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ON A TOTAL VERSION OF 1-2-3 CONJECTURE[1]

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Abstract

A total k-coloring of a graph G is a coloring of vertices and edges of G using colors of the set $\{1, \ldots, k\}$. These colors can be used to distinguish adjacent vertices of G. There are many possibilities of such a distinction. In this paper, we focus on the one by the *full sum of colors* of a vertex, i.e., the sum of the color of the vertex, the colors on its incident edges and the colors on its adjacent vertices.

This way of distinguishing vertices has similar properties to the method when we only use incident edge colors and to the corresponding 1-2-3 Conjecture.

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1. Introduction and Terminology

Let $G = (V, E)$ be a finite, undirected simple graph.

Karoński, Łuczak and Thomason introduced and investigated a coloring of the edges of a graph with positive integers so that adjacent vertices have different sums of incident edge colors [10]. More precisely, let $f : E \to \{1, 2, \ldots, k\}$ be an edge coloring of G (such a coloring is also called a *k-coloring* of G). For $x \in V$, we define

$$\sigma^e(x) := \sum_{e \ni x} f(e).$$

A k-coloring c of G is called *neighbor sum distinguishing* if $\sigma^e(x) \neq \sigma^e(y)$ whenever $xy \in E$. In other words, the vertex coloring σ^e induced by f in the above described way must be proper.

The minimum integer k for which there is a neighbor sum distinguishing coloring of a graph G will be denoted by $\chi^e(G)$. Clearly, such k does not exist when G contains K_2 as a component. Graphs without a connected component isomorphic to K_2 are called *nice graphs*.

The following elegant problem, known as the 1-2-3 *Conjecture*, was posed in [10].

Conjecture 1 [10]**.** *Let G be a nice graph. Then $\chi^e(G) \leq 3$.*

So far it is known that $\chi^e(G) \leq 5$ for any nice graph G (see [9]).

If the initial coloring f is total i.e., $f : V \cup E \to \{1, 2, \ldots, k\}$ (we call f a *total k-coloring* then), we have many possibilities to choose *palette* of colors, i.e., the distinguishing elements we take into account. For instance, cf. [12], for every vertex v we denote

$$\sigma^{ve}(v) := f(v) + \sum_{u \in N(v)} f(uv) = f(v) + \sigma^e(v),$$

where $N(v) = \{y \in V |\ vy \in E\}$ denotes (the open) neighborhood of v. Thus, $\sigma^{ve}(v)$ is the sum of incident colors of v and the color of v. We say that f is a *neighbor sum distinguishing total coloring* of G if $\sigma^{ve}(u) \neq \sigma^{ve}(v)$ for all adjacent vertices u, v in G.

Similarly as above, the minimum value of k for which there exists a neighbor sum distinguishing total coloring of a graph G will be denoted by $\chi^{ve}(G)$.

In [12] Przybyło and Woźniak posed the following problem, known as the 1-2 *Conjecture.*

Conjecture 2 [12]**.** *For every graph G, we have $\chi^{ve}(G) \leq 2$.*

In this context, the best upper bound is due to Kalkowski [8] and equals 3.

Also another possibility was considered by Flandrin *et al.* in [7] where, for $x \in V$, the authors consider the following sum

$$\sigma^{en}(x) = \sum_{e \ni x} f(e) + \sum_{y \in N(x)} f(y),$$

where f is a total k-coloring of G.

The value $\sigma^{en}(x)$ is called *an expanded sum at* x. A total k-coloring f of G is called *neighbor expanded sum distinguishing* if

$$\sigma^{en}(x) \neq \sigma^{en}(y)$$

whenever $xy \in E(G)$. The corresponding invariant, i.e., the minimum value of k for which such a neighbor expanded sum distinguishing total k-coloring of G exists is denoted by $\chi^{en}(G)$.

The following conjecture is stated in [7].

