FAIR TOTAL DOMINATION NUMBER
IN CACTUS GRAPHS

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Abstract

For $k \geq 1$, a $k$-fair total dominating set (or just kFTD-set) in a graph $G$ is a total dominating set $S$ such that $|N(v) \cap S| = k$ for every vertex $v \in V \setminus S$. The $k$-fair total domination number of $G$, denoted by $ftd_k(G)$, is the minimum cardinality of a kFTD-set. A fair total dominating set, abbreviated FTD-set, is a kFTD-set for some integer $k \geq 1$. The fair total domination number of a nonempty graph $G$, denoted by $ftd(G)$, of $G$ is the minimum cardinality of an FTD-set in $G$. In this paper, we present upper bounds for the 1-fair total domination number of cactus graphs, and characterize cactus graphs achieving equality for the upper bounds.

Keywords: fair total domination, cactus graph.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

For notation and graph theory terminology not given here, we follow [12]. Specifically, let $G$ be a graph with vertex set $V(G) = V$ of order $|V| = n$ and let $v$ be a vertex in $V$. The open neighborhood of $v$ is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and
the closed neighborhood of $v$ is $N_G[v] = \{v\} \cup N_G(v)$. If the graph $G$ is clear from the context, we simply write $N(v)$ rather than $N_G(v)$. The degree of a vertex $v$, is $\deg(v) = |N(v)|$. A vertex of degree one is called a leaf and its neighbor a support vertex. We denote the set of leaves and support vertices of a graph $G$ by $L(G)$ and $S(G)$, respectively. A strong support vertex is a support vertex adjacent to at least two leaves, and a weak support vertex is a support vertex adjacent to precisely one leaf. For a set $S \subseteq V$, its open neighborhood is the set $N(S) = \bigcup_{v \in S} N(v)$, and its closed neighborhood is the set $N[S] = N(S) \cup S$. The 2-corona 2-corona $G \times (G, E)$ of a graph $G$ is a graph obtained from $G$ by adding a path $P_2$ for every vertex $v$ and joining $v$ to a leaf of $P_2$. Note that 2-corona $G \times (G, E)$ has order $3|V(G)|$. The distance $d(u, v)$ between two vertices $u$ and $v$ in a graph $G$ is the minimum number of edges of a path from $u$ to $v$. For a subset $S$ of vertices of a graph $G$, we denote by $G[S]$ the subgraph of $G$ induced by $S$. A cactus graph is a graph such that no pair of cycles have a common edge.

A subset $S \subseteq V$ is a dominating set of $G$ if every vertex not in $S$ is adjacent to a vertex in $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. A dominating set $S$ in a graph $G$ with no isolated vertex, is a total dominating set of $G$ if every vertex in $S$ is adjacent to a vertex in $S$.

Caro et al. [1] studied the concept of fair domination in graphs. For $k \geq 1$, a k-fair dominating set, abbreviated kFD-set, in $G$ is a dominating set $S$ such that $|N(v) \cap D| = k$ for every vertex $v \in V \setminus D$. The k-fair domination number of $G$, denoted by $fd_k(G)$, is the minimum cardinality of a kFD-set. A kFD-set of $G$ of cardinality $fd_k(G)$ is called a $fd_k(G)$-set. A fair dominating set, abbreviated FD-set, in $G$ is a kFD-set for some integer $k \geq 1$. The fair domination number, denoted by $fd(G)$, of a graph $G$ that is not the empty graph is the minimum cardinality of an FD-set in $G$. An FD-set of $G$ of cardinality $fd(G)$ is called a $fd(G)$-set. A perfect dominating set in a graph $G$ is a dominating set $S$ such that every vertex in $V(G) \setminus S$ is adjacent to exactly one vertex in $S$. Hence a 1FD-set is precisely a perfect dominating set. The concept of perfect domination was introduced by Cockayne et al. in [4], and Fellows et al. [7] with a different terminology which they called semiperfect domination. This concept was further studied in, for example, [2, 3, 5, 6, 8, 9, 11].

Maravilla et al. [13] introduced the concept of fair total domination in graphs. For an integer $k \geq 1$ and a graph $G$ with no isolated vertex, a k-fair total dominating set, abbreviated kFTD-set, is a total dominating set $S \subseteq V(G)$ such that $|N(u) \cap S| = k$ for every $u \in V(G) \setminus S$. The k-fair total domination number of $G$, denoted by $ftd_k(G)$, is the minimum cardinality of a kFTD-set. A kFTD-set of $G$ of cardinality $ftd_k(G)$ is called an $ftd_k(G)$-set. A fair total dominating set, abbreviated FTD-set, in $G$ is a kFTD-set for some integer $k \geq 1$. Thus, a fair total dominating set $S$ of a graph $G$ is a total dominating set $S$ of $G$ such that
for every two distinct vertices $u$ and $v$ of $V(G) \setminus S$, $|N(u) \in S| = |N(v) \cap S|$; that is, $S$ is both a fair dominating set and a total dominating set of $G$. The fair total domination number of $G$, denoted by $ftd(G)$, is the minimum cardinality of an FTD-set. A fair total dominating set of cardinality $ftd(G)$ is called a minimum fair total dominating set or an $ftd$-set of $G$.

In [10], Volkmann and we studied fair total domination in trees and unicyclic graphs. In this paper, we study 1-fair total domination in cactus graphs. We present upper bounds for the 1-fair total domination number of cactus graphs, and characterize cactus graphs achieving equality for the upper bounds. The techniques used in this paper are similar to those presented in [9]. The following observations are easily verified.

Observation 1. Any support vertex in a graph $G$ with no isolated vertex belongs to every $k$FTD-set for each integer $k$.

Observation 2. Let $S$ be a 1FTD-set in a graph $G$, and $v$ be a vertex of degree at least two such that $v$ is adjacent to a weak support vertex $v'$. If $S$ contains a vertex $u \in N_G(v) \setminus \{v'\}$, then $v \in S$.

2. Unicyclic Graphs

A vertex $v$ of a graph is a special vertex if $\deg_G(v) = 2$ and $v$ belongs to a cycle of $G$. Let $\mathcal{H}_1$ be the class of all graphs $G$ that can be obtained from the 2-corona 2-cor($C$) of a cycle $C$ by removing precisely one support vertex $v$ and the leaf adjacent to $v$. Let $\mathcal{G}_1$ be the class of all graphs $G$ that can be obtained from a sequence $G_1, G_2, \ldots, G_s = G$, where $G_1 \in \mathcal{H}_1$, and if $s \geq 2$, then $G_{j+1}$ is obtained from $G_j$ by one of the following Operations $O_1$ or $O_2$, for $j = 1, 2, \ldots, s - 1$.

Operation $O_1$. Let $v$ be a vertex of $G_j$ with $\deg(v) \geq 2$ such that $v$ is not a special vertex. Then $G_{j+1}$ is obtained from $G_j$ by adding a path $P_3$ and joining $v$ to a leaf of $P_3$ by means of an edge.

Operation $O_2$. Let $v$ be a support vertex of $G_j$ and let $u$ be a leaf adjacent to $v$. Then $G_{j+1}$ is obtained from $G_j$ by adding a vertex $v'$ and a path $P_2$, and joining $u$ to $u'$ and $v$ to a leaf of $P_2$.

Observation 3. If $H \in \mathcal{H}_1$, then $H$ has precisely one special vertex.

Observation 4 [10]. If $G \in \mathcal{G}_1$ has order $n$, and $C$ is the cycle of $G$, then we have the following.

1. $G$ has precisely one special vertex.
2. $G$ has $(n - 1)/3$ leaves.
3. No vertex of $C$ is a support vertex.
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(4) Any vertex of C is adjacent to at most one weak support vertex of degree two.

Lemma 5 [10]. If $G \in G_1$, then every $1$FTD-set in $G$ contains every vertex of $G$ of degree at least two.

Theorem 6 [10]. If $G$ is a unicyclic graph of order $n$, then $\text{ftd}_1(G) \leq (2n+1)/3$, with equality if and only if $G = C_7$ or $G \in G_1$. 

3. Main Result

Our aim in this paper is to give an upper bound for the fair total domination number of a cactus graph $G$ in terms of the number of cycles of $G$, and then characterize all cactus graphs achieving equality for the proposed bound. For this purpose we first introduce some families of graphs. Let $\mathcal{H}_i$ and $\mathcal{G}_i$ be the families of unicyclic graphs described in Section 2. For $i = 2, \ldots, k$, we construct a family $\mathcal{H}_i$ from $\mathcal{G}_{i-1}$, and a family $\mathcal{G}_i$ from $\mathcal{H}_i$ as follows.

- Family $\mathcal{H}_i$. Let $\mathcal{H}_i$ be the family of all graphs $H_i$ such that $H_i$ can be obtained from a graph $H_1 \in \mathcal{H}_1$ and a graph $G \in \mathcal{G}_{i-1}$, by the following procedure.

