ is as maximum as possible, where $v_0 \in \{c_1, \ldots, c_r\}$ and $v_0v_1 \cdots v_d$ is the shortest path from $v_d$ to $\{c_1, \ldots, c_r\}$.

Suppose that $d = 1$. Assume that $\deg_G(c_j) = 2$, for some $j \in \{1, 2, \ldots, r\}$. Assume that $\deg_G(c_{j+1}) = 2$. Let $G' = G - c_j$. By Theorem 14, $\text{ftd}_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n+k) - 1)/3 - 4/3$. Let $S'$ be an $\text{ftd}_1(G')$-set. By Observation 1, $c_{j+2} \in S'$. If $|S' \cap \{c_{j+1}, c_{j+2}\}| = 1$, then $S'$ is a 1FTD-set for $G$ of cardinality at most $(2(n+k) - 1)/3 - 4/3$, a contradiction. Thus assume that $|S' \cap \{c_{j+1}, c_{j+2}\}| = 2$. Then $\{c_j\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n+k) - 1)/3 - 1/3$, a contradiction. Thus assume that $|S' \cap \{c_{j+1}, c_{j+2}\}| = 0$. Then $\{c_{j+1}\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n+k) - 1)/3 - 1/3$, a contradiction. Thus $\deg_G(c_{j+1}) \geq 3$. Similarly, $\deg_G(c_{j-1}) \geq 3$. Clearly, $c_{j+1} \neq c_0$ or $c_{j-1} \neq c_0$. Assume, without loss of generality, that $c_{j+1} \neq c_0$. Then $c_{j+1}$ is a support vertex of $G$. Let $G' = G - c_j$. Then by Theorem 14, $\text{ftd}_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n+k) - 1)/3 - 4/3$. Let $S'$ be an $\text{ftd}_1(G')$-set. By Observation 1, $c_{j+1} \in S'$. If $c_{j-1} \notin S'$, then $S'$ is a 1FTD-set for $G$ of cardinality at most $(2(n+k) - 1)/3 - 4/3$, a contradiction. Thus assume that $c_{j-1} \in S'$. Then $\{c_j\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n+k) - 1)/3 - 1/3$, a contradiction. We thus obtain that $\deg(c_j) \geq 3$, for $1 \leq j \leq r$. Let $G^* = G - c_0c_1c_2c_r$. Let $G^*_1$ be the component of $G^*$ containing $c_r$, and $G^*_2$ be the component of $G^*$ containing $c_0$. Clearly, $S' = \{c_1, c_2, \ldots, c_r\}$ is a 1FTD-set for $G^*_1$ of cardinality at most $n(G^*_1)/2$. Let $G^*_3 = G[V(G^*_2) \cup \{c_1\}]$. By Theorem 14, $\text{ftd}_1(G^*_3) \leq (2(n(G^*_3) + k) - 1)/3$. Let $S''$ be an $\text{ftd}_1(G^*_3)$-set. By Observation 1, $c_0 \in S''$. Clearly, $S' \cup S''$ is a 1FTD-set for $G$ and so $\text{ftd}_1(G) \leq (2(n(G^*_3) + k - 1) - 1)/3 + n(G^*_1)/2 < (2(n+k) - 1)/3$, a contradiction.

Thus assume that $d \geq 2$.