Conjecture 3 [7]**.** *For every graph G, we have $\chi^{en}(G) \leq 2$.*

In this paper we consider the seemingly last remaining extension of the concept of Karoński, Łuczak and Thomason towards total colorings, namely, we would like to distinguish vertices by *full sums*, defined for a vertex x by

$$\sigma^{ven}(x) := f(x) + \sum_{e \ni x} f(e) + \sum_{y \in N(x)} f(y),$$

where f is a total k-coloring of G. We call f a *neighbor full sum distinguishing total k-coloring* of G.

The corresponding parameter, expressing the minimum k admitting existence of such a coloring is denoted by $\chi^{ven}(G)$. It is easy to see that no such desired distinguishing coloring exists if a graph is not nice.

In the following sections, we provide some arguments in favor of the following conjecture.

Conjecture 4. *Let G be a nice graph. Then $\chi^{ven}(G) \leq 3$.*

Remark 5. It is easy to observe some similarities between 1-2-3 Conjecture (Conjecture 1) and just formulated Conjecture 4.

Firstly, it is easy to see that $\chi^e(K_3) = \chi^{ven}(K_3) = 3$. Thus, in general, we need three colors in order to distinguish adjacent vertices in both cases. Secondly, the notion of "nice" is the same for the two variants (i.e., neither the parameter χ^e nor the parameter χ^{ven} exist for graphs which are not nice).

On the other hand, there are also similarities between 1-2 Conjecture (Conjecture 2) and Conjecture 3. In particular, the both corresponding parameters are well defined for all graphs.

Remark 6. In all of the above problems, we considered general colorings and color sums. The problems differed in the considered *palettes*, i.e., elements of the graph, which we took into account. If we limit ourselves to the elements "close" to the given vertex, we have three options: the vertex itself, incident edges and neighboring vertices.

By denoting (symbolically) these three options by v (vertex), e (edges) and n (neighbors) we have the following seven options when it comes to palettes. Denote the palette by $\mathcal{P}$. If $\mathcal{P} = \{v\}$, then the coloring corresponds to the usual proper coloring of the vertices of a graph and has been intensively studied since the beginning of the graph theory. The cases $\mathcal{P} = \{e\}$, $\mathcal{P} = \{v, e\}$ and $\mathcal{P} = \{e, n\}$ are discussed above. The case $\mathcal{P} = \{n\}$ corresponds to the so-called *lucky labellings* and was introduced in [4]. In turn the case of $\mathcal{P} = \{v, n\}$ is studied in [1].

Therefore, the case that we are considering in this paper ($\mathcal{P} = \{v, e, n\}$) is the last, natural and not yet studied case. This is an additional motivation for our research.

Remark 7. The four problems described above are only a modest part of a family of problems concerning distinguishing vertices of a graph by coloring the edges, or vertices. In addition to the consideration of different palettes, one can consider different types of colorings (proper or general), one can distinguish all or only neighboring vertices, and the distinction can take into account not only the sums of colors but also multisets or sets. Sometimes, the notation taking into account all these elements is used. In this notation, parameters described above would be denoted by gndi_Σ, tgndi_Σ, egndi_Σ, fgndi_Σ, respectively. Because all colorings we consider are general, we distinguish only neighboring vertices and we always do this by means of sums, we apply somewhat simpler notation. See Seamone [13] for a survey.

2. Paths and Cycles

We use Bondy and Murty's book [2] for terminology and notation not defined here.

