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# HAMILTONIAN NORMAL CAYLEY GRAPHS

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**Abstract**

A variant of the Lovász Conjecture on hamiltonian paths states that *every finite connected Cayley graph contains a hamiltonian cycle.* Given a finite group $G$ and a connection set $S$, the Cayley graph $Cay(G, S)$ will be called *normal* if for every $g \in G$ we have that $g^{-1}Sg = S$. In this paper we present some conditions on the connection set of a normal Cayley graph which imply the existence of a hamiltonian cycle in the graph.

**Keywords:** Cayley graph, hamiltonian cycle, normal connection set.

**2010 Mathematics Subject Classification:** 05C45, 05C99.

## 1. Introduction

Let $G$ be a finite group. A subset $S \subseteq G$ will be called symmetric if $S = S^{-1}$. Given a symmetric subset $S \subseteq G \setminus \{e\}$ (with $e$ the identity of $G$), the Cayley graph $Cay(G, S)$ is the graph with vertex set $G$ and a pair $\{\alpha, \beta\}$ is an edge of $Cay(G, S)$ if and only if there is $s \in S$ such that $\alpha = \beta s$ (since $S$ is symmetric, observe that $s^{-1} \in S$ and $\beta = \alpha s^{-1}$). A Cayley graph $Cay(G, S)$ will be called *normal* if for every $\alpha \in G$, $\alpha^{-1}S\alpha = S$. In the literature there is another definition of normal Cayley graph, which is different from the one used in this paper, that said that a Cayley graph on a group $G$ is normal if the right regular representation

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of the group $G$ is normal in the full automorphism group of the graph (see, for instance [15, 19]).

The problem of finding hamiltonian cycles in graphs is a difficult problem, and since 1969 has received a great attention by the Lovász Conjecture which states that every vertex-transitive graph has a hamiltonian path. A variant of the Lovász Conjecture on hamiltonian paths states that every finite connected Cayley graph contains a hamiltonian cycle (see, for instance [1, 3, 14, 18]). In particular, there are several works on the existence of hamiltonian cycles in Cayley graphs generated by two elements (see, for instance [6–10, 12, 20]).

In this paper we present the following results.

**Theorem 1.** *Let $G$ be a finite non-abelian simple group such that $\langle \delta_1, \delta_2 \rangle = G$. If $Cay(G, S)$ is a normal Cayley graph with $\{\delta_1, \delta_2\} \subseteq S$, then $Cay(G, S)$ contains a hamiltonian cycle.*

**Theorem 2.** *Let $G$ be a finite group, $G = G_0 \trianglerighteq G_1, \ldots, G_{l-1} \trianglerighteq G_l$ be a composition series of $G$ and let $\{\delta_0, \ldots, \delta_{l+1}\} \subseteq G$ such that, for each $0 \leq i \leq l$, $G_i/G_{i+1} = \langle \delta_i G_{i+1}, \delta_{i+1} G_{i+1} \rangle$. If $Cay(G, S)$ is a normal Cayley graph with $\{\delta_0, \ldots, \delta_{l+1}\} \subseteq S$, then $Cay(G, S)$ contains a hamiltonian cycle.*

Observe that the normal Cayley graphs with vertex set a group generated by two elements have girth 4. The results are obtained via a generalization of known methods for hamiltonicity of Cayley graphs of girth 4 (see [5, 11, 13, 14]). For general concepts, we may refer the reader to [2, 16].

## 2. Notation and Previous Results

In order to prove the main theorems, we need some definitions and previous results.

**Theorem 3** [17]**.** *Let $G$ be a simple, non-abelian and finite group. $G$ can be generated by two elements.*

In all this section let $G = \langle \delta_1, \delta_2 \rangle$ be a simple, non-abelian and finite group and $Cay(G, S)$ be a normal Cayley graph with connection set $S$ such that $\{\delta_1, \delta_2\} \subseteq S$. Let $G_0 = \langle \delta_1 \rangle$, and let

$$\mathcal{P} = \{a_0 G_0 \cup a_1 G_0, \ldots, a_n G_0\}$$

be the partition of $G$ in cosets induced by the subgroup $G_0$ (with $a_0$ the identity element of $G$). For each $0 \leq i \leq n$, $C(a_i G_0)$ will denote the subdigraph of $Cay(G, S)$ induced by the set of vertices $a_i G_0$. Given two isomorphic vertex disjoint subgraphs $H$ and $H'$ of $Cay(G, S)$, we will say that $H$ and $H'$ are *attached* if there is an isomorphism $\Psi$ between $H$ and $H'$ such that for every $x \in V(H)$, $\{x, \Psi(x)\}$ is an edge of $Cay(G, S)$.