Procedure A. Let $w_0 \in V(H_1)$ be a vertex of degree at least two of $H_1$ such that $w_0$ is adjacent to a weak support vertex $w'_0$, and $w \in V(G_{i-1})$ be a vertex of degree at least two of $G_{i-1}$ such that $w$ is adjacent to a weak support vertex $w'$ of degree two. We remove $w'_0$, the leaf adjacent to $w'_0$, $w'$ and the leaf adjacent to $w'$, and then identify the vertices $w_0$ and $w$.

- Family $\mathcal{G}_i$. Let $\mathcal{G}_i$ be the family of all graphs $G$ that can be obtained from a sequence $G_1, G_2, \ldots, G_s = G$, where $G_1 \in \mathcal{H}_i$, and if $s \geq 2$, then $G_{j+1}$ is obtained from $G_j$ by one of the Operations $\mathcal{O}_1$ or $\mathcal{O}_2$, described in Section 2, for $j = 1, 2, \ldots, s-1$.

Note that $\mathcal{H}_i \subseteq \mathcal{G}_i$, for $i = 1, 2, \ldots, k$. Figure 1 demonstrates the construction of the family $\mathcal{G}_k$.

![Figure 1. Construction of the family $\mathcal{G}_k$.](image)

We will prove the following.
Theorem 7. If $G$ is a cactus graph of order $n \geq 5$ with $k \geq 1$ cycles, then $\text{ftd}_1(G) \leq (2(n + k) - 1)/3$, with equality if and only if $G = C_7$ or $G \in G_8$.

4. Preliminary Results and Observations

4.1. Notation

We call a vertex $w$ in a cycle $C$ of a cactus graph $G$ a special cut-vertex if $w$ belongs to a shortest path from $C$ to a cycle $C' \neq C$. We call a cycle $C$ in $G$, a leaf-cycle if $C$ contains exactly one special cut-vertex. In the cactus graph presented in Figure 2, $v_i$ is a special cut-vertex, for $i = 1, 2, \ldots, 8$. Moreover, $C_j$ is a leaf-cycle for $j = 1, 2, 3$.

Figure 2. $C_i$ is a leaf-cycle for $i = 1, 2, 3$ and $v_j$ is a special cut-vertex for $j = 1, 2, \ldots, 8$.

Observation 8. Every cactus graph with at least two cycles contains at least two leaf-cycles.

4.2. Properties of the family $G_k$

The following observation can be proved by a simple induction on $k$.

Observation 9. If $G \in G_k$ is a cactus graph of order $n$, then we have the following.

1. No cycle of $G$ contains a support vertex. Furthermore, any cycle of $G$ contains precisely one special vertex.

2. If a vertex $v$ of $G$ belongs to a cycle of $G$, then $v$ is adjacent to at most one weak support vertex of degree two.

3. $|L(G)| = (n + 1)/3 - 2k/3$.

4. If a vertex $v$ of $G$ belongs to at least two cycles of $G$, then $v$ is not adjacent to a weak support vertex, and $v$ belongs to precisely two cycles of $G$. 
**Proof.** Let $G \in G_k$ be a cactus graph of order $n$. To show (1), (2) or (3), we prove by an induction on $k$, that we call first-induction. For the base step, if $k = 1$, then $G \in G_1$, and the result follows by Observation 4. Assume the result holds for all graphs $G' \in G_{k'}$ with $k' < k$. Now consider the graph $G \in G_k$, where $k > 1$. Clearly, $G$ is obtained from a sequence $G_1, G_2, \ldots, G_l = G$, of cactus graphs such that $G_1 \in \mathcal{H}_k$, and if $l \geq 2$, then $G_{l+1}$ is obtained from $G_i$ by one of the Operations $O_1$ or $O_2$ for $i = 1, 2, \ldots, l - 1$. We prove by an induction on $l$, that we call second-induction. For the base step of the second-induction, let $l = 1$. Thus $G \in \mathcal{H}_k$. By the construction of graphs in the family $\mathcal{H}_k$, there are graphs $H \in \mathcal{H}_1$ and $G' \in G_{k-1}$ such that $G$ is obtained from $H$ and $G'$ by Procedure $A$. It is easy to see that the base step of the second-induction holds. Assume that the result (for the second-induction) holds for $2 \leq l' < l$. Now let $G = G_l$. Clearly, $G$ is obtained from $G_{l-1}$ by applying one of the Operations $O_1$ or $O_2$. It is easy to see that the result holds.

The proof for (4) is similarly verified. 

**Observation 10.** Let $G \in G_k$ be obtained from a sequence $G_1, G_2, \ldots, G_s = G$ ($s \geq 2$) such that $G_1 \in \mathcal{H}_1$ and $G_{j+1}$ is obtained from $G_j$ by one of the Operations $O_1$ or $O_2$ or Procedure $A$, for $j = 1, 2, \ldots, s - 1$. If $v$ is a vertex of $G$ belonging to two cycles of $G$, then there is an integer $i \in \{2, 3, \ldots, s\}$ such that $G_i$ is obtained from $G_{i-1}$ by applying Procedure $A$ on the vertex $v$ using a graph $H \in \mathcal{H}_1$, such that $v$ belongs to a cycle of $G_{i-1}$.

**Observation 11.** Assume that $G \in G_k$ and $v \in V(G)$ is a vertex of degree four belonging to two cycles. Let $D_1$ and $D_2$ be the components of $G - v$, $G_1^v$ be the graph obtained from $G[D_1 \cup \{v\}]$ by joining $v$ to a leaf of a path $P_2$, and $G_2^v$ be the graph obtained from $G[D_2 \cup \{v\}]$ by joining $v$ to a leaf of a path $P_2$. Then there exists an integer $k' < k$ such that $G_1^v \in G_{k'}$ or $G_2^v \in G_{k'}$.

**Proof.** Let $G \in G_k$. Then $G$ is obtained from a sequence $G_1, G_2, \ldots, G_s = G$ ($s \geq 2$) such that $G_1 \in \mathcal{H}_1$ and $G_{j+1}$ is obtained from $G_j$ by one of the Operations $O_1$ or $O_2$ or Procedure $A$, for $j = 1, 2, \ldots, s - 1$. Note that $s \geq k$. We define the $j$-th Procedure-Operation or just $PO_j$ as one of the Operation $O_1$, Operation $O_2$, or Procedure $A$ that can be applied to obtain $G_{j+1}$ from $G_j$. Thus $G$ is obtained from $G_1$ by Procedure-Operations $PO_1, PO_2, \ldots, PO_{s-1}$.

Let $v$ be a vertex of $G$ of degree four belonging to two cycles of $G$, and $D_1$ and $D_2$ be the components of $G - v$. By Observation 10, there is an integer $i \in \{2, 3, \ldots, s\}$ such that $G_i$ is obtained from $G_{i-1}$ by applying Procedure $A$ on the vertex $v$ using a graph $H \in \mathcal{H}_1$. Note that $v$ is adjacent to a weak support vertex $v'$ of $G_{i-1}$. Let $v''$ be the leaf of $v'$ in $G_{i-1}$ that is removed in Procedure $A$. Clearly, either $V(G_{i-1}) \cap D_1 \neq \emptyset$ or $V(G_{i-1}) \cap D_2 \neq \emptyset$. Without loss of generality, assume that $V(G_{i-1}) \cap D_1 \neq \emptyset$. Among $PO_1, PO_{i+1}, \ldots, PO_{s-1}$, let $PO_{r_1}, PO_{r_2}, \ldots, PO_{r_t}$, be those procedure-operations applied on a vertex of $D_1$. 

Note that $i \leq t \leq s - 1$. Let $G_{r_0} = G_{i-1}$ and $G_{r_l+1}$ be obtained from $G_{r_l}$ by $PO_{l+1}$, for $l = 0, 1, 2, \ldots, t-1$. Clearly, by an induction on $t$, we can deduce that there is an integer $k^* < k$ such that $G_{r_l} \in \mathcal{G}_{k^*}$. Note that $G_{r_l} = G^*_r$.

**Lemma 12.** If $G \in \mathcal{G}_k$, then every 1FTD-set in $G$ contains each vertex of $G$ of degree at least two.

**Proof.** Let $G \in \mathcal{G}_k$, and $S$ be a 1FTD-set in $G$. We prove by an induction on $k$, that we call first-induction, that $S$ contains every vertex of $G$ of degree at least two. For the base step, if $k = 1$, then $G \in \mathcal{G}_1$, and the result follows by Lemma 5. Assume the result holds for all graphs $G' \in \mathcal{G}_{k'}$ with $k' < k$. Now consider the graph $G \in \mathcal{G}_k$, where $k > 1$. Clearly, $G$ is obtained from a sequence $G_1, G_2, \ldots, G_t = G$, of cactus graphs such that $G_1 \in \mathcal{H}_k$, and if $l \geq 2$, then $G_{r_l+1}$ is obtained from $G_l$ by one of the Operations $O_1$ or $O_2$ for $i = 1, 2, \ldots, l-1$.