Proposition 8. *Let P_n denote the path of order n. If $n \geq 4$, then $\chi^{ven}(P_n) = 2$, and $\chi^{ven}(P_3) = 1$.*

Proof. Denote by $x_1, \ldots, x_n$ the consecutive vertices of the path P_n, $n \geq 4$. By *odd (even) vertices* we mean the vertices with odd (even) indices, respectively. Let us put $f(x_1) = f(x_3) = \cdots = 1$ on odd vertices, and $f(x_2) = f(x_4) = \cdots = 2$ on even vertices, and $f(x_i x_{i+1}) = 1$ for $i = 1, \ldots, n-1$. Then, it is easy to see that $\sigma^{ven}(x_1) = \sigma^{ven}(x_n) = 4$, $\sigma^{ven}(x) = 7$ for all inner odd vertices, $\sigma^{ven}(x) = 6$

for all inner even vertices. Thus, all adjacent vertices are distinguished. Clearly, in the case of P_3, one color is enough. ■

Proposition 9. *Let C_n denote the cycle of order n. For $n \geq 4$, $\chi^{ven}(C_n) = 2$ and $\chi^{ven}(C_3) = 3$.*

Proof. If n is even then coloring vertices and edges of the cycle as in the case of paths, we get a coloring distinguishing neighbors by full sums. So, let n be odd and denote by $x_0, x_1, \ldots, x_{n-1}$ the consecutive vertices of the cycle C_n, $n \geq 5$.

We define f as follows:
$f(x_0) = 1$,
$f(x_1) = 2$,
$f(x_i) = 1$ for i odd, $i \neq 1$,
$f(x_i) = 2$ for i even, $2 \leq i \leq n-1$,
$f(x_i x_{i+1}) = 2$ if $i \in \{0, 1\}$,
$f(x_i x_{i+1}) = 1$ for all remaining edges.

It can be verified that $\sigma^{ven}(x_0) = 8$, $\sigma^{ven}(x_1) = 9$, $\sigma^{ven}(x_2) = 8$, $\sigma^{ven}(x_i) = 7$ for all remaining odd vertices, and $\sigma^{ven}(x_i) = 6$ for all remaining even vertices. Thus, the above defined function is a neighbor full sum distinguishing total 2-coloring of C_n. For $n = 3$, it is easy to see that two colors do not suffice to get such a coloring, while three are enough. Indeed, it is sufficient to put 1 on vertices and 1,2,3 on edges of C_3. ■

3. BIPARTITE GRAPHS

Observe that if we color all edges and vertices with 1, then for every vertex v we obtain $\sigma^{ven}(v) = 2 \times d(v) + 1$. If there are no adjacent vertices of the same degree in G, then $\chi^{ven}(G) = 1$.

Theorem 10. *If G is a connected bipartite graph of order $n \geq 3$, then $\chi^{ven}(G) \leq 2$.*

Proof. Let x be a vertex of the maximum degree Δ of the graph $G = (V, E)$. By Propositions 8 and 9, we may assume that $d(x) \geq 3$. We consider a spanning tree T of a graph G obtained by the BFS-algorithm (breadth-first search) rooted at x. First, we define a total coloring f for all vertices and edges of G except these edges which belong to T, and only just then choose colors for these remaining edges. At the end we might also be forced to make some alterations concerning x and its incident edges.

We thus start by setting $f(v) = 1$ for every vertex $v \in V \backslash N(x)$ and $f(v) = 2$ whenever $v \in N(x)$. Moreover, we put $f(e) = 1$ for every edge $e \in E \backslash E(T)$. Note that all edges incident with x belong to $E(T)$.

Next, we will be consecutively assigning colors to edges of T in such a way that all vertices on spheres of even radius (greater than 0) centered in x will have even full sums (in G) and all vertices on spheres of odd radius centered in x will have odd full sums (note that the distances of any vertex $v \in V$ from x are the same in G and T). For this goal, we will be analyzing consecutive vertices of T from subsequent spheres centered in x with decreasing radiuses. We thus start from (all) vertices on the sphere with the largest radius (each such vertex is a leaf of T).

In general, suppose y ($y \neq x$) is a consecutive vertex we are about to process, and that it belongs to $S_k(x)$, the *sphere of radius* k centered in x i.e., the set of vertices of G distant by k from x. Observe that the facts that T is a BFS-tree and G is bipartite imply that every edge $e \in E$ joins two consecutive spheres centered in x i.e., the vertices of each sphere are independent. Then all edges $yu \in E(G)$ with $u \in S_{k+1}(x)$, if there are any, are already colored. Let v be the unique predecessor of y in T, hence $v \in S_{k-1}(x)$ and yv is the only yet uncolored edge incident with y. Then, we color the edge yv with either 1 or 2 in such a way that $\sigma^{ven}(y)$ (in G) is even if k is even, or odd otherwise.