**Claim 3.** If $d \geq 3$, then $G \in \mathcal{G}_k$.

**Proof.** Assume that $d \geq 3$. By Claim 1, $\deg_G(v_{d-1}) = 2$. Assume first that $\deg_G(v_{d-2}) \geq 3$. Then $v_{d-2}$ is a support vertex. Let $G' = G - \{v_{d-1}, v_d\}$. By Theorem 14, $\text{ftd}_1(G') \leq (2(n(G') + k - 1)/3 = (2(n+k) - 1)/3 - 4/3$. Let $S'$ be an $\text{ftd}_1(G')$-set. By Observation 1, $v_{d-2} \in S'$. Then $\{v_{d-1}\} \cup S'$ is a 1FTD-set in $G$, and so $\text{ftd}_1(G) \leq (2(n+k) - 1)/3 - 1/3$, a contradiction. Thus assume that $v_{d-2}$ is not a support vertex of $G$. Let $x \neq v_{d-1}, v_{d-3}$ be a support vertex of $G$ such that $x \in N(v_{d-2})$. By the choice of the path $v_0v_1 \cdots v_d$, (the part “$\deg(v_{d-1})$
is as maximum as possible"), \(\deg_G(x) = 2\). Let \(y\) be the leaf adjacent to \(x\), and \(G' = G - \{v_d, v_{d-1}, y\}\). By Theorem 14, \(\text{ftd}_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2\). Assume that \(\text{ftd}_1(G') < (2(n(G') + k) - 1)/3\). Let \(S'\) be an \(\text{ftd}_1(G')\)-set. By Observation 1, \(v_{d-2} \in S'\), since \(v_{d-2}\) is a support vertex of \(G'\). Then \(\{v_{d-1}, x\} \cup S'\) is a \(1\)FTD-set in \(G\) and so \(\text{ftd}_1(G) < (2(n + k) - 1)/3\), a contradiction. Thus \(\text{ftd}_1(G') = (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2\). By the choice of \(G\), \(G' \in \mathcal{G}_k\). Thus \(G\) is obtained from \(G'\) by Operation \(\mathcal{O}_2\), and so \(G \in \mathcal{G}_k\).

Assume that \(\deg_G(v_{d-2}) = 2\). We consider the following cases.

Case 1. \(d \geq 4\). Suppose that \(\deg_G(v_{d-3}) = 2\). Let \(G' = G - \{v_d, v_{d-1}, v_{d-2}, v_{d-3}\}\). By Theorem 14, \(\text{ftd}_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 8/3\). Let \(S'\) be an \(\text{ftd}_1(G')\)-set. If \(v_{d-4} \in S'\), then \(\{v_{d-1}, v_d\} \cup S'\) is a \(1\)FTD-set in \(G\) and so \(\text{ftd}_1(G) \leq (2(n + k) - 1)/3 - 2/3\), a contradiction. Thus \(v_{d-4} \notin S'\). Then \(\{v_{d-2}, v_{d-1}\} \cup S'\) is a \(1\)FTD-set in \(G\) and so \(\text{ftd}_1(G) \leq (2(n + k) - 1)/3 - 2/3\), a contradiction. We deduce that \(\deg_G(v_{d-3}) \geq 3\). Let \(G' = G - \{v_d, v_{d-1}, v_{d-2}\}\). By Theorem 14, \(\text{ftd}_1(G') \leq (2(n(G') + k) - 1)/3\). Assume that \(\text{ftd}_1(G') < (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2\). Let \(S'\) be an \(\text{ftd}_1(G')\)-set. If \(v_{d-3} \in S'\), then \(\{v_{d-1}, v_{d-2}\} \cup S'\) is a \(1\)FTD-set in \(G\) and so \(\text{ftd}_1(G) < (2(n + k) - 1)/3\), a contradiction. Thus \(v_{d-3} \notin S'\). Then \(\{v_{d-1}, v_d\} \cup S'\) is a \(1\)FTD-set in \(G\) and so \(\text{ftd}_1(G) < (2(n + k) - 1)/3\), a contradiction. We thus obtain that \(\text{ftd}_1(G') = (2(n(G') + k) - 1)/3\). By the choice of \(G\), \(G' \in \mathcal{G}_k\). Since \(d \geq 4\), \(v_{d-3}\) is not a special vertex of \(G'\). Thus \(G\) is obtained from \(G'\) by Operation \(\mathcal{O}_1\), and so \(G \in \mathcal{G}_k\).