We prove by an induction on $l$, that we call second-induction, that $S$ contains every vertex of $G$ of degree at least two.

For the base step of the second-induction, let $l = 1$. Thus $G \in \mathcal{H}_k$. By the construction of graphs in the family $\mathcal{H}_k$, there are graphs $H \in \mathcal{H}_1$ and $G' \in \mathcal{G}_{k-1}$ such that $G$ is obtained from $H$ and $G'$ by Procedure A. Clearly, $H$ is obtained from the 2-corona $2\text{-}cor(C)$ of a cycle $C$, by removing precisely one support vertex $v$ and the leaf adjacent to $v$ of $2\text{-}cor(C)$.

Let $C = c_0c_1 \cdots c_r c_0$ be the cycle of $H$, where $c_0$ is a vertex of degree at least two of $H$ that is adjacent to a weak support vertex $c'_0$, and let $c'_0$ and its leaf (that we call $c''_0$) be removed according to Procedure A. By Observation 3, $H$ has precisely one special vertex. Let $c_t$ be the special vertex of $H$. Let $w \in V(G')$ be a vertex of degree at least two of $G'$ that is adjacent to a weak support vertex $w'$, and let $w'$ and its leaf (that we call $w''$) be removed according to Procedure A.

First we show that $\{c_1, c_r\} \cap S \neq \emptyset$. Clearly, $S \cap \{c_{t-1}, c_t, c_{t+1}\} \neq \emptyset$, since $\deg_H(c_t) = 2$. Assume that $c_t \in S$. Since at least one of $c_{t-1}$ or $c_{t+1}$ is adjacent to a weak support vertex, by Observation 2, $\{c_{t-1}, c_t, c_{t+1}\} \cap S \neq \emptyset$. By applying Observation 2, we obtain that $\{c_1, c_r\} \cap S \neq \emptyset$, since any vertex of $\{c_1, \ldots, c_r\}\{c_t\}$ is adjacent to a weak support vertex of $G$. Thus assume that $c_t \notin S$. Then $\{c_{t-1}, c_{t+1}\} \cap S \neq \emptyset$, and so $\{c_1, c_r\} \cap S \neq \emptyset$, since any vertex of $\{c_1, \ldots, c_r\}\{c_t\}$ is adjacent to a weak support vertex of $G$. Hence, $\{c_1, c_r\} \cap S \neq \emptyset$. If $c_0 \notin S$, then $S \cup \{w', w''\}$ is a 1FTD-set for $G'$, and thus by the first-inductive hypothesis, $S'$ contains $w = c_0$, a contradiction. Thus $c_0 \in S$. By Observation 2, $V(C) \subseteq S$, since any vertex of $\{c_1, \ldots, c_r\}\{c_t\}$ is adjacent to a weak support vertex of $G$. Thus $S \cap V(G')$ is a 1FTD-set for $G'$. By the first-inductive hypothesis, $(S \cap V(G')) \cup \{w', w''\}$ contains every vertex of $G'$ of degree at least two. Consequently, $S$ contains every vertex of $G$ of degree at least two. We conclude that the base step of the second-induction holds.
Assume that the result (for the second-induction) holds for $2 \leq l' < l$. Now let $G = G_1$. Clearly, $G$ is obtained from $G_{l-1}$ by applying one of the Operations $O_1$ or $O_2$.

Assume that $G$ is obtained from $G_{l-1}$ by applying Operation $O_2$. Let $x$ be a support vertex of $G_{l-1}$ and let $x'$ be a leaf adjacent to $x$. Let $G$ be obtained from $G_{l-1}$ by adding a vertex $u'$ and a path $P_2 = y_1y_2$, joining $x'$ to $u'$ and joining $x$ to $y_1$, according to Operation $O_2$. By Observation 1, $x', y_1 \in S$ and so $x \in S$. Thus $S'\{y_1\}$ is a 1FTD-set for $G_{l-1}$. By the second-inductive hypothesis, $S$ contains all vertices of $G_{l-1}$ of degree at least two. Consequently, $S$ contains every vertex of $G_k$ of degree at least two.

Next assume that $G$ is obtained from $G_{l-1}$ by applying Operation $O_1$. Let $P_3 = x_1x_2x_3$ be a path and $x_1$ be joined to $y \in V(G_{l-1})$, where $\deg_{G_{l-1}}(y) \geq 2$ and $y$ is not a special vertex of $G_{l-1}$, according to Operation $O_2$. By Observation 1, $x_2 \in S$. Observe that $\{x_1, x_3\} \cap S \neq \emptyset$. If $x_1 \notin S$, then $x_3 \in S$ and $y \notin S$. Then $S'\{x_2, x_3\}$ is a 1FTD-set for $G_{l-1}$ that does not contains $y$, a contradiction by the second-inductive hypothesis. Thus assume that $x_1 \in S$. Suppose that $y \notin S$. Clearly, $N_{G_{l-1}}(y) \cap S = \emptyset$. Assume that there exists a component $G_1'$ of $G_{l-1} - y$ such that $|V(G_1') \cap N_{G_{l-1}}(y)| = 1$. Then clearly $S' = (S \cap V(G_{l-1})) \cup V(G_1')$ is a 1FTD-set for $G_{l-1}$, and by the second-inductive hypothesis, $S'$ contains every vertex of $G_{l-1}$ of degree at least two. Thus $y \in S'$, and so $y \in S$, a contradiction. Next assume that every component of $G_{l-1} - y$ has at least two vertices in $N_{G_{l-1}}(y)$. Since $y$ is a non-special vertex of $G_{l-1}$, $y$ belongs to at least two cycles of $G_{l-1}$. By Observation 9(4), $y$ belongs to exactly two cycles of $G_{l-1}$. Thus $\deg_{G_{l-1}}(y) = 4$. By Observation 11, $G_{l-1} - y$ has exactly two components $D_1$ and $D_2$. Let $G^*$ be a graph obtained from $D_1 \cup \{y\}$ or $D_2 \cup \{y\}$ by adding a path $P_2 = yy'$ to $y$. Then there exists $k' \leq k$ such that $G^* \in G_{k'}$. Evidently, $S^* = (S \cap V(G^*)) \cup \{y', y''\}$ is a 1FTD-set for $G^*$, and so by the first-inductive hypothesis, $S^*$ contains every vertex of $G^*$ of degree at least two (since $G^* \in G_{k'}$). Thus $y \in S^*$, and so $y \in S$, a contradiction. We conclude that $y \in S$. Observe that $S \cap V(G_{l-1})$ is a 1FTD-set for $G_{l-1}$, and so by the second-inductive hypothesis, $S \cap V(G_{l-1})$ contains every vertex of $G_{l-1}$ of degree at least two. Consequently, $S$ contains every vertex of $G$ of degree at least two.

As a consequence of Observation 9(3) and Lemma 12, we obtain the following.

**Corollary 13.** If $G \in G_k$ is a cactus graph of order $n$, then $V(G) \setminus L(G)$ is the unique $ftd_1(G)$-set.

In what follows, we present an upper bound for the 1-fair domination number of a cactus graph in terms of the order and the number of cycles.

**Theorem 14.** If $G$ is a cactus graph of order $n \geq 4$ with $k \geq 1$ cycles, then $ftd_1(G) \leq (2(n + k) - 1)/3$. 

Proof. The result follows by Theorem 6 if \( k = 1 \). Thus assume that \( k \geq 2 \). Suppose to the contrary that \( ftd_1(G) > (2(n(G) + k) - 1)/3 \). Assume that \( G \) has the minimum order, and among all such graphs, we may assume that the size of \( G \) is minimum. Let \( C_1, C_2, \ldots, C_k \) be the \( k \) cycles of \( G \). Let \( C_i \) be a leaf-cycle of \( G \), where \( i \in \{1, 2, \ldots, k\} \). Let \( C_i = c_0c_1 \cdots c_0, \) where \( c_0 \) is the special cut-vertex of \( G \). Suppose that \( G \) has a strong support vertex \( u \), and \( u_1, u_2 \) are leaves adjacent to \( u \). Let \( G_0 = G - u_1 \). By the choice of \( G \), \( ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2/3 \). Let \( S' \) be an \( ftd_1(G') \)-set. By Observation 1, \( u \in S' \). Clearly, \( S' \) is a 1FTD-set in \( G \) and so \( ftd_1(G) \leq (2(n + k) - 1)/3 - 2/3 \), a contradiction. We deduce that every support vertex of \( G \) is adjacent to precisely one leaf.