**Lemma 4.** *For every* $0 \le i, j \le n$, $C(a_iG_0) \cong C(a_jG_0)$. *Moreover, for every* $0 \le i \le n$ *and* $\delta \in \{\delta_1, \delta_2\}$, $C(a_iG_0)$ *and* $C(\delta a_iG_0)$ *are attached.*

***Proof.*** Given $a_i, a_j$ let $\Phi : a_iG_0 \to a_jG_0$ be defined, for each $g \in G_0$, as $\Phi(a_ig) = a_jg$. If $\Phi(a_ig) = \Phi(a_ig_1)$ then $a_jg = a_jg_1$, so $g = g_1$. Therefore $\Phi$ is injective and since all cosets have the same cardinality, $\Phi$ is bijective. If $a_ig_1$ and $a_ig_2$ are adjacent in $C(a_iG_0)$, then $g_1^{-1}a_i^{-1}a_ig_2 = g_1^{-1}g_2 \in S$. Therefore

$$\Phi(a_ig_1)^{-1}\Phi(a_ig_2) = g_1^{-1}a_j^{-1}a_jg_2 = g_1^{-1}g_2 \in S$$

and then $\Phi(a_ig_1)$ and $\Phi(a_ig_2)$ are adjacent in $C(a_jG_0)$, and the first part of the lemma follows. For the second part, let $a \in a_iG_0$ and $\delta a \in \delta a_iG_0$. Clearly the map $a \to \delta a$ define an isomorphism between $C(a_iG_0)$ and $C(\delta a_iG_0)$ and since $S$ is normal, $a^{-1}\delta a \in S$, therefore $\{a, \delta a\}$ is an edge in $Cay(G, S)$ (see Figure 1), and the lemma follows. ∎

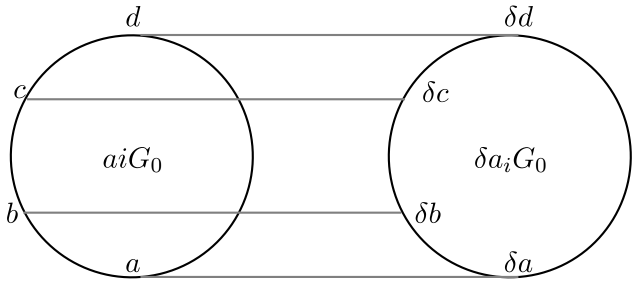


Figure 1

As a *word on* $\{\delta_1, \delta_2\}$ we will understand a product $s_1s_2 \cdots s_{n-1}s_n$ of powers of $\delta_1$ and $\delta_2$, where two consecutive elements in the product are not powers of the same elements, that is to say, if $s_i \in \langle a \rangle$ then $s_{i+1}, s_{i-1} \notin \langle a \rangle$. The *length of a word* $s_1s_2 \cdots s_{n-1}s_n$ is $n$. Since $G = \langle \delta_1, \delta_2 \rangle$, it follows that for each $\alpha \in G$, $\alpha = s_1s_2 \cdots s_{n-1}s_n$ for some word $s_1s_2 \cdots s_{n-1}s_n$ on $\{\delta_1, \delta_2\}$. For each $\alpha \in G$, let $\ell(\alpha)$ be the minimum length of a word on $\{\delta_1, \delta_2\}$ such that $\alpha = s_1s_2 \cdots s_{n-1}s_n$.

Let $\mathcal{H}_0 = \{G_0\}$ and, for each $k \ge 1$, let $\mathcal{H}_k = \{a_iG_0 \in \mathcal{P} : \ell(a_i) = k\}$.

Given a coset $a_iG_0 \in \mathcal{P}$, let $L[a_iG_0] = \{a_iG_0\} \cup \{\delta^r a_iG_0 \in \mathcal{P} : \delta \in \{\delta_1, \delta_2\}, r \ge 1\}$. Observe that if $\ell(a_i) = k$, then for every $aG_0 \in L[a_iG_0] \setminus \{a_iG_0\}$, $\ell(a) = k+1$, and the number of cosets in the set $\{\delta^r a_iG_0 \in \mathcal{P} : \delta \in \{\delta_1, \delta_2\}, r \ge 1\}$ depends on the commutativity of the words $\delta a_1\delta_1^j, \delta^2 a_1\delta_1^j, \ldots, \delta^{n-1}a_1\delta_1^j$ for $j \ge 1$.