We continue in the manner described above until we have colored all edges of T. Note that afterwards, the only possible conflicts between full sums of adjacent vertices in G are between x and some of its neighbors. Then either f already meets our requirements, or we have the same full sum in x and in some $y' \in N(x)$ (possibly in more than one). In the latter case however, by our construction, we have

$$\begin{aligned}
\sigma^{ven}(x) &= f(x) + \sum_{u \in N(x)} f(u) + \sum_{e \ni x} f(e) \\
&\geq f(x) + f(y') + 2(\Delta - 1) + (\Delta - 1) + f(xy') \\
&= f(y') + f(x) + (\Delta - 1) + 2(\Delta - 1) + f(xy') \\
&\geq f(y') + \sum_{u \in N(y')} f(u) + \sum_{e \ni y'} f(e) = \sigma^{ven}(y'),
\end{aligned} \tag{1}$$

and hence $\sigma^{ven}(x) = \sigma^{ven}(y')$ implies that all edges incident with x, except possibly for xy', must be colored with 1. Then, we change colors of the vertex x and all its incident edges from 1 to 2 and vice versa. As a result, the parity of the full sum at every neighbor of x will not change, while the full sum of x will get larger than the ones corresponding to its neighbors, since afterwards x (of degree $\Delta \geq 3$) will have all neighbors colored with 2 and at most one incident edge colored with 1, hence the obtained $f : V \cup E \to \{1, 2\}$ will distinguish neighboring vertices of G by full sums. ■

The proof above provides an algorithm in which the parities of full sums in the independent sets making up a bipartition of G are different, with one possible exception — a root of a BFS-tree. We can observe that such exception might be unavoidable, as indeed there are bipartite graphs for which there does not exist a total coloring f inducing odd full sums in one set of the bipartition and even full sums in the other one. Namely, let $G = (X, Y; E)$ be a bipartite graph with both sets X and Y of odd orders. And let every vertex of G have odd degree. Then, the sum of all full sums in G is even since

$$\sum_{v \in X \cup Y} \sigma^{ven}(v) = 2 \sum_{e \in E} f(e) + \sum_{v \in X \cup Y} (d(v) + 1) f(v).$$

On the other hand however, the sum of all full sums in G must be odd, if we assume that all full sums are even in X and odd in Y, or vice versa.

An analogous algorithm works even more simply in the case of χ^{en}.

Theorem 11. *If G is a bipartite graph, then $\chi^{en}(G) \leq 2$.*

Proof. For $n \leq 2$, the thesis is straightforward. Otherwise, we use the same initial coloring of the vertices, and then the same algorithm of coloring the edges as in the proof of Theorem 10, but taking into account the parities of $\sigma^{en}(y)$ instead of $\sigma^{ven}(y)$ for every $y \in V(G) \backslash \{x\}$ (we do not have to assume that $\Delta(G) \geq 3$ either). Then, by similar estimations as in (1), the sum at x is larger than the sums at its neighbors, and thus all adjacent vertices are distinguished by expanded sums in G. ■

Remark 12. Note that in the case of bipartite graphs, there is a significant difference between the χ^{ven} parameter and the simplest χ^{e} parameter. Namely, in the case of bipartite graphs, we know that χ^{e} is not more than three. The problem of characterization of graphs for which $\chi^{e} = 2$ was discussed, among others in the papers ([5, 11]) and was finally resolved at work ([14]).