Case 2. \(d = 3\). Clearly, \(\deg(v_0) \geq 3\). We show that \(\deg(v_0) \geq 4\). Suppose that \(\deg(v_0) = 3\). Let \(G' = G - \{v_1, v_2, v_3\}\). By Theorem 14, \(\text{ftd}_1(G') \leq (2(n(G') + k) - 1)/3\). Assume that \(\text{ftd}_1(G') = (2(n(G') + k) - 1)/3\). By the choice of \(G\), \(G' \in \mathcal{G}_k\). By Observation 9(1), \(v_0\) is the unique special vertex of \(G'\), since \(\deg_G(v_0) = 2\). We show that \(\deg_G(x) = 3\) for each \(x \in \{c_1, \ldots, c_r\} \setminus \{v_0\}\). Assume that \(\deg_G(c_j) \geq 4\) for some \(c_j \in \{c_1, \ldots, c_r\} \setminus \{v_0\}\). If there is a vertex \(w \in V(G) \setminus V(C_1)\) such that \(d(w, C_1) = d(w, c_j) = 3\), then \(w\) can play the same role of \(v_d\), and thus \(\deg(v_j) = 3\), a contradiction. Thus there is no vertex \(w \in V(G) \setminus V(C_1)\) such that \(d(w, C_1) = d(w, c_j) = 3\). Thus any vertex of \(N(u_j) \setminus V(C_1)\) is a leaf or a weak support vertex. Assume that \(N(c_j) \setminus V(C_1)\) contains \(t_1\) leaves and \(t_2\) support vertices, where \(t_1 + t_2 \geq 2\). By Observation 9(1), \(t_1 = 0\), since \(G' \in \mathcal{G}_k\). Thus \(t_2 \geq 2\). Let \(z_1\) and \(z_2\) be two weak support vertices in \(N(c_j) \setminus V(C_1)\). Let \(z'_1\) and \(z'_2\) be the leaves adjacent to \(z_1\) and \(z_2\), respectively. (We switch for a while to \(G'\)). Let \(G'' = G - \{z_1, z'_1, z'_2\}\). By Theorem 14, \(\text{ftd}_1(G'') \leq (2(n(G'') + k) - 1)/3\). Suppose that \(\text{ftd}_1(G'') = (2(n(G'') + k) - 1)/3\). By the choice of \(G\), \(G'' \in \mathcal{G}_k\). Clearly, \(\deg_G(c_j) \geq 3\), since \(v_0\) is the unique special vertex of \(G'\), a contradiction (by Observation 9(1)). Thus \(\text{ftd}_1(G'') < (2(n(G'') + k) - 1)/3 = (2(n + k) - 1)/3 - 2\). Let \(S''\) be a \(1\)FTD-set of \(G''\). By Observation 1, \(c_j \in S''\). Then \(S'' \cup \{z_1, z_2\}\) is a \(1\)FTD-set of \(G\). Thus \(\text{ftd}_1(G) < (2(n + k) - 1)/3\), a contradiction. We
deduce that $\deg_{G'}(c_i) = 3$ for each $c_i \in \{c_1, \ldots, c_r\} \setminus \{v_0\}$. Thus $\deg_{G'}(c_i) = 3$ for each $1 \leq i \leq r$. Note that by Observation 9(1), $c_i$ is not a support vertex, for each $i$ with $1 \leq i \leq r$ in $G'$, since $G' \in \mathcal{G}_k$. (We switch for a while to $G$).

Let $F = \bigcup_{i=1}^r (N[c_i]) \setminus \{c_0, \ldots, c_r\}$. Clearly, $|F| = r$, since $\deg_{G'}(c_i) = 3$ for each $c_i \in \{c_1, \ldots, c_r\} \setminus \{v_0\}$ and $\deg_{G'}(v_0) = 3$. Let $F' = \{u_1, u_2, \ldots, u_r\}$. Clearly $\deg_{G}(u_i) \geq 2$, for each $i$ with $1 \leq i \leq r$, since $c_i$ is not a support vertex for $1 \leq i \leq r$ in $G'$. By Claim 2, $u_i$ is not a strong support vertex of $G$, for $1 \leq i \leq r$. If $u_i$ is adjacent to a support vertex $u_i' \in V(G) \setminus V(C_i)$, for some integer $i$, then since the leaf of $u_i'$ can play the role of $v_3$, we obtain that $\deg(u_i) = 2$. Since $\deg_{G}(u_i) \geq 2$ for each $i$ with $1 \leq i \leq r$, we find that $\deg_G(u_i) = 2$ for each $i$ with $1 \leq i \leq r$.

