Discussiones Mathematicae
Graph Theory 39 (2019) 757–769
doi:10.7151/dmgt.2188

ON THE RESTRICTED SIZE RAMSEY NUMBER INVOLVING A PATH P_3

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Abstract

For any pair of graphs G and H, both the size Ramsey number $\hat{r}(G,H)$ and the restricted size Ramsey number $r^*(G,H)$ are bounded above by the size of the complete graph with order equals to the Ramsey number $r(G,H)$, and bounded below by $e(G)+e(H)-1$. Moreover, trivially, $\hat{r}(G,H) \leq r^*(G,H)$. When introducing the size Ramsey number for graph, Erdős *et al.* (1978) asked two questions; (1) Do there exist graphs G and H such that $\hat{r}(G,H)$ attains the upper bound? and (2) Do there exist graphs G and H such that $\hat{r}(G,H)$ is significantly less than the upper bound?

In this paper we consider the restricted size Ramsey number $r^*(G,H)$. We answer both questions above for $r^*(G,H)$ when $G = P_3$ and H is a connected graph.

Keywords: restricted size Ramsey number, path, connected graph, star.

2010 Mathematics Subject Classification: 05C55, 05D10.

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1. Introduction

Let G be a graph, where the vertex set, edge set, order, size, minimum degree, maximum degree, and its complement are $V(G)$, $E(G)$, $v(G)$, $e(G)$, $\delta(G)$, $\Delta(G)$, and $\overline{G}$, respectively. The degree of a vertex v in G is denoted by $d(v)$. If H is a subgraph of G, then $G - H$ is a graph obtained from G by deleting the edges of H [13]. Thus, $V(G - H) = V(G)$. For further terminologies in graphs, please see [6].

A graph F is called a graph *arrowing* a pair of graphs G and H, denoted by $F \rightarrow (G, H)$, if any 2-coloring (say red and blue) of the edges of F contains a red G or a blue H. In 1978, Erdős *et al.* in [9] introduced the question of how few edges in such an arrowing graph can be. The *size Ramsey number* $\hat{r}(G, H)$ is the smallest number of edges that an arrowing graph can have. If the order of the arrowing graph is equal to the Ramsey number $r(G, H)$, the smallest number of edges in the arrowing graph is called the *restricted size Ramsey number* $r^*(G, H)$. The *Ramsey number* $r(G, H)$ itself is the smallest number r such that K_r is an arrowing graph for a pair of graphs G and H. For a diagonal case, when $G = H$, we write $\hat{r}(G)$ and $r^*(G)$ instead of $\hat{r}(G, G)$ and $r^*(G, G)$, respectively.

To find the exact values for the (restricted) size Ramsey number for a pair of graphs is a challenging but difficult problem even for small order graphs. The complete list of the (restricted) size Ramsey numbers for all pairs of graphs of order at most four with no isolates can be found in [13], and for the size and the restricted size Ramsey numbers for all pairs of forest graphs of order at most five with no isolates can be found in [18]. For further results about the (restricted) size Ramsey number for graphs can be found in [2, 3, 12]. The current results on the exact values of the (restricted) size Ramsey number for graphs can be found in [16, 19, 21–23], and on the bounds of the (restricted) size Ramsey number involving paths in [7, 8, 16, 17].

The (restricted) size Ramsey number for any pair of graphs G and H meet the following inequalities.

$$e(G) + e(H) - 1 \leq \hat{r}(G, H) \leq r^*(G, H) \leq \binom{r(G, H)}{2}. \tag{1}$$

The first inequality was given by Erdős and Faudree in [10]. The diagonal version of these bounds for the size Ramsey number was given by Harary and Miller in [15].

When introducing the size Ramsey number for graphs, Erdős *et al.* in [9] proposed two preliminary questions.

(i) Do there exist graphs G and H such that $\hat{r}(G, H)$ attains the upper bound?

(ii) Do there exist o-sequences?

An o-sequence is defined to give the precise meaning to the idea that $\hat{r}(G,H)$ is 'significantly' less than $\binom{r(G,H)}{2}$. For diagonal case, a sequence of graphs $\{G_n\}$ is called o-sequence if $\hat{r}(G_n) = o\binom{r(G_n)}{2}$.

Erdős *et al.* [9] provided a positive answer to the first question by showing that $\hat{r}(K_m, K_n) = r^*(K_m, K_n) = \binom{r(K_m,K_n)}{2}$ for all values of m and n. This result is due to Chvátal (by personal communication). It is also true for some pairs of small graphs, namely, (P_3, C_4), (C_4, C_4), and $(C_4, K_4 - e)$ (see [11, 13]).