4. Regular Graphs and Graphs with $\chi = 3$

In [3] Chartrand *et al.* considered a general coloring of the edges of a graph with the elements of $[k] := \{1, 2, \ldots, k\}$. Such a coloring is an *irregular assignment* if, for any two vertices x, y of G, the sum of colors of edges incident to x differs from the sum of colors of edges incident to y. The *irregularity strength* of a graph G, denoted by $s(G)$, is the minimum number k such that G has an irregular assignment from the set $[k]$. Let K_n denote the complete graph on n vertices. In this case, there is no difference if we distinguish all or only neighboring vertices. Therefore, $s(K_n) = \chi^{e}(K_n)$. In [3] the authors showed that $s(K_n) = 3$, for $n \geq 3$. So, if we start with the appropriate coloring for edges of K_n, and then put 1 on each vertex of this graph, we obtain a neighbor full sum distinguishing total

3-coloring of K_n. Since the sum of colors of the vertices belonging to the closed neighborhood of any vertex of K_n is the same and equals n. Hence, we cannot color the vertices and the edges in a desired manner using just two colors (such a coloring would imply the existence of an irregular assignment of K_n). Thus, the following proposition holds.

Proposition 13. *For $n \geq 3$, $\chi^{ven}(K_n) = 3$.*

Let $K_r(t)$ denote a complete r-partite graph with t vertices in each class (r, $t \geq 1$). Faudree *et al.* [6] proved that if $G = K_r(t)$, $r > 2$ and $t \geq 1$, then $s(K_r(t)) = 3$. Since G is regular, it follows that G admits a neighbor full sum distinguishing total 3-coloring.

Proposition 14. *If $r \geq 3$ and $t \geq 1$, then $\chi^{ven}(K_r(t)) \leq 3$.*

In [10] the authors studied a coloring of the edges of the graph with the elements of an abelian group.

Theorem 15 [10]**.** *Let $(\Gamma, +)$ be a finite abelian group of odd order and G be a connected graph of order at least 3. If G is $|\Gamma|$ colorable, then there exists a neighbor sum distinguishing coloring of the edges of G with the elements of Γ.*

Corollary 16. *Let G be a connected graph of order at least 3. If G is $(2k+1)$-colorable for $k \geq 1$, then $\chi^{ven}(G) \leq 2k+1$.*

Proof. Suppose that G is $(2k+1)$-colorable. By Theorem 15, there exists a neighbor sum distinguishing coloring of the edges of G with the elements of the group $\mathbb{Z}_{2k+1}$. Afterwards, put 0 on every vertex of G. It can be easily seen that a total coloring with the elements of $\mathbb{Z}_{2k+1}$ obtained in this manner distinguishes neighbors by full sums. It suffices now to replace 0 by $2k+1$ and apply the addition in $\mathbb{N}$ in order to obtain the desired total coloring. ∎

Remark 17. Since we showed that $\chi^{ven}(G) = 2$ for a connected bipartite graph G having at least three vertices and we will prove that this index is less than or equal to 5 for any connected graph G of order at least 3, the above corollary is interesting only for the class of graphs with chromatic number 3.

5. Split Graphs

Let $G = (S \cup K, E)$ be a split graph, i.e., a graph such that S is a stable set (possibly empty) and K is a clique on $k \geq 1$ vertices. We assume that K is maximal, i.e., every vertex in S is not adjacent with at least one vertex in K.

Proposition 18. *If G is a split graph of order $n \geq 3$, then $\chi^{ven}(G) \leq 3$.*

Proof. First of all, we may always suppose that $|K| \geq 2$ and every vertex of S has degree at least 1.

We also assume that K is maximal, i.e., every vertex in S is not connected to at least one vertex in K.

If $|K| = 2$, i.e., $K = \{u, v\}$ with $uv \in E(G)$, then $|S| \geq 1$ and each vertex of S is adjacent either with u or with v exclusively. Without loss of generality, we suppose that $d(u) \leq d(v)$. Then we color all the edges incident to v, except uv, by 2, every vertex and all the edges incident to u by 1. Then,

$\sigma^{ven}(v) = 3(d(v) - 1) + 3,$

$\sigma^{ven}(u) = 2(d(u) - 1) + 3.$

Thus, $\sigma^{ven}(v) > \sigma^{ven}(u)$ and $\chi^{ven}(G) \leq 2$.