Let $F' = \bigcup_{i=1}^r N(u_i) \setminus \{c_0, \ldots, c_r\}$. Clearly, $|F'| = r$, since $\deg_{G}(u_i) = 2$, for each $u_i \in \{u_1, \ldots, u_r\}$. Let $F'' = \{u'_1, u'_2, \ldots, u'_r\}$. By the choice of the path $v_0 v_1 \cdots v_d$, (the part “$\deg(v_{d-1})$ is as maximum as possible”), $\deg(u'_i) \leq 2$, for $1 \leq i \leq r$. Let $F'_1 = \{u'_1 \in F'' | \deg_G(u'_1) = 1\}$ and $F'_2 = F' - F'_1$. Then every vertex of $F'_2$ is a weak support vertex. Since $v_1 \in F'_2$, we have $|F'_2| \geq 1$. Let $G^* = G - c_0 c_1 - c_0 c_r$, and $G_1^*$ and $G_2^*$ be the components of $G^*$, where $c_1 \in V(G_1^*)$. By Theorem 14, $\text{ftd}_{d_1}(G_2^*) \leq (2(n(G_2^*) + k - 1) - 1)/3$. Clearly, $n(G_2^*) = n(G) - 3r - |F'_2|$. Let $S_2^*$ be an $\text{ftd}_{d_1}(G_2^*)$-set. If $c_0 \notin S_2^*$, then $S_2^* \cup F \cup F''$ is a $1\text{FTD}$-set for $G$. Thus $\text{ftd}_{d_1}(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + 2r = (2(n(G) - 3r - |F'_2| + k - 1) - 1)/3 + 2r$ and so $\text{ftd}_{d_1}(G) < (2(n + k) - 1)/3$, a contradiction. Thus $c_0 \in S_2^*$. If $|F'_2| = 1$, then $S_2^* \cup V(C_1) \cup F \cup \{v_2\}$ is a $1\text{FTD}$-set for $G$ and thus $\text{ftd}_{d_1}(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + 2r + 1 = (2(n(G) - 3r - |F'_2| + k - 1) - 1)/3 + 2r + 1 < (2(n + k) - 1)/3$, a contradiction. Thus assume that $|F'_2| \geq 2$. Let $\{u'_1, u'_r\} \subseteq F'_2$ (assume without loss of generality that $t < t'$) such that $\deg_{G}(u'_i) = 1$, for $1 \leq i \leq t$ and $t' < i \leq r$. Let $u''_i$ and $u''_{t'}$ be the leaves of $u_t$ and $u_{t'}$, respectively. Clearly, $S_2^* \subseteq \{c_1, \ldots, c_{t-1}\} \cup \{u_1, \ldots, u_{t-1}\} \cup \{c_{t+1}, \ldots, c_r\} \cup \{u_{t+1}, \ldots, u_r\} \cup \{u_{t+1}, \ldots, u_{t-1}\} \cup \{u'_{t+1}, \ldots, u'_{r}\} \cup \{u''_{t}, u''_{r}\}$ is a $1\text{FTD}$-set for $G$ and thus $\text{ftd}_{d_1}(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + 2r = (2(n(G) - 3r - |F'_2| + k - 1) - 1)/3 + 2r + 1 < (2(n + k) - 1)/3$, a contradiction. We deduce that $\text{ftd}_{d_1}(G') < (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$. Let $S'$ be an $\text{ftd}_{d_1}(G')$-set. If $v_0 \in S'$, then $S' \cup \{v_1, v_2\}$ is a $1\text{FTD}$-set in $G$, and so $\text{ftd}_{d_1}(G) < (2(n + k) - 1)/3$, a contradiction. Thus assume that $v_0 \notin S'$. Then $S' \cup \{v_2, v_3\}$ is a $1\text{FTD}$-set in $G$ and thus $\text{ftd}_{d_1}(G) < (2(n + k) - 1)/3$, a contradiction. Thus assume that $v_0 \notin S'$. Then $S = S' \cup \{v_2, v_3\}$ is a $1\text{FTD}$-set for $G$ and thus $\text{ftd}_{d_1}(G) < (2(n + k) - 1)/3$, a contradiction. Hence, $\text{ftd}_{d_1}(G') = (2(n(G') + k) - 1)/3$. By the inductive hypothesis, $G' \in \mathcal{G}_{k-1}$. Since $\deg(v_0) \geq 4$, $v_0$ is not a special vertex of $G'$. Thus $G$ is obtained from $G'$ by Operation $O_1$ and so $G \in \mathcal{G}_k$. □
By Claim 3, we assume that $d = 2$. We show that $\deg_G(v_0) = 3$. Suppose that $\deg_G(v_0) \geq 4$. Assume that $v_0$ is a support vertex. Let $G' = G - \{v_1, v_2\}$. By Theorem 14, $\text{ftd}_1(G') \leq (2(n(G') + k - 1) - 1)/3$. Let $S'$ be an $\text{ftd}_1(G')$-set. By Observation 1, $v_0 \in S'$. Then $S' \cup \{v_{d-1}\}$ is a 1FTD-set in $G$, and so $\text{ftd}_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus assume that $v_0$ is not a support vertex of $G$. Let $x \neq v_1$ be a support vertex of $G$ such that $x \in N(v_0) \setminus V(C_1)$. By the choice of the path $v_0v_1 \cdots v_d$, (the part "$\deg(v_{d-1})$ is as maximum as possible"), $\deg_G(x) = 2$. Let $y$ be the leaf adjacent to $x$. Let $G' = G - \{v_2, v_1, y\}$. By Theorem 14, $\text{ftd}_1(G') \leq (2(n(G') + k - 1)/3$. Let $\text{ftd}_1(G') < (2(n(G') + k - 1)/3$. Let $S'$ be an $\text{ftd}_1(G')$-set. By Observation 1, $v_0 \in S'$, since $v_0$ is a support vertex of $G$. Then $\{v_1, x\} \cup S'$ is a 1FTD-set in $G$ and so $\text{ftd}_1(G') < (2(n + k) - 1)/3$, a contradiction. Thus $\text{ftd}_1(G') = (2(n(G') + k - 1)/3$. By the inductive hypothesis, $G' \in G_k$, a contradiction by Observation 9(1), since $v_0$ is a support vertex of $G'$. Thus $\deg_G(v_0) = 3$. Observe that $G$ has no strong support vertex. If $c_i$ is adjacent to a support vertex $c_i'$ of $N(c_i) \setminus V(C_1)$ for some $i$, then the leaf of $c_i'$ can play the role of $v_2$, and thus $\deg(c_i) = 3$. Thus we may assume that $\deg_G(c_i) \leq 3$ for each $i$ with $i = 1, 2, \ldots, r$. Assume that $\deg_G(c_i) = 3$ for each $i$ with $1 \leq i \leq r$.