Erdős *et al.* [9] also give a positive answer to the second question by showing that it is true for the size Ramsey number of stars and some graphs obtained by star operation. By considering the value of $\hat{r}(nK_2)$ (see [15]) and the value of $r(nK_2)$ (see [20]), we conclude that $\{nK_2\}$ is also an o-sequence. Note that the size Ramsey number of stars, $\hat{r}(K_{1,m}, K_{1,n}) = m + n - 1$, is attaining the lower bound. However, it is still open whether every sequence of graph $\{(G,H)\}$ for which $\hat{r}(G,H)$ is attaining the lower bound always belong to o-sequence. For a special case, when $G = H = P_3$, the lower and upper bounds are the same, which is $\hat{r}(P_3) = r^*(P_3) = 3$. They are equal to the size of K_3 [4, 10].

In this paper we are concerned with the restricted size Ramsey number $r^*(P_3, H)$ for any connected graph H. We characterize all connected graphs H such that $r^*(P_3, H)$ attains the upper and lower bounds. We also show that $\{(P_3, H)\}$ with $r^*(P_3, H)$ attaining the lower bound belongs to restricted o-sequence. The main results of this paper are the following.

Theorem 1. *Let H be a connected graph and $v(H) = n$.*

$$r^*(P_3, H) = \binom{r(P_3, H)}{2}$$

if and only if one of the following holds:

(a) *n is even, $n \geq 4$, and $H = K_n - \frac{n}{2}K_2$,*

(b) *n is odd, $n \geq 5$, and H is one of the following*

  (b1) *H with $\beta(\overline{H}) = \frac{n-1}{2}$ and $\Delta(H) = n - 1$,*

  (b2) *$H = K_n - \left(P_3 \cup \left(\frac{n-1}{2} - 1\right) K_2\right)$, or*

  (b3) *$H = K_n - \left(C_3 \cup \left(\frac{n-1}{2} - 1\right) K_2\right)$.*

Theorem 2. *Let H be a connected graph and $v(H) = n$.*

$$r^*(P_3, H) = e(P_3) + e(H) - 1$$

if and only if $H = K_{1,n-1}$ and n is even.

2. Preliminaries

In 1972, Chvátal and Harary [5] gave the Ramsey number for P_3 and any graph with no isolates, as stated in Theorem A. In finding $r^*(P_3, H)$, the order of graph F satisfying $F \to (P_3, H)$ will be determined by this result.

Theorem A [5]**.** *For any graph H with no isolates,*

$$r(P_3, H) = \begin{cases} v(H), & \overline{H} \text{ has 1-factor}, \\ 2v(H) - 2\beta(\overline{H}) - 1, & \text{otherwise}, \end{cases}$$

where $\beta(\overline{H})$ is the maximum number of independent edges in the complement of H.

From the definition of the (restricted) size Ramsey number, the following monotonicity property is obvious. If $G' \subseteq G$ and $H' \subseteq H$, then

$$\hat{r}(G', H') \leq \hat{r}(G, H), \tag{2}$$

and

$$r^*(G', H') \leq r^*(G, H). \tag{3}$$

Let F be a graph with all the edges colored by red and blue. Following the idea of Faudree and Sheehan in [14], we define a graph that represents the subgraph induced by the blue edges in such a coloring. A graph G_F has $V(G_F) = V(F)$ and $E(G_F)$ that consists of red edges in the coloring of F and edges in $\overline{F}$. It is important to notice that $\overline{G}_F$ is precisely the subgraph of F induced by blue edges. The notation of the graph G_F will be extensively used to prove the main results.

3. Proof of Theorem 1

The following lemmas are used to prove this theorem.

Lemma 3 [14]**.** *For a positive integer $n \geq 2$,*

$$\hat{r}(P_3, K_n) = r^*(P_3, K_n) = 2(n-1)^2.$$

Lemma 4. *Let H be a connected graph of order n such that $\beta = \beta(\overline{H})$. If $n \geq 4$ and $\beta \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$, then*

$$r^*(P_3, H) \leq \binom{2n - 2\beta - 1}{2} - 1.$$

Proof. Note that $\beta(\overline{H})$ is the maximum number of independent edges in the complement of H. For $\beta = 0$, by Theorem A we obtain $r(P_3, H) = 2n - 1$ and from Lemma 3 we have $r^*(P_3, H) = 2(n-1)^2 < \binom{2n-1}{2} - 1$.