**Lemma 5.** *If for some* $k \ge 1$, $\delta^k a_iG_0 \in L[a_iG_0]$, *then* $\delta^{k-1}a_iG_0 \in L[a_iG_0]$.

***Proof.*** Let us suppose that for some $k \ge 1$, $\delta^k a_iG_0 \in L[a_iG_0]$ and let $b \in G$ such that $\delta^{k-1}a_i \in bG_0 \in \mathcal{P}$. Thus, $\delta^{k-1}a_i = b\delta_1^j$ and therefore $\delta^k a_i = \delta b\delta_1^j$ which implies that $\delta^k a_i \in \delta bG_0 = \delta^k a_iG_0$. Hence $\delta^k a_i = \delta b$ and $\delta^{k-1}a_i = b$, and the result follows. ∎

Observe that for each $k \geq 0$, the set $\{L[a_iG_0] : a_iG_0 \in \mathcal{H}_k\}$ is a partition of $\mathcal{H}_k \cup \mathcal{H}_{k+1}$. Given a coset $a_iG_0 \in \mathcal{P}$, the subgraph of $Cay(G,S)$ induced by $L[a_iG_0]$ will be called a *leaf*.

Given a leaf $M$ induced by $L[a_iG_0] = \{a_iG_0, \delta a_iG_0, \ldots, \delta^m a_iG_0\}$ (with $\delta \in \{\delta_1, \delta_2\}$), and a pair of elements $a_i\delta_1^t, a_i\delta_1^{t+1} \in a_iG_0$, a path of $Cay(G,S)$ with vertex-set $\{a_i\delta_1^t, a_i\delta_1^{t+1}\} \cup \bigcup_{j=1}^m \delta^j a_iG_0$ which starts at $a_i\delta_1^t$, ends at $a_i\delta_1^{t+1}$ and such that for every $1 \leq j \leq m$, there is $s_j$ such that $\{\delta^j a_i\delta_1^{s_j}, \delta^j a_i\delta_1^{s_j+1}\}$ is an edge of the path $P$, will be called an $(a_i\delta_1^t, a_i\delta_1^{t+1}, M)$*-complete path.*

**Lemma 6.** *Let $M$ be a leaf of $Cay(G,S)$ induced by*

$$L[a_iG_0] = \{a_iG_0, \delta a_iG_0, \ldots, \delta^m a_iG_0\}$$

*(with $\delta \in \{\delta_1, \delta_2\}$). For every pair of elements $a_i\delta_1^t, a_i\delta_1^{t+1} \in a_iG_0$ there is an $(a_i\delta_1^t,\ a_i\delta_1^{t+1}, M)$-complete path.*

***Proof.*** From Lemma 4 we see that any two "consecutive" subgraphs of the leaf, $C(\delta^t a_iG_0)$ and $C(\delta^{t+1}a_iG_0)$, are attached, and again, by Lemma 4, each subgraph of the leaf is isomorphic to $C(G_0)$, which is a cycle of the form $(e, \delta_1, \delta_1^2, \ldots, \delta_1^n = e)$. Since $G = \langle \delta_1, \delta_2 \rangle$, from here it follows that for each $1 \leq k \leq m$,

$$\left(\delta^k a_i, \delta^k a_i\delta_1, \delta^k a_i\delta_1^2, \ldots, \delta^k a_i\delta_1^n = \delta^k a_i\right)$$

is a hamiltonian cycle of $C(\delta^k a_iG_0)$.

Let $a_i\delta_1^t, a_i\delta_1^{t+1} \in a_iG_0$ and $\delta \in \{\delta_1, \delta_2\}$. To simplify the notation, let $\alpha_0 = a_i\delta_1^t$, $\beta_0 = a_i\delta_1^{t+1}$, $\epsilon_0 = a_i\delta_1^{t-1}$, and for each $1 \leq k \leq m$, let $\alpha_k = \delta^k a_i\delta_1^t$, $\beta_k = \delta^k a_i\delta_1^{t+1}$ and $\epsilon_k = \delta^k a_i\delta_1^{t-1}$.

*Case* 1. $n$ is even (see Figure 2).