Let $F = \bigcup_{i=1}^{r} (N(c_i) \setminus \{c_0, \ldots, c_r\})$. Clearly, $|F| = r$, since $\deg_G(c_i) = 3$, for each $c_i \in \{c_1, \ldots, c_r\}$. Let $F = \{u_1, u_2, \ldots, u_r\}$. Clearly, $\deg_G(u_i) \leq 2$, for $1 \leq i \leq r$, since $G$ has no strong support vertex. Let $F' = \{u_i \mid \deg_G(u_i) = 2\}$. Clearly, $v_1 \in F'$. Let $F''$ be the set of leaves of $F'$. Clearly, $v_2 \in F''$. Let $G^* = G - c_0c_1 - c_0c_r$. Let $G_1^*$ be the component of $G^*$ containing $c_r$ and $G_2^*$ be the component of $G^*$ containing $c_0$. Assume that $F = F'$. Thus $n(G_1^*) = 3r$, since $d = 2$. Further, $n(G_2^*) = n - 3r$. By Theorem 14, $\text{ftd}_1(G_2^*) \leq (2(n(G_2^*) + k - 1) - 1)/3$. Let $S''$ be an $\text{ftd}_1(G_2^*)$-set. If $c_0 \in S''$, then $S'' \cup V(C_1) \cup F$ is a 1FTD-set for $G$ and so $\text{ftd}_1(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + 2r = (2(n + k - 1) - 1)/3 + 2r = (2(n + k - 1) - 1)/3$, a contradiction. Thus $c_0 \in S''$. Then $S'' \cup F \cup F'$ is a 1FTD-set for $G$ and so $\text{ftd}_1(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + 2r = (2(n + k - 1) - 1)/3 + 2r = (2(n + k - 1) - 1)/3$, a contradiction. We conclude that $F \neq F'$. Let $|F'| = r'$. Clearly, $1 \leq r' < r$, since $v_1 \in F'$. Thus $n(G_1^*) = 2r + r'$. Then $n(G_2^*) = n - (2r + r')$. Let $G_3^* = G[V(G_2^*) \cup \{c_1\}]$. Then $n(G_3^*) = n - (2r + r') + 1$. By Theorem 14, $\text{ftd}_1(G_3^*) \leq (2(n(G_3^*) + k - 1) - 1)/3$. Let $S''$ be an $\text{ftd}_1(G_3^*)$-set. By Observation 1, $c_0 \in S''$ and so $S'' \cup V(C_1) \cup F'$ is a 1FTD-set for $G$. Thus $\text{ftd}_1(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + r + r' = (2(n - (2r + r') + 1 + k - 1) - 1)/3 + r + r' = (2(n + k - 1) - 1)/3 + r + r' = (2(n + k) - 1)/3 - 4/3$. Let $S'$ be an $\text{ftd}_1(G')$-set. By Observation 1, $c_j \in S'$. If