For $1 \le \beta \le \lfloor \frac{n}{2} \rfloor - 1$, by Theorem A we obtain $r(P_3, H) = 2n - 2\beta - 1$. Note that for any β, there will be a plenty of non-isomorphic graphs H such that $\beta(\overline{H}) = \beta$. However, such a graph H must satisfy $H \subseteq K_n - \beta K_2$. By (3) we have $r^*(P_3, H) \le r^*(P_3, K_n - \beta K_2)$. Therefore, we only need to show that $r^*(P_3, K_n - \beta K_2) \le \binom{2n-2\beta-1}{2} - 1$.

Now, let $F = K_{2n-2\beta-1} - K_2$. We will show that $F \to (P_3, K_n - \beta K_2)$ for any β, where $1 \le \beta \le \lfloor \frac{n}{2} \rfloor - 1$. To do so, consider any 2-coloring of the edges of F having no red P_3. The graph G_F will be isomorphic to a subgraph of either $P_4 \cup (n-\beta-3)K_2 \cup K_1$ or $P_3 \cup (n-\beta-2)K_2$. Since $K_{n-\beta} \subseteq K_n - \beta K_2$, we will show that we can construct $K_n - \beta K_2$ in $\overline{G}_F$ by constructing $K_{n-\beta}$ first.

If $G_F \subseteq P_4 \cup (n-\beta-3)K_2 \cup K_1$, then in $\overline{G}_F$ there is a $K_{n-\beta}$ induced by two nonadjacent vertices from P_4 (one of degree 1 and one of degree 2), vertex of K_1, and $n-\beta-3$ non-adjacent vertices in $(n-\beta-3)K_2$. Furthermore, in G_F there are at least $n-\beta-2$ (one is from P_4) nonadjacent vertices of degree 1 (or of degree $2n-2\beta-3$ in $\overline{G}_F$) which are not in $K_{n-\beta}$. Since for $n \ge 4$ it is true that $n-\beta-2 \ge \beta$ and $2n-2\beta-3 \ge n-\beta-1$, we can extend this $K_{n-\beta}$ to $K_n - \beta K_2$ in $\overline{G}_F$.

If $G_F \subseteq P_3 \cup (n-\beta-2)K_2$, then in $\overline{G}_F$ there is a $K_{n-\beta}$ induced by two nonadjacent vertices from P_3 and $n-\beta-2$ vertices each from K_2 in $(n-\beta-2)K_2$. Furthermore, in G_F there are at least $n-\beta-2$ nonadjacent vertices of degree 1 (or of degree $2n-2\beta-3$ in $\overline{G}_F$) which are not in $K_{n-\beta}$. Since for $n \ge 4$ it is true that $n-\beta-2 \ge \beta$ and $2n-2\beta-3 \ge n-\beta-1$, we can extend this $K_{n-\beta}$ to $K_n - \beta K_2$ in $\overline{G}_F$.

Since $\overline{G}_F$ is exactly the subgraph of F induced by blue edges, $F \to (P_3, K_n - \beta K_2)$. Thus $r^*(P_3, K_n - \beta K_2) \le \binom{2n-2\beta-1}{2} - 1$ for any $1 \le \beta \le \lfloor \frac{n}{2} \rfloor - 1$. ■

Next, we examine $r^*(P_3, H)$ when $r(P_3, H) = n$. In Lemma 5 we give the first conditions for a connected graph H with $r(P_3, H) = n$, so that $r^*(P_3, H)$ attains the upper bound of (1). Odd stars, wheels, and fans meet the conditions, so we also obtain $r^*(P_3, H)$ for H are those graphs.

Lemma 5. *Let H be a connected graph of order n such that $\beta(\overline{H}) = \frac{n-1}{2}$, and $\Delta(H) = n - 1$ where n is odd and $n \ge 3$.*

$$r^*(P_3, H) = \binom{n}{2}.$$

Proof. Since $\beta(\overline{H}) = \frac{n-1}{2}$, by Theorem A we obtain $r(P_3, H) = n$. As a consequence, $r^*(P_3, H) \le \binom{n}{2}$.

Now, let F be a graph with $v(F) = n$ and $e(F) = \binom{n}{2} - 1$. The only possible F is isomorphic to $K_n - K_2$. Since n is odd, we can give a 2-coloring of the edges of F such that $\frac{n-1}{2}$ independent edges which are involving only one vertex from non-edge are red and the remaining edges are blue. In this 2-coloring, there is no red P_3 and the maximum degree of the subgraph of F induced by blue edges is $n-2$. Since $\Delta(H) = n-1$, we cannot have a blue H. Thus $F \nrightarrow (P_3, H)$ and $r^*(P_3, H) \geq \binom{n}{2}$. ■

The maximal graph H with $v(H) = n$ and $\beta(\overline{H}) = \lfloor \frac{n}{2} \rfloor$ is $K_n - \lfloor \frac{n}{2} \rfloor K_2$, which is the graph obtained by deleting a maximal matching from K_n. In Lemma 6 we consider this graph. Note that if n is even, then the graph H is a cocktail party graph.