Assume that \( \deg_G(v_j) = 2 \) for each \( j = 1, 2, \ldots, r \). Let \( G' = G - c_2 \). Then by the choice of \( G \), \( ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2/3 \). Let \( S' \) be an \( ftd_1(G') \)-set. By Observation 1, \( c_0 \in S' \). If \( |S' \cap \{c_1, c_3\}| = 1 \), then \( S' \) is a 1FTD-set for \( G \) cardinality at most \( (2(n + k) - 1)/3 - 2/3 \), a contradiction. Thus assume that \( |S' \cap \{c_1, c_3\}| = 0 \). Then \( \{c_2\} \cup S' \) is a 1FTD-set in \( G \) of cardinality at most \( (2(n + k) - 1)/3 - 2/3 \), a contradiction. Thus assume that \( |S' \cap \{c_1, c_3\}| = 0 \). Now \( \{c_1\} \cup S' \) is a 1FTD-set in \( G \) of cardinality at most \( (2(n + k) - 1)/3 - 2/3 \), a contradiction. We deduce that \( \deg_G(c_i) \geq 3 \) for some \( i \in \{1, 2, \ldots, r\} \).

Let \( v_d \) be a leaf of \( G \) such that \( d(v_d, C_1 - c_0) \) is as maximum as possible, the shortest path from \( v_d \) to \( C_i \) does not contain \( c_0 \) and \( \deg(v_d-1) \) is as maximum as possible, where \( v_d-1 \) is the neighbor of \( v_d \) on the shortest path from \( v_d \) to a vertex \( v_0 \in C_i \).

Assume that \( d \geq 3 \). Observe that \( \deg_G(v_d-1) = 2 \), since \( G \) has no strong support vertex. Assume that \( \deg_G(v_d-2) = 2 \). Let \( G' = G - \{v_d, v_d-1, v_d-2\} \). By the choice of \( G \), \( ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2 \). Let \( S' \) be an \( ftd_1(G') \)-set. If \( v_d-3 \in S' \), then \( \{v_{v_d-1}, v_d-2\} \cup S' \) is a 1FTD-set in \( G \) and so \( ftd_1(G) \leq (2(n + k) - 1)/3 - 2 \), a contradiction. If \( v_d-3 \notin S' \), then \( \{v_{v_d-1}, v_d\} \cup S' \) is a 1FTD-set in \( G \) and so \( ftd_1(G) \leq (2(n + k) - 1)/3 - 2 \), a contradiction. Thus assume that \( \deg_G(v_d-2) \geq 3 \). Assume that \( v_d-2 \) is a support vertex. Let \( G' = G - \{v_d-1, v_d\} \). By the choice of \( G \), \( ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 4 \). Let \( S' \) be an \( ftd_1(G') \)-set. By Observation 1, \( v_d-2 \in S' \). Then \( \{v_{v_d-1}, v_d\} \cup S' \) is a 1FTD-set in \( G \) and so \( ftd_1(G) \leq (2(n + k) - 1)/3 - 4 \), a contradiction. Thus assume that \( v_d-2 \) is not a support vertex of \( G \). Let \( x \neq v_d-1, v_d-3 \) be a support vertex of \( G \) such that \( x \in N(v_d-2) \). By the choice of the path \( v_0v_1 \cdots v_d \), (the part “\( \deg(v_{d-1}) \) is as maximum as possible”), \( \deg_G(x) = 2 \). Let \( y \) be the leaf adjacent to \( x \) and \( G' = G - \{v_d, v_d-1, y\} \). By the choice of \( G \), \( ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2 \). Let \( S' \) be an \( ftd_1(G') \)-set. By Observation 1, \( v_d-2 \in S' \), since \( v_d-2 \) is a support vertex of \( G' \). Thus \( \{v_{d-1}, x\} \cup S' \) is a 1FTD-set in \( G \) and so \( ftd_1(G) \leq (2(n + k) - 1)/3 - 2 \), a contradiction.
Next assume that $d = 2$. Assume that $\deg_G(c_i) = 2$ for some $i \in \{1, 2, \ldots, r\}$. Let $\deg_G(c_j) = 2$. Assume that $\deg_G(c_{j+1}) = 2$. Let $G' = G - c_j$. Then by the choice of $G$, \(ftd_1(G') \le (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3\). Let $S'$ be an \(ftd_1(G')\)-set. By Observation 1, $c_{j+2} \in S'$. If $|S' \cap \{c_{j-1}, c_{j+1}\}| = 1$, then $S'$ is a 1FTD-set for $G$ of cardinality at most $(2(n + k) - 1)/3 - 4/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 2$. Then $\{c_j\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 1/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 0$ and so $\{c_{j+1}\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 4/3$, a contradiction. Similarly $\deg_G(c_{j-1}) \ge 3$. Clearly, $c_{j+1} \neq c_0$ or $c_{j-1} \neq c_0$. Assume, without loss of generality, that $c_{j+1} \neq c_0$. Let $c_{j+1}$ be a support vertex of $G$, and $G' = G - c_j$. Then by the choice of $G$, \(ftd_1(G') \le (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3\). Let $S'$ be an \(ftd_1(G')\)-set. By Observation 1, $c_{j+1} \in S'$. If $c_{j-1} \notin S'$, then $S'$ is a 1FTD-set for $G$ of cardinality at most $(2(n + k) - 1)/3 - 4/3$, a contradiction. Thus assume that $c_{j-1} \in S'$ and so $\{c_j\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 1/3$, a contradiction. Thus $c_{j+1}$ is not a support vertex of $G$. Let $c'_{j+1} = N(c_{j+1}) \setminus \{V(C_i)\}$. Clearly, $c'_{j+1}$ is a support vertex, since $d = 2$. Observe that $\deg_G(c'_{j+1}) = 2$, since $G$ has no strong support vertex. Let $c''_{j+1}$ be the leaf of $c'_{j+1}$. Let $G'' = G - c_j - c''_{j+1}$. By the choice of $G$, \(ftd_1(G'') \le (2(n(G'') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 2\). Let $S'$ be an \(ftd_1(G'')\)-set. By Observation 1, $c_{j+1} \in S'$, since $c_{j+1}$ is a support vertex in $G'$. If $c_{j-1} \notin S'$, then $S' \cup \{c'_{j+1}\}$ is a 1FTD-set for $G$ of cardinality at most $(2(n + k) - 1)/3 - 1$, a contradiction. Thus assume that $c_{j-1} \in S'$. Then $\{c_j, c'_{j+1}\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3$, a contradiction. Thus $\deg(c_j) \ge 3$ for $1 \le i \le r$. Let $G^* = G - c_0c_1 - c_0c_r$. Let $G_1^*$ be the component of $G^*$ containing $c_r$, and $G_2^*$ be the component of $G^*$ containing $c_0$. Let $D = S(G_1^*) \setminus \{V(C_i)\}$. Clearly, $S' = D \cup \{c_1, c_2, \ldots, c_r\}$ is a 1FTD-set for $G^*_1$ of cardinality at most $2n(G_1^*)/3$. Let $G_3^* = G[V(G_2^*) \cup \{c_1\}]$. By the choice of $G$, \(ftd_1(G_3^*) \le (2(n(G_3^*) + k - 1) - 1)/3\). Let $S''$ be an \(ftd_1(G_3^*)\)-set. By Observation 1, $c_0 \in S''$. Clearly, $S'' \cup S'$ is a 1FTD-set for $G$ and so \(ftd_1(G) \le (2(n(G_2^*) + k - 1) - 1)/3 + 2n(G_1^*)/3 = (2(n + k) - 1)/3\), a contradiction.

Now assume that $d = 1$. Assume that $\deg_G(c_i) = 2$ for some $i \in \{1, 2, \ldots, r\}$. Let $\deg_G(c_j) = 2$. Assume that $\deg_G(c_{j+1}) = 2$. Let $G' = G - c_j$. By the choice of $G$, \(ftd_1(G') \le (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3\). Let $S'$ be an \(ftd_1(G')\)-set. By Observation 1, $c_{j+2} \in S'$. If $|S' \cap \{c_{j-1}, c_{j+1}\}| = 1$, then $S'$ is a 1FTD-set for $G$ of cardinality at most $(2(n + k) - 1)/3 - 4/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 2$. Then $\{c_j\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 4/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 0$. Then $\{c_{j+1}\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 1/3$, a contradiction. Thus $\deg_G(c_{j+1}) \ge 3$. Similarly, $\deg_G(c_{j-1}) \ge 3$. Clearly, $c_{j+1} \neq c_0$ or $c_{j-1} \neq c_0$. Assume, without loss of general-
If we prove by an induction on \(1\)

**Proof.** Suppose that \(G / n\) for all cactus graphs \(G\) \(n\) with \(n\) \(k\) \(2\). Assume that the size of \(G\) \(S\) \(G\) \(c\) and \(G\) \(G\) \(c\) 

Clearly, \(S' = \{c_1, c_2, \ldots, c_r\}\) is a 1FTD-set for \(G\) of cardinality at most \(n(G_1^*)/2\).

Let \(G_3^* = G[V(G_2^* \cup \{c_1\})].\) By the choice of \(G\), \(ftd_1(G_3^*) \leq (2(n(G_3^*) + k - 1) - 1)/3).\) Let \(S''\) be an \(ftd_1(G_3^*)\)-set. By Observation 1, \(c_0 \in S''.\) Clearly, \(S'' \cup S''\) is a 1FTD-set for \(G\) and so \(ftd_1(G) \leq (2(n(G_3^*) + k - 1) - 1)/3 + n(G_1^*)/2 < (2(n + k) - 1)/3), a contradiction.