Now, we suppose that $|K| \geq 3$. If $|S| = 0$, then G is a complete graph with at least 3 vertices. Then, by Proposition 13, $\chi^{ven}(G) = 3$.

Now, suppose that $|S| \geq 1$. First, we order the vertices $v_1, \ldots, v_k$ of K in such a way that $i > j$ implies that $d(v_i) \geq d(v_j)$. Then, we color the edges and vertices of G_K, the subgraph induced by K, in such a way that $\chi^{ven}(G_K) = 3$ and that $i > j$ implies $\sigma^{ven}(v_i) > \sigma^{ven}(v_j)$ (in G_K). Finally, we put 1 on the vertices of S and the edges between S and K. It is easy to see that if $x \in S$ and $y \in K$, $\sigma^{ven}(x) < \sigma^{ven}(y)$. Therefore, this coloring is a neighbor full sum distinguishing 3-coloring. ■

Remark 19. It is easy to find split graphs which admit a neighbor full sum distinguishing total 2-coloring.

For example, let G be the graph formed by adding a pendant edge to K_3. Then, $\chi^{ven}(G) = 2$. Note that $\chi^{ven}(K_3) = 3$, as well as, for the remaining complete graphs K_n, $n > 3$.

Question 20. *Does there exist a simple characterization of split graphs which admit a neighbor full sum distinguishing total* 2*-coloring*?

6. $\chi^{ven} \leq 5$

As already mentioned, in [9], Kalkowski *et al.* showed that for every graph G without components isomorphic to K_2 there exists a coloring of the edges of G with the elements of $\{1, \ldots, 5\}$ such that the resulting vertex coloring σ^e of G is proper. This implies at once the following corollary.

Corollary 21. *If G is a connected regular graph of order at least* 3*, then* $\chi^{ven}(G) \leq 5$.

Using the approach from [9] we will show that the same holds for all connected graphs of order at least 3. As one may start constructing a total coloring e.g.

by choosing first arbitrary admissible vertex colors, this follows by the following lemma, within which f denotes a total coloring of G.

Lemma 22. *Given any connected graph $G = (V, E)$ of order at least 3 and any set $\{f(v) : v \in V\}$ of integers assigned to its vertices, there exist $f(e) \in \{1, 2, 3, 4, 5\}$ for $e \in E$ such that f is a neighbor sum distinguishing total coloring of G.*

Proof. Suppose we are given a connected graph $G = (V, E)$ of order at least 3 and a set $\{f(v) : v \in V\}$ of integers. Order the vertices of G into a sequence $v_1, v_2, \ldots, v_n$ so that $d(v_n) \geq 2$ and for every vertex v_i with $i < n$ there exists an edge $v_iv_j \in E$ with $j > i$. Every such edge will be called a *forward edge* of v_i, while v_j is a *forward neighbor* of v_i, and the other way round, i.e., v_iv_j and v_i will be referred to as a *backward edge* and a *backward neighbor*, respectively, of v_j. At every step of the coloring algorithm described below, we will denote by $f_T(e)$ and $\sigma_T(v)$ the up-to-date color of an edge $e \in E$ and total sum of a vertex $v \in V$, respectively. Initially we assign color 3 to every edge of G (i.e., initially $f_T(e) = 3$ for $e \in E$). With every consecutive vertex v_i in the sequence we will now subsequently associate a two element set

$$W_i \in \mathcal{W} := \{\{a, a+2\} : a \equiv 0 \pmod 4 \ \text{ or } \ a \equiv 1 \pmod 4\}$$

disjoint with the corresponding sets associated with its backward neighbors, and make sure that ever since defining such set W_i, the total sum $\sigma_T(v_i)$ will always belong to W_i for every vertex $v_i \in V$ (thus assuring its distinction from the total sums of the backward neighbors of v_i) — a possible exceptions to this rule will only be admitted while analyzing the last vertex in the sequence (see details below). To achieve this goal we will allow:

(i) subtracting or adding 2 (or doing nothing) to the color of every backward edge v_kv_i of v_i so that $\sigma_T(v_k) \in W_k$ afterwards,

(ii) subtracting or adding 1 to the color of the *first forward edge* of v_i,

i.e., a forward edge v_iv_j of v_i with the least j. While applying the rules above we will additionally require that after analyzing v_i (and committing admissible alterations of colors of the edges incident with v_i) and choosing appropriate W_i:

(iii) if $c_T(v_iv_j) = 2$ then, $\sigma_T(v_i) = \min W_i$, while if $c_T(v_iv_j) = 4$ then, $\sigma_T(v_i) = \max W_i$, where v_iv_j is the first forward edge of v_i.

Note that the rules (i)–(iii) guarantee that the colors of all edges will belong to the set $\{1, 2, 3, 4, 5\}$ at the end of our algorithm.

Suppose we begin our algorithm. For the first vertex v_1 we do not need to introduce any alterations, hence we have $\sigma_T(v_1) = f(v_1) + 3d(v_1)$, and we choose (the only) 2-element set from $\mathcal{W}$ to which $\sigma_T(v_1)$ belongs and set it as W_1.

Now, assume that we are about to analyze a vertex v_i with $1 < i < n$ and so far all our requirements are fulfilled and all rules have been obeyed. Denote the number of backward neighbors of v_i by d. Observe that since v_i has at least one forward neighbor, then via admissible alterations consistent with (i) and (ii), we may obtain $2d+3$ distinct sums at v_i, which are consecutive integers. At most 2 of these cannot be achieved consistently with (iii). Thus, we are left with a set of at least $2d+1$ options for $\sigma_T(v_i)$, containing elements (not necessarily both) from at least $d+1$ 2-element sets from $\mathcal{W}$. At least one of these 2-element sets is not associated with any of d backward neighbors of v_i. We arbitrarily choose one such set to be W_i, and perform alterations on the edges incident with v_i consistent with (i)–(iii) so that $\sigma_T(v_i) \in W_i$ afterwards. We continue in the same manner until we reach the last vertex in the sequence.

Finally, we analyze v_n. By (i) we may obtain $d(v_n)+1$ distinct sums at v_n, say $a, a+2, \ldots, a+2d(v_n)$. If $a \in \{2,3\} \pmod 4$, we perform the admissible alterations consistent with (i) so that $\sigma_T(v_n) = a$. Then however, $\sigma_T(u) \in \{0,1\} \pmod 4$ for every (backward) neighbor u of v_n, which thus cannot be in conflict with v_n. Therefore, we may assume that $a \in \{0,1\} \pmod 4$, and that not all neighbors of v_n are associated with the same 2-element list (if the later was not true, i.e., if the total sums of all neighbors of v_n belonged to the same $W \in \mathcal{W}$, then as $d(v_n) \geq 2$, we would have at least 3 available options for $\sigma_T(v_n)$, at least one of which would not belong to W, hence we could again distinguish v_n from all its neighbors). Let v_l be a neighbor of v_n with $\sigma_T(v_l) \in W' \in \mathcal{W}$ such that $W' \cap \{a, a+2\} = \emptyset$. Then we apply (i) (if necessary) to all backward edges of v_n so that each of them except for $v_l v_n$ has the smaller of the two admissible colors and $v_l v_n$ — the larger one. Then $\sigma_T(v_n) = a+2$ is distinct from all total sums of the neighbors of v_n. At the end, setting for all edges $e \in E$, $f(e) = f_T(e)$, we obtain a desired total coloring f of G. ∎

Corollary 23. *If G is a connected graph of order at least 3, then $\chi^{ven}(G) \leq 5$.*

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