Lemma 6. *For $n \geq 3$,*

$$r^*(P_3, K_n - \left\lfloor \frac{n}{2} \right\rfloor K_2) = \binom{n}{2}.$$

Proof. If n is odd, then $\beta(\overline{H}) = \frac{n-1}{2}$ and $\Delta(H) = n-1$. From Lemma 5 we have $r^*(P_3, H) = \binom{n}{2}$.

If n is even, then H is an $(n-2)$-regular graph. Since $\beta(\overline{H}) = \frac{n}{2}$ ($\overline{H}$ has 1-factor), by Theorem A we obtain $r(P_3, H) = n$. As a consequence, $r^*(P_3, H) \leq \binom{n}{2}$.

Now, let F be a graph with $v(F) = n$ and $e(F) = \binom{n}{2} - 1$. The only possible F is isomorphic to $K_n - K_2$. Since n is even, we can give a 2-coloring of the edges of F such that $\frac{n}{2}$ independent edges are red and the remaining edges are blue. In this 2-coloring, there is no red P_3 and in the subgraph of F induced by blue edges there are exactly two vertices of degree less than $n-2$. Since H is an $(n-2)$-regular graph, we cannot have a blue H. Thus $F \nrightarrow (P_3, H)$ and $r^*(P_3, H) \geq \binom{n}{2}$. ■

Since $r^*\left(P_3, K_n - \lfloor \frac{n}{2} \rfloor K_2\right)$ is still attaining the upper bound of (1), we will go further by considering the graphs obtained by removing one more edge from $K_n - \lfloor \frac{n}{2} \rfloor K_2$.

Lemma 7. *Let $H_1 = K_n - \lfloor \frac{n}{2} \rfloor K_2$ and $H = H_1 - e$ for $n \geq 4$. Then,*

$$r^*(P_3, H) = \begin{cases} \binom{n}{2} - 1, & n \text{ is even}, \\ \binom{n}{2}, & n \text{ is odd}. \end{cases}$$

Proof. Since $v(H) = n$ and $\beta(\overline{H}) = \lfloor \frac{n}{2} \rfloor$, by Theorem A we obtain $r(P_3, H) = n$. We divide the proof into two cases based on the parity of n.

Case 1. n is even. The graph H will be isomorphic to $K_n - \left(P_4 \cup \left(\frac{n}{2} - 2\right) K_2\right)$. Let $F = K_n - K_2$. We will show that $F \to (P_3, H)$. To do so, consider any 2-coloring of F having no red P_3. The graph G_F will be a subgraph of $P_4 \cup \left(\frac{n}{2} - 2\right) K_2$. As a consequence $\overline{G}_F$ must contain H. Thus $r^*(P_3, H) \leq \binom{n}{2} - 1$.

Now, let F be a graph with $v(F) = n$ and $e(F) = \binom{n}{2} - 2$. Thus, F must be isomorphic to either $K_n - P_3$ or $K_n - 2K_2$. We will show that $F \nrightarrow (P_3, H)$. First, consider $F = K_n - P_3$. There is exactly one vertex in F of degree $n - 3$. We can color an edge incident to this vertex by red and the remaining edges by blue. In this 2-coloring, there is no red P_3 in F and the induced subgraph of F by blue edges will contain a vertex of degree $n-4$. Since $\delta(H) = n-3$, we cannot have a blue H in F.

Consider $F = K_n - 2K_2$. There are exactly four vertices in F of degree $n-2$. We can color independent edges incident to these four vertices by red and the remaining edges by blue. In this 2-coloring, there is no red P_3 in F, and the induced subgraph of F by blue edges will contain at least four vertices of degree $n - 3$. Since in H there are exactly two vertices of minimum degree, which is $n - 3$, we cannot have a blue H. Thus, in any cases we have that $F \nrightarrow (P_3, H)$ and so $r^*(P_3, H) \geq \binom{n}{2} - 1$.

Case 2. n is odd. In this case, the graph H must be isomorphic to either $K_n - \left(P_4 \cup \left(\frac{n-1}{2} - 2\right) K_2\right)$ or $K_n - \left(P_3 \cup \left(\frac{n-1}{2} - 1\right) K_2\right)$. If $H = K_n - \left(P_4 \cup \left(\frac{n-1}{2} - 2\right) K_2\right)$, then $\beta(\overline{H}) = \frac{n-1}{2}$ and $\Delta(H) = n - 1$. From Lemma 5 we have $r^*(P_3, H) = \binom{n}{2}$.