It is evident that for the cycle \(C_7\) the equality of the bound given in Theorem 14 holds.

**Theorem 15.** If \(G \neq C_7\) is a cactus graph of order \(n \geq 5\) with \(k \geq 1\) cycles, then \(ftd_1(G) = (2(n + k) - 1)/3\) if and only if \(G \in G_k\).

**Proof.** We prove by an induction on \(k\) to show that any cactus graph \(G \neq C_7\) of order \(n \geq 5\) with \(k \geq 1\) cycles and \(ftd_1(G) = (2(n + k) - 1)/3\) belongs to \(G_k\).

The base step of the induction follows by Theorem 6. Assume the result holds for all cactus graphs \(G' \neq C_7\) with \(k' < k\) cycles. Now let \(G \neq C_7\) be a cactus graph of order \(n\) with \(k \geq 2\) cycles and \(ftd_1(G) = (2(n + k) - 1)/3\). Suppose to the contrary that \(G \notin G_k\). Assume that \(G\) has the minimum order, and among all such graphs, assume that the size of \(G\) is minimum.

**Claim 1.** Every support vertex of \(G\) is weak support vertex.

**Proof.** Suppose that \(G\) has a strong support vertex \(u\), and assume that \(u_1\) and \(u_2\) are two leaves adjacent to \(u\). Let \(G' = G - u_1\), and \(S'\) be an \(ftd_1(G')\)-set. By Observation 1, \(u \in S'\). By Theorem 14, \(|S'| \leq (2(n(G') + 2) - 1)/3 = (2(n + k) - 1)/3 - 2/3\). Clearly, \(S'\) is a 1FTD-set for \(G\) of cardinality at most \((2(n + k) - 1)/3 - 2/3\), a contradiction.

By Observation 8, \(G\) has at least two leaf-cycles. Let \(C_1 = c_0c_1 \cdots c_rc_0\) be a leaf-cycle of \(G\), where \(c_0\) is a special cut-vertex of \(G\). Let \(G_1'\) be the component of \(G - c_0c_1 - c_0c_r\) containing \(c_1\).

**Claim 2.** \(V(G_1') \neq \{c_1, \ldots, c_r\}\).

**Proof.** Suppose that \(V(G_1') = \{c_1, \ldots, c_r\}\). Then \(deg_G(c_i) = 2\), for each \(i = 1, 2, \ldots, r\). Let \(G' = G - c_2\). By Theorem 14, \(ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3\). Let \(S'\) be an \(ftd_1(G')\)-set. By Observation 1,
$c_0 \in S'$. If $|S' \cap \{c_1, c_3\}| = 1$, then $S'$ is a $1$FTD-set for $G$ of cardinality at most $(2(n + k) - 1)/3 - 4/3$, a contradiction. Thus assume that $|S' \cap \{c_1, c_3\}| = 2$. Then $\{c_2\} \cup S'$ is a $1$FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 1/3$, a contradiction. Thus assume that $|S' \cap \{c_1, c_3\}| = 0$. Then $\{c_1\} \cup S'$ is a $1$FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 1/3$, a contradiction. 

Let $v_d \in V(G'_1) \backslash \{c_1, \ldots, c_r\}$ be a leaf of $G'_1$ at maximum distance from $\{c_1, \ldots, c_r\}$, and assume that $\text{deg}(v_d-1)$ is as maximum as possible, $\text{deg}_G(v_0)$ is as maximum as possible, and $\text{deg}_G(v_1)$ is as maximum as possible, where $v_0 \in \{c_1, \ldots, c_r\}$ and $v_0v_1 \cdots v_d$ is the shortest path from $v_d$ to $\{c_1, \ldots, c_r\}$. 

Suppose that $d = 1$. Assume that $\text{deg}_G(c_j) = 2$, for some $j \in \{1, 2, \ldots, r\}$. Assume that $\text{deg}_G(c_{j+1}) = 2$. Let $G' = G - c_j$. By Theorem 14, $\text{ftd}_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n+k) - 1)/3 - 4/3$. Let $S'$ be an $\text{ftd}_1(G')$-set. By Observation 1, $c_{j+2} \in S'$. If $|S' \cap \{c_{j-1}, c_{j+1}\}| = 1$, then $S'$ is a $1$FTD-set for $G$ of cardinality at most $(2(n + k) - 1)/3 - 4/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 2$. Then $\{c_j\} \cup S'$ is a $1$FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 1/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 0$. Then $\{c_{j+1}\} \cup S'$ is a $1$FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 1/3$, a contradiction. Thus $\text{deg}_G(c_{j+1}) \geq 3$. Similarly, $\text{deg}_G(c_{j-1}) \geq 3$. Clearly, $c_{j+1} \neq c_0$ or $c_{j-1} \neq c_0$. Assume, without loss of generality, that $c_{j+1} \neq c_0$. Then $c_{j+1}$ is a support vertex of $G$. Let $G'' = G - c_j$. Then by Theorem 14, $\text{ftd}_1(G'') \leq (2(n(G'') + k - 1) - 1)/3 = (2(n+k) - 1)/3 - 4/3$. Let $S''$ be an $\text{ftd}_1(G'')$-set. By Observation 1, $c_{j+1} \in S''$. If $c_{j-1} \notin S''$, then $S''$ is a $1$FTD-set for $G$ of cardinality at most $(2(n + k) - 1)/3 - 4/3$, a contradiction. Thus assume that $c_{j-1} \in S''$. Then $\{c_j\} \cup S''$ is a $1$FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 1/3$, a contradiction. We thus obtain that $\text{deg}(c_j) \geq 3$, for $1 \leq j \leq r$. Let $G^* = G - c_0c_1c_2c_r$. Let $G_1^*$ be the component of $G^*$ containing $c_r$, and $G_2^*$ be the component of $G^*$ containing $c_0$. Clearly, $S' = \{c_1, c_2, \ldots, c_r\}$ is a $1$FTD-set for $G_1^*$ of cardinality at most $n(G_1^*)/2$. Let $G_3^* = G[V(G_2^*) \cup \{c_1\}]$. By Theorem 14, $\text{ftd}_1(G_3^*) \leq (2(n(G_3^*) + k - 1) - 1)/3$. Let $S''$ be an $\text{ftd}_1(G_3^*)$-set. By Observation 1, $c_0 \in S''$. Clearly, $S' \cup S''$ is a $1$FTD-set for $G$ and so $\text{ftd}_1(G) \leq (2(n(G^*) + k - 1) - 1)/3 + n(G_3^*)/2 < (2(n+k) - 1)/3$, a contradiction.

Thus assume that $d \geq 2$.

**Claim 3.** If $d \geq 3$, then $G \in \mathcal{G}_k$.

**Proof.** Assume that $d \geq 3$. By Claim 1, $\text{deg}_G(v_{d-1}) = 2$. Assume first that $\text{deg}_G(v_{d-2}) \geq 3$. Assume that $v_{d-2}$ is a support vertex. Let $G' = G - \{v_{d-1}, v_d\}$. By Theorem 14, $\text{ftd}_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n+k) - 1)/3 - 4/3$. Let $S'$ be an $\text{ftd}_1(G')$-set. By Observation 1, $v_{d-2} \in S'$. Then $\{v_{d-1}\} \cup S'$ is a $1$FTD-set in $G$, and so $\text{ftd}_1(G) \leq (2(n+k) - 1)/3 - 1/3$, a contradiction. Thus assume that $v_{d-2}$ is not a support vertex of $G$. Let $x \neq v_{d-1}, v_{d-3}$ be a support vertex of $G$ such that $x \in N(v_{d-2})$. By the choice of the path $v_0v_1 \cdots v_d$, (the part “$\text{deg}(v_{d-1})$
is as maximum as possible”), $\deg_G(x) = 2$. Let $y$ be the leaf adjacent to $x$, and $G' = G - \{v_d, v_{d-1}, y\}$. By Theorem 14, \( ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2 \). Assume that \( ftd_1(G') < (2(n(G') + k) - 1)/3 \). Let $S'$ be an $ftd_1(G')$-set. By Observation 1, $v_{d-2} \in S'$, since $v_{d-2}$ is a support vertex of $G'$. Then $\{v_{d-1}, x\} \cup S'$ is a 1FTD-set in $G$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus $ftd_1(G') = (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$. By the choice of $G, G' \in G_k$. Thus $G$ is obtained from $G'$ by Operation $O_2$, and so $G \in G_k$.

Assume that $\deg_G(v_{d-2}) = 2$. We consider the following cases.