Claim 4. No vertex of $C_1 - c_0$ is a support vertex.

Proof. Let $c_j$ be a support vertex of $G$. Assume that $c_{j+1}$ is a special vertex. Let $G' = G - c_{j+1}$. Then by Theorem 14, $\text{ftd}_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$. Let $S'$ be an $\text{ftd}_1(G')$-set. By Observation 1, $c_j \in S'$. If
$c_{j+2} \notin S'$, then $S'$ is a 1FTD-set for $G$ of cardinality at most $(2(n+k)-1)/3-4/3$ and so $ftd_1(G) < (2(n+k)-1)/3$, a contradiction. Thus $c_{j+2} \in S'$. Then $\{c_{j+1}\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n+k)-1)/3-1/3$ and so $ftd_1(G) < (2(n+k)-1)/3$, a contradiction. Thus $\deg_G(c_{j+1}) \neq 2$. Note that $c_i$ is a special vertex of $G$. Assume without loss of generality that $j < t$. Let $c_j$ be a support vertex of $G$ and $c_t$ be a special vertex of $G$, where $j < j' < t' \leq t$, and among such vertices choose $c_j$ and $c_t$ such that $c_i$ is neither a support vertex nor a special vertex of $G$ for each $i$ with $j' < i < t'$. Let $u_i \in N(c_i) \setminus V(C_1)$ for $j' < i < t'$. Clearly, $\deg_G(u_i) = 2$ for $j' < i < t'$, since $G$ has no strong support vertex. Let $G^* = G - c_jc_{j+1} - c_tc_{t+1}$. Let $G^*_1$ be the component of $G^*$ containing $c_j$ and $G^*_2$ be the component of $G^*$ containing $c_t$. Clearly, $n(G^*_2) = 3(t' - j' - 1) + 1$. Thus $n(G^*_1) = n - (3(t' - j' - 1) + 1)$.

By Theorem 14, $ftd_1(G^*_1) \leq (2(n(G^*_1)+k-1)-1)/3$. Let $S'$ be an $ftd_1(G^*_1)$-set. By Observation 1, $c_j \in S'$. Assume that $c_{j+1} \notin S'$. Then $S' \cup \{c_{j+1}, c_{j+2}, \ldots, c_{t-1}\} \cup \{u_{j+1}, u_{j+2}, \ldots, u_{t-1}\}$ is a 1FTD-set in $G$ of cardinality at most $(2(n(G^*_1)+k-1)-1)/3 + 2(t' - j' - 1) = (2(n-3(t' - j' - 1)+1)+k-1)-1)/3 + 2(t' - j' - 1) = (2(n+k)-1)/3 - 4/3$ and so $ftd_1(G) < (2(n+k)-1)/3$, a contradiction. Thus $c_{j+1} \in S'$. Then $S' \cup \{c_{j+1}, c_{j+2}, \ldots, c_{t-1}\}$ is a 1FTD-set in $G$ of cardinality at most $(2(n(G^*_1)+k-1)-1)/3 + 2(t' - j' - 1) + 1 = (2(n-3(t' - j' - 1)+1)+k-1)-1)/3 + 2(t' - j' - 1) = (2(n+k)-1)/3 - 1/3$ and so $ftd_1(G) < (2(n+k)-1)/3$, a contradiction. □

Claim 5. If $\deg_G(c_j) = 2$ for some $j$ with $1 \leq j \leq r$, then $\deg_G(c_{j+1}) = 3$ and $\deg_G(c_{j-1}) = 3$.