Let $H = K_n - \left(P_3 \cup \left(\frac{n-1}{2} - 1\right) K_2\right)$. Since $r(P_3, H) = n$, $r^*(P_3, H) \leq \binom{n}{2}$. Now, let F be a graph with $v(F) = n$ and $e(F) = \binom{n}{2} - 1$. Thus, the graph F must be isomorphic to $K_n - K_2$. There are exactly two vertices in F of degree $n-2$. Now, color the independent edges incident to these two vertices by red and the remaining edges by blue. In this 2-coloring, there is no red P_3 in F and the induced subgraph of F by blue edges will contain two vertices of degree $n - 3$. Since H contains exactly one vertex of degree $n - 3$ (minimum), we cannot have a blue H in F. Thus $F \nrightarrow (P_3, H)$ and $r^*(P_3, H) \geq \binom{n}{2}$. ∎

We see in Lemma 7 that $r^*\left(P_3, K_n - \left\lfloor \frac{n}{2} \right\rfloor K_2 - e\right)$ still attains the upper bound of (1) if n is odd. So, we can consider its subgraph H by deleting one edge and $\Delta(H) < n - 1$. (If n is odd and $\Delta(H) = n - 1$, then $r^*(P_3, H)$ has been given in Lemma 5). We get four different graphs, namely, $H_1 = K_n - \left(C_3 \cup \left(\frac{n-1}{2} - 1\right) K_2\right)$ $(n \geq 5)$, $H_2 = K_n - \left(P_5 \cup \left(\frac{n-1}{2} - 2\right) K_2\right)$ $(n \geq 5)$, $H_3 = K_n - \left(P_4 \cup P_3 \cup \left(\frac{n-1}{2} - 3\right) K_2\right)$ $(n \geq 7)$, and $H_4 = K_n - \left(S_{1,3} \cup \left(\frac{n-1}{2} - 2\right) K_2\right)$ $(n \geq 5)$, where $S_{k,n}$ is a graph obtained from $K_{1,n}$ by subdividing one edge k times [1].

The amalgamation of graphs H_1 and H_2, denoted by $Amal(H_1, H_2)$, is formed by identifying a vertex from H_1 to a vertex from H_2.

Lemma 8. *Let* $H_1 = K_n - \left(C_3 \cup \left(\frac{n-1}{2} - 1\right) K_2\right)$ $(n \geq 5)$*,* $H_2 = K_n - \left(P_5 \cup \left(\frac{n-1}{2} -2\right)K_2\right)$ $(n \geq 5)$*,* $H_3 = K_n - \left(P_4 \cup P_3 \cup \left(\frac{n-1}{2} - 3\right) K_2\right)$ $(n \geq 7)$*, and* $H_4 = K_n - \left(S_{1,3} \cup \left(\frac{n-1}{2} - 2\right) K_2\right)$ $(n \geq 5)$*. For odd* n

$$r^*(P_3, H_1) = \binom{n}{2}, \; r^*(P_3, H_2) = r^*(P_3, H_3) = r^*(P_3, H_4) = \binom{n}{2} - 1.$$

Proof. Since $v(H_i) = n$, n is odd, and $\beta(\overline{H_i}) = \frac{n-1}{2}$, by Theorem A we obtain $r(P_3, H_i) = n$ for $1 \leq i \leq 4$.

Case 1. $H_1 = K_n - \left(C_3 \cup \left(\frac{n-1}{2} - 1\right)K_2\right)$. Since $r(P_3, H_1) = n$, $r^*(P_3, H_1) \leq \binom{n}{2}$. Now, let F be a graph with $v(F) = n$ and $e(F) = \binom{n}{2} - 1$. Then, F must be isomorphic to $K_n - K_2$. Consider a 2-coloring of the edges of F having no red P_3 such that the graph G_F is $P_4 \cup \left(\frac{n-1}{2} - 2\right)K_2 \cup K_1$. In this 2-coloring, G_F is not a subgraph of $C_3 \cup \left(\frac{n-1}{2} - 1\right)K_2$ and so $\overline{G_F}$ contains no H_1. Thus $F \nrightarrow (P_3, H_1)$ and $r^*(P_3, H_1) \geq \binom{n}{2}$.