**Case 1.** $d \geq 4$. Suppose that $\deg_G(v_{d-3}) = 2$. Let $G' = G - \{v_d, v_{d-1}, v_{d-2}, v_{d-3}\}$. By Theorem 14, \( ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 8/3 \). Let $S'$ be an $ftd_1(G')$-set. If $v_{d-4} \in S'$, then $\{v_{d-1}, v_d\} \cup S'$ is a 1FTD-set in $G$ and so $ftd_1(G) \leq (2(n + k) - 1)/3 - 2/3$, a contradiction. Thus $v_{d-4} \notin S'$. Then $\{v_{d-2}, v_{d-1}\} \cup S'$ is a 1FTD-set in $G$ and so $ftd_1(G) \leq (2(n + k) - 1)/3 - 2/3$, a contradiction. We deduce that $\deg_G(v_{d-3}) \geq 3$. Let $G' = G - \{vd, v_{d-1}, v_{d-2}\}$. By Theorem 14, \( ftd_1(G') \leq (2(n(G') + k) - 1)/3 \). Assume that $ftd_1(G') < (2(n(G') + k) - 1)/3 - 1 = (2(n + k) - 1)/3 - 2$. Let $S'$ be an $ftd_1(G')$-set. If $v_{d-3} \in S'$, then $\{v_{d-1}, v_{d-2}\} \cup S'$ is a 1FTD-set in $G$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus $v_{d-3} \notin S'$. Then $\{v_{d-1}, v_d\} \cup S'$ is a 1FTD-set in $G$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. We thus obtain that $ftd_1(G') = (2(n(G') + k) - 1)/3$. By the choice of $G, G' \in G_k$. Since $d \geq 4$, $v_{d-3}$ is not a special vertex of $G'$. Thus $G$ is obtained from $G'$ by Operation $O_1$, and so $G \in G_k$.

**Case 2.** $d = 3$. Clearly, $\deg(v_0) \geq 3$. We show that $\deg(v_0) \geq 4$. Suppose that $\deg(v_0) = 3$. Let $G' = G - \{v_1, v_2, v_3\}$. By Theorem 14, \( ftd_1(G') \leq (2(n(G') + k) - 1)/3 \). Assume that $ftd_1(G') = (2(n(G') + k) - 1)/3$. By the choice of $G, G' \in G_k$. By Observation 9(1), $v_0$ is the unique special vertex of $G'$, since $\deg_G(v_0) = 2$. We show that $\deg_G(x) = 3$ for each $x \in \{c_1, \ldots, c_r\} \setminus \{v_0\}$. Assume that $\deg_G(c_j) \geq 4$ for some $c_j \in \{c_1, \ldots, c_r\} \setminus \{v_0\}$. If there is a vertex $w \in V(G) \setminus V(C_1)$ such that $d(w, C_1) = d(w, c_j) = 3$, then $w$ can play the same role of $v_0$, and thus $\deg(v_0) = 3$, a contradiction. Thus there is no vertex $w \in V(G) \setminus V(C_1)$ such that $d(w, C_1) = d(w, c_j) = 3$. Thus any vertex of $N(v_0) \setminus V(C_1)$ is a leaf or a weak support vertex. Assume that $N(c_j) \setminus V(C_1)$ contains $t_1$ leaves and $t_2$ support vertices, where $t_1 + t_2 \geq 2$. By Observation 9(1), $t_1 = 0$, since $G' \in G_k$. Thus $t_2 \geq 2$. Let $z_1$ and $z_2$ be two weak support vertices in $N(c_j) \setminus V(C_1)$. Let $z_1'$ and $z_2'$ be the leaves adjacent to $z_1$ and $z_2$, respectively. (We switch for a while to $G'$). Let $G'' = G - \{z_1, z_1', z_2, z_2\}$. By Theorem 14, \( ftd_1(G'') \leq (2(n(G'') + k) - 1)/3 \). Suppose that $ftd_1(G'') = (2(n(G'') + k) - 1)/3$. By the choice of $G, G'' \in G_k$. Clearly, $\deg_G(c_j) \geq 3$, since $v_0$ is the unique special vertex of $G'$, a contradiction (by Observation 9(1)). Thus $ftd_1(G'') < (2(n(G'') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$. Let $S''$ be a 1FTD-set of $G''$. By Observation 1, $c_j \in S''$. Then $S'' \cup \{z_1, z_2\}$ is a 1FTD-set of $G$. Thus $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. We
deduce that \( \text{deg}_{G'}(c_i) = 3 \) for each \( c_i \in \{ c_1, \ldots, c_r \} \setminus \{ v_0 \} \). Thus \( \text{deg}_{G'}(c_i) = 3 \) for each \( 1 \leq i \leq r \) in \( G' \), since \( G' \in \mathcal{G}_k \). (We switch for a while to \( G \)).

Let \( F = \bigcup_{i=1}^{r} \langle N[c_i] \rangle \cup \{ c_0, \ldots, c_r \} \). Clearly, \( |F| = r \), since \( \text{deg}_{G'}(c_i) = 3 \) for each \( c_i \in \{ c_1, \ldots, c_r \} \setminus \{ v_0 \} \) and \( \text{deg}_{G'}(v_0) = 3 \). Let \( F = \{ u_1, u_2, \ldots, u_r \} \). Clearly \( \text{deg}_G(u_i) \geq 2 \), for each \( i \) with \( 1 \leq i \leq r \), since \( c_i \) is not a support vertex for \( 1 \leq i \leq r \) in \( G' \). By Claim 2, \( u_i \) is not a strong support vertex of \( G \), for \( 1 \leq i \leq r \). If \( u_i \) is adjacent to a support vertex \( u'_i \in V(G) \setminus V(C_1) \), for some integer \( i \), then since the leaf of \( u'_i \) can play the role of \( v_3 \), we obtain that \( \text{deg}(u_i) = 2 \). Since \( \text{deg}_{G}(u_i) \geq 2 \) for each \( 1 \leq i \leq r \), we find that \( \text{deg}_G(u_i) = 2 \) for each \( i \) with \( 1 \leq i \leq r \).

Let \( F' = \bigcup_{i=1}^{r} N(u_i) \cup \{ c_0, \ldots, c_r \} \). Clearly, \( |F'| = r \), since \( \text{deg}_{G'}(u_i) = 2 \), for each \( u_i \in \{ u_1, \ldots, u_r \} \). Let \( F' = \{ u'_1, u'_2, \ldots, u'_r \} \). By the choice of the path \( v_0v_1 \cdots v_d \) (the part “\( \text{deg}(v_{d-1}) \) is as maximum as possible”), \( \text{deg}_{G'}(u'_i) \leq 2 \), for \( 1 \leq i \leq r \). Let \( F'_i = \{ u'_i \in F' | \text{deg}_G(u'_i) = 1 \} \) and \( F'_i = F' - F'_i \). Then every vertex of \( F'_2 \) is a weak support vertex. Since \( v_1 \in F'_2 \), we have \( |F'_2| \geq 1 \).

Let \( G^* = G - c_0c_1 - c_0c_r \), and \( G_1^* \) and \( G_2^* \) be the components of \( G^* \), where \( c_1 \in V(G_1^* \). By Theorem 14, \( ftd_1(G_2^*) \leq (2(n(G_2^*) + k - 1))/3 \). Clearly, \( n(G_2^*) = n(G) - 3r - |F_2| \). Let \( S_2^* \) be an \( ftd_1(G_2^*) \)-set. If \( c_0 \notin S_2^* \), then \( S_2^* \cup F \cup F' \) is a 1FTD-set for \( G \). Thus \( ftd_1(G) \leq (2(n(G_2^*) + k - 1))/3 + 2r = (2(n(G) - 3r - |F_2'| + k - 1))/3 + 2r \) and so \( ftd_1(G) \leq (2(n(k) - 1))/3 \), a contradiction.

Thus \( c_0 \in S_2^* \). If \( |F_2'| = 1 \), then \( S_2^* \cup V(C_1) \cup F \cup \{ v_2 \} \) is a 1FTD-set for \( G \) and thus \( ftd_1(G) \leq (2(n(G_2^*) + k - 1))/3 + 2r + 1 = (2(n(G) - 3r - |F_2'| + k - 1))/3 + 2r + 1 < (2(n(k) - 1))/3 \), a contradiction. Thus assume that \( |F_2'| \geq 2 \).

Let \( \{ u'_t, u'_t \} \subseteq F_2' \) (assume without loss of generality that \( t < t' \)) such that \( \text{deg}_G(u'_t) = 1 \), for \( 1 \leq t \leq t' < i \leq r \). Let \( u''_t \) be the leaves of \( u_t \) and \( u'_t \) respectively. Clearly, \( S_2^* \cup \{ c_1, c_1, c_1 \} \cup \{ u_1, \ldots, u_{t-1} \} \cup \{ c_t \} \cup \{ u_{t+1}, \ldots, u_r \} \cup \{ u_{t+1}, \ldots, u_{r-1} \} \cup \{ u'_t, u''_t \} \cup \{ u''_t \} \) is a 1FTD-set for \( G \) and thus \( ftd_1(G) \leq (2(n(G_2^*) + k - 1))/3 + 2r = (2(n(G) - 3r - |F_2'| + k - 1))/3 + 2r + 1 < (2(n(k) - 1))/3 \), a contradiction. We deduce that \( ftd_1(G') \leq (2(n(G') + k - 1))/3 = (2(n(k) - 1))/3 - 2 \). Let \( S' \) be an \( ftd_1(G') \)-set.