Proof. Assume that $\deg_G(c_j) = \deg_G(c_{j+1}) = 2$, for some $j$ with $1 \leq j \leq r$, and among such vertices choose $c_j$ such that $\deg_G(c_{j-1}) = 3$. Let $G' = G - c_j$. Then by Theorem 14, $ftd_1(G') \leq (2(n(G^*_1)+k-1)-1)/3 = (2(n+k)-1)/3 - 4/3$. Let $S'$ be an $ftd_1(G')$-set. By Observation 1, $c_{j+2} \in S'$. If $|S' \cap \{c_{j-1}, c_{j+1}\}| = 1$, then $S'$ is a 1FTD-set for $G$ of cardinality at most $(2(n+k)-1)/3 - 4/3$ and so $ftd_1(G) < (2(n+k)-1)/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 2$. Then $\{c_j\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n+k)-1)/3 - 1/3$ and so $ftd_1(G) < (2(n+k)-1)/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 0$. Then $\{c_{j+1}\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n+k)-1)/3 - 1/3$ and so $ftd_1(G) < (2(n+k)-1)/3$, a contradiction. Thus $\deg_G(c_{j+1}) \geq 3$. Similarly $\deg_G(c_{j-1}) \geq 3$. □

Claim 6. $C_1$ has precisely one special vertex.

Proof. Let $c_{t_1}$ and $c_{t_2}$ be two special vertices of $C_1$ and among such vertices choose $c_{t_1}$ and $c_{t_2}$ such that $c_i$ is not a special vertex of $C_1$ for $t_1 < i < t_2$. By Claim 5, $t_1 + 1 < t_2$. By Claim 4, $c_i$ is not a support vertex for $t_1 < i < t_2$. Let $u_i \in N(c_i) \setminus V(C_1)$, for $t_1 < i < t_2$. Clearly, $\deg_G(u_i) = 2$, for $t_1 < i < t_2$. Let
Let $v'_1$ be the leaf adjacent to $v_i$, for $t_1 < i < t_2$, and $G^* = G - c_{t_1}c_{t_1+1} - c_{t_2}c_{t_2+1}$. Let $G'_1$ be the component of $G^*$ containing $c_{t_1}$, and $G'_2$ be the component of $G^*$ containing $c_{t_2}$. Clearly, $n(G^*_2) = 3(t_2 - t_1) + 1$. Then $n(G''_1) = n - (3(t_2 - t_1) + 1)$. By Theorem 14, $ftd(G'_1) \leq (2(n(G''_1) + k - 1) - 1)/3$. Let $S'$ be an $ftd(G'_1)$-set. By Observation 1, $c_{t_1-1} \in S'$. Assume that $\{c_{t_1}, c_{t_2+1}\} \cap S' = \emptyset$. Then $S' \cup \{c_{t_1}, c_{t_1+1}, \ldots, c_{t_2-1}\}$ is a 1FTD-set in $G$ of cardinality at most $(2(n(G''_1) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) + (3(t_2 - t_1) + 1)(k - 1)/3 + 2(t_2 - t_1 - 1) + 1 = (2(n + k) - 1)3/3 - 3/3$ and so $ftd(G) < (2(n + k) - 1)/3$, a contradiction.