Case 2. $H_2 = K_n - \left(P_5 \cup \left(\frac{n-1}{2} - 2\right)K_2\right)$, $H_3 = K_n - \left(P_4 \cup P_3 \cup \left(\frac{n-1}{2} - 3\right)K_2\right)$, or $H_4 = K_n - \left(S_{1,3} \cup \left(\frac{n-1}{2} - 2\right)K_2\right)$. If $n = 5$, then Silaban *et al.* have shown that $r^*(P_3, H_2) = r^*(P_3, H_4) = \binom{n}{2} - 1 = 9$ [23].

For $n \geq 7$, let $F = K_n - K_2$. We will show that $F \to (P_3, H_i)$ for $2 \leq i \leq 4$. To do so, consider any 2-coloring of F that have no red P_3. Then G_F will be isomorphic to a subgraph of either $P_3 \cup \left(\frac{n-1}{2} - 1\right)K_2$ or $P_4 \cup K_1 \cup \left(\frac{n-1}{2} - 2\right)K_2$. Since each of $P_5 \cup \left(\frac{n-1}{2} - 2\right)K_2$, $P_4 \cup P_3 \cup \left(\frac{n-1}{2} - 3\right)K_2$, and $S_{1,3} \cup \left(\frac{n-1}{2} - 2\right)K_2$ is a subgraph of both $P_3 \cup \left(\frac{n-1}{2} - 1\right)K_2$ and $P_4 \cup K_1 \cup \left(\frac{n-1}{2} - 1\right)K_2$, $\overline{G_F}$ must contain H_i for $2 \leq i \leq 4$. Thus $F \to (P_3, H_i)$ and $r^*(P_3, H_i) \leq \binom{n}{2} - 1$ for $2 \leq i \leq 4$.

Now, let F be a graph with $v(F) = n$ and $e(F) = \binom{n}{2} - 2$. Then, F must be isomorphic to either $K_n - P_3$ or $K_n - 2K_2$. We will show that $F \nrightarrow (P_3, H_i)$ for $2 \leq i \leq 4$ for such a graph F.

If $F = K_n - P_3$, then F must contain exactly one vertex of degree $n - 3$ and two vertices of degree $n - 2$. Now, color three independent edges incident to these three vertices by red so that G_F contains an $Amal(P_4, P_3)$ in a vertex of P_4 of degree 2 and leaf of P_3. All the remaining edges are colored by blue. In this 2-coloring, there is no red P_3. Since $Amal(P_4, P_3) \nsubseteq P_5 \cup \left(\frac{n-1}{2} - 2\right)K_2$, $Amal(P_4, P_3) \nsubseteq P_4 \cup P_3 \cup \left(\frac{n-1}{2} - 3\right)K_2$, and $Amal(P_4, P_3) \nsubseteq S_{1,3} \cup \left(\frac{n-1}{2} - 2\right)K_2$, we cannot have a blue H_i for $2 \leq i \leq 4$. Thus $F \nrightarrow (P_3, H_i)$ for $2 \leq i \leq 4$.

If $F = K_n - 2K_2$, then F contains exactly four vertices of degree $n - 2$. Now, color three independent edges incident to these four vertices of degree $n - 2$ by red such that G_F contains a P_6 and the remaining edges by blue. In this 2-coloring, there is no red P_3. Since $P_6 \nsubseteq P_5 \cup \left(\frac{n-1}{2} - 2\right)K_2$, $P_6 \nsubseteq P_4 \cup P_3 \cup \left(\frac{n-1}{2} - 3\right)K_2$, and $P_6 \nsubseteq S_{1,3} \cup \left(\frac{n-1}{2} - 2\right)K_2$, we cannot have a blue H_i for $2 \leq i \leq 4$. Thus $F \nrightarrow (P_3, H_i)$.

Hence $r^*(P_3, H_i) \geq \binom{n}{2} - 1$ for $2 \leq i \leq 4$. ∎

Since $r^*\big(P_3, K_n - \big(C_3 \cup \big(\frac{n-1}{2} - 1\big)K_2\big)\big) = \binom{n}{2}$, then we can consider the subgraph of $K_n - \big(C_3 \cup \big(\frac{n-1}{2} - 1\big)K_2\big)$ by deleting one edge for n being odd. This subgraph will be isomorphic to either $H_1 = K_n - \big(Amal(P_3, C_3) \cup \big(\frac{n-1}{2} - 2\big)K_2\big)\big)$ for $n \geq 5$ or $H_2 = K_n - \big(C_3 \cup P_4 \cup \big(\frac{n-1}{2} - 3\big)K_2\big)\big)$ for $n \geq 7$ where $Amal(P_3, C_3)$ is the amalgamation of a leaf in P_3 with a vertex in C_3. Since $K_n - \big(Amal(P_3, C_3) \cup \big(\frac{n-1}{2} - 2\big)K_2\big)\big) \subseteq K_n - \big(S_{1,3} \cup \big(\frac{n-1}{2} - 2\big)K_2\big)$ and $K_n - \big(C_3 \cup P_4 \cup \big(\frac{n-1}{2} - 3\big)K_2\big)\big) \subseteq P_4 \cup P_3 \cup \big(\frac{n-1}{2} - 3\big)K_2\big)$, we have Corollary 9.