If \( v_0 \in S' \), then \( S' \cup \{ v_1, v_2 \} \) is a 1FTD-set in \( G \), and so \( ftd_1(G) < (2(n(k) - 1))/3 \), a contradiction. Thus assume that \( v_0 \notin S' \). Then \( S' \cup \{ v_2, v_3 \} \) is a 1FTD-set in \( G \) and thus \( ftd_1(G) < (2(n(k) - 1))/3 \), a contradiction. Thus assume that \( v_0 \notin S' \). Then \( S' \cup \{ v_2, v_3 \} \) is a 1FTD-set for \( G \) and thus \( ftd_1(G) < (2(n(k) - 1))/3 \), a contradiction. Hence, \( ftd_1(G') = (2(n(G') + k - 1))/3 \). By the inductive hypothesis, \( G' \in \mathcal{G}_k \). Since \( \text{deg}(v_0) \geq 4 \), \( v_0 \) is not a special vertex of \( G' \). Thus \( G \) is obtained from \( G' \) by Operation \( O_1 \) and so \( G \in \mathcal{G}_k \). □
By Claim 3, we assume that \( d = 2 \). We show that \( \deg_G(v_0) = 3 \). Assume that \( v_0 \) is a support vertex. Let \( G' = G - \{v_1, v_2\} \). By Theorem 14, \( ftd_1(G') \leq (2(n(G') + k) - 1)/3 \). Let \( S' \) be an \( ftd_1(G') \)-set. By Observation 1, \( v_0 \in S' \). Then \( S' \cup \{v_{d-1}\} \) is a 1FTD-set in \( G \), and so \( ftd_1(G) < (2(n + k) - 1)/3 \), a contradiction. Thus assume that \( v_0 \) is not a support vertex of \( G \). Let \( x \neq v_1 \) be a support vertex of \( G \) such that \( x \in N(v_0) \setminus V(C_1) \). By the choice of the path \( v_0v_1 \cdots v_d \), (the part “\( \deg(v_{d-1}) \) is as maximum as possible”), \( \deg_G(x) = 2 \). Let \( y \) be the leaf adjacent to \( x \). Let \( G' = G - \{v_2, v_1, y\} \). By Theorem 14, \( ftd_1(G') \leq (2(n(G') + k) - 1)/3 \). Let \( ftd_1(G') < (2(n(G') + k) - 1)/3 \). Let \( S' \) be an \( ftd_1(G') \)-set. By Observation 1, \( v_0 \in S' \), since \( v_0 \) is a support vertex of \( G' \). Then \( \{v_1, x\} \cup S' \) is a 1FTD-set in \( G \) and so \( ftd_1(G') < (2(n + k) - 1)/3 \), a contradiction. Thus \( ftd_1(G') = (2(n(G') + k) - 1)/3 \). By the inductive hypothesis, \( G' \in G_k \), a contradiction by Observation 9(1), since \( v_0 \) is a support vertex of \( G' \). Thus \( \deg_G(v_0) = 3 \). Observe that \( G \) has no strong support vertex. If \( c_j \) is adjacent to a support vertex \( c_i' \) of \( N(c_i) \setminus V(C_1) \) for some \( i \), then the leaf of \( c_i' \) can play the role of \( v_2 \), and thus \( \deg(c_i) = 3 \). Thus we may assume that \( \deg_G(c_i) \leq 3 \) for each \( i \) with \( i = 1, 2, \ldots, r \). Assume that \( \deg_G(c_i) = 3 \) for each \( i \) with \( 1 \leq i \leq r \).

Let \( F = \bigcup_{i=1}^{r} (N(c_i) \setminus \{c_0, \ldots, c_r\}) \). Clearly, \( |F| = r \), since \( \deg_G(c_i) = 3 \), for each \( c_i \in \{c_0, \ldots, c_r\} \). Let \( F = \{u_1, u_2, \ldots, u_r\} \). Clearly, \( \deg_G(u_i) \leq 2 \), for \( 1 \leq i \leq r \), since \( G \) has no strong support vertex. Let \( F' = \{u_i | \deg_G(u_i) = 2\} \). Clearly, \( v_1 \in F' \). Let \( F'' \) be the set of leaves of \( F' \). Clearly, \( v_2 \in F'' \). Let \( G^* = G - c_0c_1 - c_0c_r \). Let \( G_1^* \) be the component of \( G^* \) containing \( c_r \), and \( G_2^* \) be the component of \( G^* \) containing \( c_0 \). Assume that \( F = F' \). Thus \( n(G_1^*) = 3r \), since \( d = 2 \). Further, \( n(G_2^*) = n - 3r \). By Theorem 14, \( ftd_1(G_2^*) \leq (2(n(G_2^*) + k - 1) - 1)/3 \). Let \( S'' \) be an \( ftd_1(G_2^*) \)-set. If \( c_0 \in S'' \), then \( S'' \cup V(C_1) \cup F \) is a 1FTD-set for \( G \) and so \( ftd_1(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + 2r = (2(n - 3r + k - 1) - 1)/3 \), a contradiction. Thus \( c_0 \in S'' \). Then \( S'' \cup F'' \cup F \) is a 1FTD-set for \( G \) and so \( ftd_1(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + 2r = (2(n - 3r + k - 1))/3 + 2r = (2(n + k - 1))/3 \), a contradiction. We conclude that \( F \neq F' \). Let \( |F'| = r' \). Clearly, \( 1 \leq r' < r \), since \( v_1 \in F' \). Thus \( n(G_1^{*'}) = 2r + r' \). Then \( n(G_2^{*'}) = n - (2r + r') \). Let \( G_3^{*'} = G[V(G_2^*) \cup \{c_1\}] \). Then \( n(G_3^{*'}) = n - (2r + r') + 1 \). By Theorem 14, \( ftd_1(G_3^{*'}) \leq (2(n(G_3^{*'}) + k - 1) - 1)/3 \). Let \( S'' \) be an \( ftd_1(G_3^{*'}) \)-set. By Observation 1, \( c_0 \in S'' \) and so \( S'' \cup V(C_1) \cup F' \) is a 1FTD-set for \( G \). Thus \( ftd_1(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + r + r' = (2(n - (2r + r') + 1 + k - 1))/3 + r + r' = (2(n + k) - 1 + r + r')/3 < (2(n + k))/3 \), a contradiction. Therefore \( \deg_G(c_i) = 2 \) for some \( 1 \leq t \leq r \).

**Claim 4.** No vertex of \( C_1 - c_0 \) is a support vertex.

**Proof.** Let \( c_j \) be a support vertex of \( G \). Assume that \( c_{j+1} \) is a special vertex. Let \( G' = G - c_{j+1} \). Then by Theorem 14, \( ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3 \). Let \( S' \) be an \( ftd_1(G') \)-set. By Observation 1, \( c_j \in S' \). If
c_{j+2} \notin S'$, then $S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n+k)−1)/3−4/3$ and so $ftd_1(G) < (2(n+k)−1)/3$, a contradiction. Thus $c_{j+2} \in S'$. Then \( \{c_{j+1}\} \cup S' \) is a 1FTD-set in $G$ of cardinality at most $(2(n+k)−1)/3−1/3$ and so $ftd_1(G) < (2(n+k)−1)/3$, a contradiction. Thus $deg_G(c_{j+1}) \neq 2$. Note that $c_i$ is a special vertex of $G$. Assume without loss of generality that $j < t$. Let $c_j'$ be a support vertex of $G$ and $c_i'$ be a special vertex of $G$, where $j < i' < t' \leq t$, and among such vertices choose $c_j'$ and $c_i'$ such that $c_i$ is neither a support vertex nor a special vertex of $G$ for each $i$ with $j' < i < t'$. Let $u_i \in N(c_i) \setminus V(C_1)$ for $j' < i < t'$. Clearly, $deg_G(u_i) = 2$ for $j' < i < t'$, since $G$ has no strong support vertex. Let $G^* = G - c_j' c_{j+1}' - c_i' c_{i+1}'$. Let $G^*_1$ be the component of $G^*$ containing $c_j'$ and $G^*_2$ be the component of $G^*$ containing $c_i'$. Clearly, $n(G^*_2) = 3(t' − j' − 1) + 1$. Thus $n(G^*_1) = n = (3(t' − j' − 1) + 1)$.