Thus $\{c_{t_1}, c_{t_2+1}\} \cap S' \neq \emptyset$. If $\{c_{t_1}, c_{t_2+1}\} \subseteq S'$, then $S' \cup \{c_{t_1+1}, c_{t_1+2}, \ldots, c_{t_2}\} \cup \{u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_2-1}\}$ is a 1FTD-set in $G$ of cardinality at most $2(n(G''_1) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) + 1 = (2(n - (3(t_2 - t_1 - 1) + 1)(k - 1) - 1)/3 + 2(t_2 - t_1 - 1) + 1 = (2(n + k) - 1)3/3 - 3/3$. Then $ftd(G) < (2(n + k) - 1)/3$, a contradiction. Thus assume that $c_{t_2+1} \in S'$ and $c_{t_1} \notin S'$. Then $S' \cup \{u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_2-1}\} \cup \{u'_1, u'_2, \ldots, u'_{t_2-1}\}$ is a 1FTD-set in $G$ of cardinality at most $2(n(G''_1) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) = (2(n - (3(t_2 - t_1 - 1) + 1)(k - 1) - 1)/3 + 2(t_2 - t_1 - 1) = (2(n + k) - 1)3/3 - 4/3$ and so $ftd(G) < (2(n + k) - 1)/3$, a contradiction.

By Claims 4 and 6, $c_i$ is not a support vertex and is not a special vertex, for $i \in \{1, 2, \ldots, t - 1, t + 1, \ldots, r\}$. Let $u_i \in N(c_i) \setminus V(C_1)$, for $i \in \{1, 2, \ldots, t - 1, t + 1, \ldots, r\}$. Clearly, $\deg_{G''}(u_i) = 2$, for $i \in \{1, 2, \ldots, t - 1, t + 1, \ldots, r\}$.

Let $G''_1$ be the component of $G - c_0c_1 - c_0c_r$ that contains $c_1$, $G''_2$ be the component of $G - c_0c_1 - c_0c_r$ that contains $c_0$, and $G^*$ be a graph obtained from $G''_2$ by adding a path $p_2 = x_1x_2$ and joining $c_0$ to $x_1$. Clearly, $n(G^*) = n - (3r - 2) + 2$. By Theorem 14, $ftd(G^*) \leq (2(n(G''_2) + k - 1) - 1)/3$. Suppose that $ftd(G^*) < (2(n(G''_2) + k - 1) - 1)/3$. Let $S^*$ be an $ftd(G^*)$-set. By Observation 1, $x_1 \in S^*$. If $c_0 \in S^*$, then $S^* \cup \{c_1, c_2, \ldots, c_r\} \cup \{u_1, u_2, \ldots, u_{t_1-1}, u_{t_1}, u_{t_1+1}, \ldots, u_r\}$ is a 1FTD-set in $G$. Thus $ftd(G) < (2(n(G''_2) + k - 1) - 1)/3 + 2r - 1 - 1 = (2(n - (3r - 2) + 2 + k - 1) - 1)/3 + 2r - 2 = (2(n + k) - 1)/3$, a contradiction. Thus $c_0 \notin S^*$. Then $\{x_1, x_2\} \cup \{c_1, \ldots, c_{t_1-1}\} \cup \{u_1, \ldots, u_{t_1-1}\} \cup \{u'_{t_1+1}, \ldots, u_r\}$ is a 1FTD-set in $G$. Thus $ftd_1(G) < (2(n(G''_2) + k - 1) - 1)/3 + 2(r - 1) - 2 = (2(n - (3r - 2) + 2 + k - 1) - 1)/3 + 2r - 4 = (2(n + k) - 1)/3 - 2$, a contradiction. Thus assume that $t = 1$. Then $S^* \setminus \{x_1, x_2\} \cup \{c_2, \ldots, c_r\} \cup \{u_2, \ldots, u_r\}$, of a 1FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 2$ and so $ftd_1(G) < (2(n + k) - 1)/3 - 2$, a contradiction. Thus $ftd_1(G^*) < (2(n(G''_2) + k - 1) - 1)/3$. By the inductive hypothesis, $G^* \in \mathcal{G}_{k-1}$. Let $G'_1$ be the graph obtained from $G'G''_1 \cup \{c_0\}$ by adding a path $p_2 = x'_1x'_2$.
and joining $c_0$ to $x'_1$. Clearly, $G^*_1 \in H_1$. Thus $G$ is obtained from $G^* \in G_{k-1}$ and $G^*_1 \in H_1$ by Procedure A. Consequently, $G \in H_k \subseteq G_k$.

For the converse, by Corollary 13, $V(G) \setminus L(G)$ is the unique $ftd_1(G)$-set. Now Observation 9 implies that $ftd_1(G) = (2(n + k) - 1)/3$.

Acknowledgements

We would like to thank the referee(s) for many helpful comments.

References