Corollary 9. *Let $H_1 = K_n - \big(Amal(P_3, C_3) \cup \big(\frac{n-1}{2} - 2\big)K_2\big)$ $(n \geq 5)$ and $H_2 = K_n - \big(C_3 \cup P_4 \cup \big(\frac{n-1}{2} - 3\big)K_2\big)$ $(n \geq 7)$ where n is odd. Then,*

$$r^*(P_3, H_1) = \begin{cases} 8, & n = 5, \\ \begin{pmatrix} n \\ 2 \end{pmatrix} - 1, & n \geq 7, \end{cases} \quad \text{and } r^*(P_3, H_2) = \begin{pmatrix} n \\ 2 \end{pmatrix} - 1.$$

Now, we are ready to prove Theorem 1.

Proof. $(\Leftarrow)$ If n is even and $H = K_n - \frac{n}{2}K_2$, then by Theorem A we have $r(P_3, H) = n$ and the result follows by Lemma 6. Now, let us consider the case if n is odd. If $\beta(\overline{H}) = \frac{n-1}{2}$, $\Delta(H) = n-1$, then by Theorem A we have $r(P_3, H) = n$ and the result follows by Lemma 5.

If $K_n - \big(P_3 \cup \big(\frac{n-1}{2} - 1\big)K_2\big)$, then by Theorem A we have $r(P_3, H) = n$ and the result follows by Lemma 7. If $K_n - \big(C_3 \cup \big(\frac{n-1}{2} - 1\big)K_2\big)$, then by Theorem A we have $r(P_3, H) = n$ and the result follows by Lemma 8.

$(\Rightarrow)$ Suppose to the contrary that H is not one of (a)–(b) and $r^*(P_3, H) = \binom{r(P_3,H)}{2}$. If $\beta(\overline{H}) \leq \lfloor \frac{n}{2} \rfloor - 1$, then by Theorem A we have $r(P_3, H) = 2n - 2\beta - 1$ and by Lemma 4 we have $r^*(P_3, H) < \binom{r(P_3,H)}{2}$.

So our next consideration is for the case if $\beta(\overline{H}) = \lfloor \frac{n}{2} \rfloor$. By Theorem A $r(P_3, H) = n$. If n is even and $H \neq K_n - \frac{n}{2}K_2$, then $H \subseteq K_n - (P_4 \cup \big(\frac{n}{2} - 2)K_2\big)$. By Lemma 7 and (3) we have $r^*(P_3, H) < \binom{r(P_3,H)}{2}$.

If n is odd, $\Delta(H) < n - 1$, $H \neq K_n - \big(P_3 \cup \big(\frac{n-1}{2} - 1\big)K_2\big)$, and $H \neq K_n - \big(C_3 \cup \big(\frac{n-1}{2} - 1\big)K_2\big)$, then $H \subseteq K_n - \big(P_5 \cup \big(\frac{n-1}{2} - 2\big)K_2\big)$ or $H \subseteq K_n - \big(P_4 \cup P_3 \cup \big(\frac{n-1}{2} - 3\big)K_2\big)$ or $H \subseteq K_n - \big(S_{1,3} \cup \big(\frac{n-1}{2} - 2\big)K_2\big)$.

If $H \subseteq K_n - \big(P_5 \cup \big(\frac{n-1}{2} - 2\big)K_2\big)$ or $H \subseteq K_n - \big(P_4 \cup P_3 \cup \big(\frac{n-1}{2} - 3\big)K_2\big)$ or $H \subseteq K_n - \big(S_{1,3} \cup \big(\frac{n-1}{2} - 2\big)K_2\big)$, then by Lemma 8 and (3) we have $r^*(P_3, H) < \binom{r(P_3,H)}{2}$. ■

4. Proof of Theorem 2

The following lemmas are used to prove Theorem 2.

Lemma 10 [9]**.** *For $m, n \geq 1$,*

$$\hat{r}(K_{1,m}, K_{1,n}) = m + n - 1.$$

Clearly, $F = K_{1,m+n-1}$ satisfies that $F \rightarrow (K_{1,m}, K_{1,n})$.