By Theorem 14, $ftd_1(G^*_1) ≤ (2(n(G^*_1) + k)−1)/3$. Let $S'$ be an $ftd_1(G^*_1)$-set. By Observation 1, $c_j' \in S'$. Assume that $c_{i+1}' \notin S'$. Then $S' \cup \{c_{j+1}', c_{j+2}', \ldots, c_{i-1}'\} \cup \{u_{j+1}', u_{j+2}', \ldots, u_{i-1}'\}$ is a 1FTD-set in $G$ of cardinality at most $(2(n(G^*_1) + k)−1)/3 + 2(t' − j' − 1) = (2(n − (3(t' − j' − 1) + k)−1))/3 + 2(t' − j' − 1) = (2(n + k)−1)/3 + 2(t' − j' − 1) = (2(n + k)−1)/3 − 1/3$ and so $ftd_1(G) < (2(n + k)−1)/3$, a contradiction.

Claim 5. If $deg_G(c_j) = 2$ for some $j$ with $1 \leq j \leq r$, then $deg_G(c_{j+1}) = 3$ and $deg_G(c_{j−1}) = 3$.

**Proof.** Assume that $deg_G(c_j) = deg_G(c_{j+1}) = 2$, for some $j$ with $1 \leq j \leq r$, and among such vertices choose $c_j$ such that $deg_G(c_{j−1}) = 3$. Let $G' = G − c_j$. Then by Theorem 14, $ftd_1(G') ≤ (2(n(G') + k)−1)/3 = (2(n + k)−1)/3 − 4/3$. Let $S'$ be an $ftd_1(G')$-set. By Observation 1, $c_{j+2}' \in S'$. If $|S' \cap \{c_{j−1}, c_{j+1}\}| = 1$, then $S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n + k)−1)/3 − 4/3$ and so $ftd_1(G) < (2(n + k)−1)/3$, a contradiction. Thus assume that $|S' \cap \{c_{j−1}, c_{j+1}\}| = 2$. Then $\{c_j\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n + k)−1)/3 − 1/3$ and so $ftd_1(G) < (2(n + k)−1)/3$, a contradiction. Thus assume that $|S' \cap \{c_{j−1}, c_{j+1}\}| = 0$. Then $\{c_{j+1}\} \cup S'$ is a 1FTD-set in $G$ of cardinality at most $(2(n + k)−1)/3 − 1/3$ and so $ftd_1(G) < (2(n + k)−1)/3$, a contradiction. Thus $deg_G(c_{j+1}) ≥ 3$. Similarly $deg_G(c_{j−1}) ≥ 3$.

Claim 6. $C_1$ has precisely one special vertex.

**Proof.** Let $c_{t_1}$ and $c_{t_2}$ be two special vertices of $C_1$ and among such vertices choose $c_{t_1}$ and $c_{t_2}$ such that $c_i$ is not a special vertex of $C_1$ for $t_1 < i < t_2$. By Claim 5, $t_1 + 1 < t_2$. By Claim 4, $c_i$ is not a support vertex for $t_1 < i < t_2$. Let $u_i \in N(c_i) \setminus V(C_1)$, for $t_1 < i < t_2$. Clearly, $deg_G(u_i) = 2$, for $t_1 < i < t_2$. Let
Let $u'_i$ be the leaf adjacent to $u_i$, for $t_1 < i < t_2$, and $G^* = G - c_1, c_{t_1+1} - c_{t_2}, c_{t_2+1}$. Let $G'_1$ be the component of $G^*$ containing $c_{t_1}$, and $G'_2$ be the component of $G^*$ containing $c_{t_2}$. Clearly, $n(G'_1) = 3(t_2 - t_1) + 1$. Then $n(G'_2) = n - (3(t_2 - t_1) + 1)$. By Theorem 14, $ftd(G'_1) \leq (2(n(G'_1) + k - 1) - 1)/3$. Let $S'$ be an $ftd(G'_1)$-set. By Observation 1, $c_{t_1-1} \in S'$. Assume that $\{c_{t_1}, c_{t_2+1}\} \cap S' = \emptyset$. Then $S' \cup \{c_{t_1}, c_{t_1+1}, \ldots, c_{t_2-1}\} \cup \{u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_2-1}\}$ is a 1FTD-set in $G$ of cardinality at most $(2(n(G'_1) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) + 1 = (2(n - (3(t_2 - t_1) + 1) - 1)/3 + 2(t_2 - t_1 - 1) + 1 = (2(n + k) - 1)/3 - 1/3$ and so $ftd(G) < (2(n + k) - 1)/3$, a contradiction.

Thus $\{t_1, c_{t_2+1}\} \cap S' \neq \emptyset$. If $c_{t_1}, c_{t_2+1} \subseteq S'$, then $S' \cup \{c_{t_1}, c_{t_1+2}, \ldots, c_{t_2}\} \cup \{u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_2-1}\}$ is a 1FTD-set in $G$ of cardinality at most $(2(n(G'_1) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) + 1 = (2(n - (3(t_2 - t_1) + 1) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) + 1 = (2(n + k) - 1)/3$. If $ftd(G) < (2(n + k) - 1)/3$, a contradiction. Thus $\{c_{t_1}, c_{t_2+1}\} \nsubseteq S'$, then $S' \cup \{u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_2-1}\} \cup \{u'_{t_1}, u'_{t_1+2}, \ldots, u'_{t_2}\}$ is a 1FTD-set in $G$ of cardinality at most $(2(n(G'_1) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) = (2(n - (3(t_2 - t_1) + 1) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) = (2(n + k) - 1)/3 - 3/3$ and so $ftd(G) < (2(n + k) - 1)/3$, a contradiction.

By Claims 4 and 6, $c_i$ is not a support vertex and is not a special vertex, for $i \in \{1, 2, \ldots, t-1, t+1, \ldots, r\}$. Let $u_i \in N(c_i) \setminus V(C_1)$, for $i \in \{1, 2, \ldots, t-1, t+1, \ldots, r\}$. Clearly, $\deg_{C_1}(u_i) = 2$, for $i \in \{1, 2, \ldots, t-1, t+1, \ldots, r\}$.

Let $G''_1$ be the component of $G - c_0c_1 - c_0c_r$ that contains $c_1$, $G''_2$ be the component of $G - c_0c_1 - c_0c_r$ that contains $c_0$, and $G^*$ be a graph obtained from $G''_2$ by adding a path $p_2 = x_1x_2$ and joining $c_0$ to $x_1$. Clearly, $n(G^*) = n - (3r - 2) + 2$. By Theorem 14, $ftd(G^*) \leq (2(n(G^*) + k - 1) - 1)/3$. Suppose that $ftd(G^*) < (2(n(G^*) + k - 1) - 1)/3$. Let $S^*$ be an $ftd(G^*)$-set. By Observation 1, $x_1 \in S^*$. If $c_0 \in S^*$, then $S^* \cup \{c_1, c_2, \ldots, c_r\} \cup \{u_1, u_2, \ldots, u_{t-1}, u_{t+1}, \ldots, u_r\} \setminus \{x_1, x_2\}$ is a 1FTD-set in $G$. Thus $ftd(G) < (2(n(G^*) + k - 1) - 1)/3 + 2r - 1 - 1 = (2(n - (3r - 2) + 2 + k - 1) - 1)/3 + 2r - 2 = (2(n + k) - 1)/3$, a contradiction. Thus $c_0 \notin S^*$. Then $\{x_1, x_2\} \cup \{u_1, u_2, \ldots, u_{t-1}, u_{t+1}, \ldots, u_r\} \cup \{u'_{t+1}, \ldots, u'_r\}$ is a 1FTD-set in $G$. Thus $ftd(G) < (2(n(G^*) + k - 1) - 1)/3 + 2(r - 1) - 2 = (2(n - (3r - 2) + 2 + k - 1) - 1)/3 + 2r - 4 = (2(n + k) - 1)/3 - 2$, a contradiction. Thus assume that $t = 1$. Then $S^* \setminus \{x_1, x_2\} \cup \{c_2, \ldots, c_r\} \cup \{u_2, \ldots, u_r\}$, is a 1FTD-set in $G$ of cardinality at most $(2(n + k) - 1)/3 - 2$ and so $ftd(G) < (2(n + k) - 1)/3 - 2$, a contradiction. Thus $ftd(G^*) < (2(n + k) - 1)/3 - 3$. By the inductive hypothesis, $G^* \in G_{k-1}$. Let $G'_1$ be the graph obtained from $G[G'_1 \cup \{c_0\}]$ by adding a path $p_2 = x'_1x'_2$
and joining $c_0$ to $x'_1$. Clearly, $G^*_1 \in \mathcal{H}_1$. Thus $G$ is obtained from $G^* \in \mathcal{G}_{k-1}$ and $G^*_1 \in \mathcal{H}_1$ by Procedure A. Consequently, $G \in \mathcal{H}_k \subseteq \mathcal{G}_k$.

For the converse, by Corollary 13, $V(G) \setminus L(G)$ is the unique $ftd_1(G)$-set. Now Observation 9 implies that $ftd_1(G) = (2(n + k) - 1)/3$.

**Acknowledgements**

We would like to thank the referee(s) for many helpful comments.

**References**


Received 29 August 2018

Revised 3 April 2019

Accepted 3 April 2019