Lemma 11. *Let H be a connected graph. If $r^*(P_3, H) = e(P_3) + e(H) - 1$, then H is a star.*

Proof. Let H be a connected graph. Suppose $e(H) = m$. Since $r^*(P_3, H) = e(P_3) + e(H) - 1 = m + 1$, there is a graph F with $e(F) = m + 1$ such that $F \rightarrow (P_3, H)$. It means that any 2-coloring of the edges of F having no red P_3 must contain a blue H. As a consequence, there is at most one independent edge in F. Thus, the only possible F is either K_3 or $K_{1,m+1}$. If $F = K_3$, then H must be $K_{1,2}$. If $F = K_{1,m+1}$, then H must be a subgraph of $K_{1,m}$. In both cases, H is a star. ∎

Lemma 12. *For integer $n \geq 2$,*

$$r^*(P_3, K_{1,n-1}) = \begin{cases} n, & n \textit{ is even}, \\ \binom{n}{2}, & n \textit{ is odd}. \end{cases}$$

Proof. We know $v(K_{1,n-1}) = n$. If n is odd, then the assertion holds from Lemma 5 since $\beta(\overline{K}_{1,n-1}) = \frac{n-1}{2}$ and $\Delta(K_{1,n-1}) = n - 1$. If n is even, then $\beta(\overline{K}_{1,n-1}) = \frac{n}{2} - 1$. By Theorem A we obtain $r(P_3, K_{1,n-1}) = n+1$. From Lemma 10 we have $\hat{r}(P_3, K_{1,n-1}) = n$ and the graph $F = K_{1,n}$ satisfies $F \rightarrow (P_3, K_{1,n-1})$. Since $v(K_{1,n}) = n + 1 = r(P_3, K_{1,n-1})$, we have $r^*(P_3, K_{1,n-1}) = \hat{r}(P_3, K_{1,n-1}) = n$. ∎

As given in Lemma 12, if n is odd, then $r^*(P_3, K_{1,n-1}) = \binom{n}{2}$. In fact, this value is the upper bound of (1). On the other hand, Lemma 10 gives that $\hat{r}(P_3, K_{1,n-1}) = n$, the lower bound of (1). This is an example of a pair of graphs for which the size Ramsey number attains the lower bound of (1) while the restricted size Ramsey number attains the upper bound of (1). If n is even, then we have a different phenomenon. Both the size and the restricted size Ramsey number, $\hat{r}(P_3, K_{1,n-1}) = r^*(P_3, K_{1,n-1}) = n$ are attaining the lower bound.

Now, we show the proof of Theorem 2.

Proof. ($\Leftarrow$) Let $H = K_{1,n-1}$ and n be even. By Lemma 12 we have $r^*(P_3, H) = n = e(P_3) + e(H) - 1$.

($\Rightarrow$) Let H be a connected graph, $v(H) = n$ and $r^*(P_3, H) = e(P_3) + e(H) - 1$. According to Lemma 11, H must be a star. By Lemma 12 we have $H = K_{1,n-1}$ and n is even. ∎

5. Restricted o-Sequence

Erdős *et al.* [9] defined o-sequence for diagonal case of the size Ramsey number. We can adopt the concept for non-diagonal restricted size Ramsey number and we call it as a *restricted o-sequence*. Then, we have Corollary 13.

Corollary 13. *For even* n *the sequence* $\{(P_3, K_{1,n-1})\}$ *is a restricted o-sequence.*

Proof. Since $v(K_{1,n-1}) = n$, and by Theorem A we have $r(P_3, K_{1,n-1}) = n+1$ if n is even, by Lemma 12 we obtain (if n is even),

$$\lim_{n\to\infty} \frac{r^*(P_3, K_{1,n-1})}{\binom{r(P_3, K_{1,n-1})}{2}} = \lim_{n\to\infty} \frac{n}{\binom{n+1}{2}} = 0.$$

■

In the introduction we have discussed that $\{(K_{1,m}, K_{1,n})\}$, some graphs obtained by star operation, and $\{nK_2\}$ belong to o-sequence. However, in Corollary 13 we showed that if $m = 2$, then $\{(K_{1,m}, K_{1,n})\}$ belongs to restricted o-sequence only if $n+1$ is even.

Acknowledgements

The authors would like to thank the referees for their careful reading of the manuscript and valuable suggestions. This research was supported by the Research Grant "Penelitian Dasar Unggulan Perguruan Tinggi", Ministry of Research, Technology, and Higher Education, Indonesia.

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Received 12 January 2018
Revised 11 October 2018
Accepted 11 October 2018