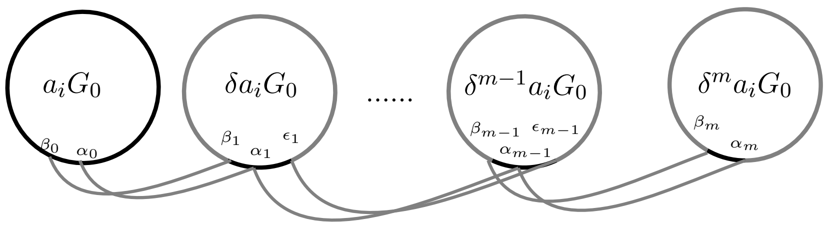


Figure 2

Let

$$\begin{aligned} P = \big(&\alpha_0, \alpha_1, \ldots, \alpha_m = \delta^m a_i\delta_1^t, \delta^m a_i\delta_1^{t-1}, \ldots, \left(\delta^m a_i\delta_1^{t+1} = \beta_m\right), \\ &\left(\beta_{m-1} = \delta^{m-1}a_i\delta_1^{t+1}\right), \delta^{m-1}a_i\delta_1^{t+2}, \ldots, \left(\delta^{m-1}a_i\delta_1^{t-1} = \epsilon_{m-1}\right), \\ &\left(\epsilon_{m-2} = \delta^{m-2}a_i\delta_1^{t-1}\right), \delta^{m-2}a_i\delta_1^{t-2}, \ldots, \left(\delta^{m-2}a_i\delta_1^{t+1} = \beta_{m-2}\right), \ldots, \\ &\left(\epsilon_1 = \delta a_i\delta_1^{t-1}\right), \delta a_i\delta_1^{t-2}, \ldots, \left(\delta a_i\delta_1^{t+1} = \beta_1\right), \beta_0\big). \end{aligned}$$

*Case* 2. $n$ is odd (see Figure 3).

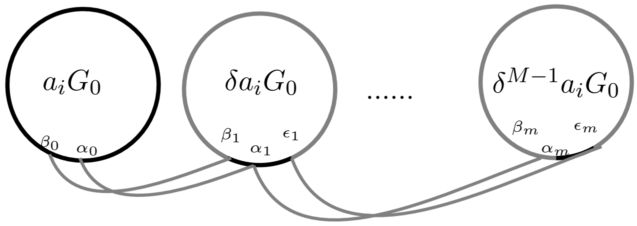


Figure 3

Let

$$\begin{aligned}P = \big(&\alpha_0, \alpha_1, \ldots, \alpha_m = \delta^m a_i \delta_1^t, \delta^m a_i \delta_1^{t+1}, \ldots, \big(\delta^m a_i \delta_1^{t-1} = \epsilon_m\big),\\ &\big(\epsilon_{m-1} = \delta^{m-1} a_i \delta_1^{t-1}\big), \delta^{m-1} a_i \delta_1^{t-2}, \ldots, \big(\delta^{m-1} a_i \delta_1^{t+1} = \beta_{m-1}\big),\\ &\big(\beta_{m-2} = \delta^{m-2} a_i \delta_1^{t+1}\big), \delta^{m-2} a_i \delta_1^{t+2}, \ldots, \big(\delta^{m-2} a_i \delta_1^{t-1} = \epsilon_{m-2}\big), \ldots,\\ &\big(\epsilon_1 = \delta a_i \delta_1^{t-1}\big), \delta a_i \delta_1^{t-2}, \ldots, \big(\delta a_i \delta_1^{t+1} = \beta_1\big), \beta_0\big).\end{aligned}$$

From here the result follows. ■

## 3. THE PROOFS OF THE MAIN RESULTS

***Proof of Theorem* 1.** Let $G = \langle \delta_1, \delta_2 \rangle$ be a non-abelian simple group and $Cay(G,S)$ be a normal Cayley graph with $\{\delta_1, \delta_2\} \subseteq S$. Let $G_0 = \langle \delta_1 \rangle$, and let $\mathcal{P} = \{G_0, a_1G_0, \ldots, a_nG_0\}$ be the partition of $G$ in cosets induced by the subgroup $G_0$.

Let $\mathcal{H}_0 = \{G_0\}$ and, for each $k \geq 1$, let $\mathcal{H}_k = \{a_iG_0 \in \mathcal{P} : \ell(a_i) = k\}$. Since $G$ is finite, it follows that for some $p \geq 1$, $G = \bigcup_{j=0}^{p} \left( \bigcup_{A \in \mathcal{H}_j} A \right)$.

We will prove the result by showing, by induction on $k$, that for every $k \geq 1$ the subgraph of $Cay(G,S)$ induced by

$$\bigcup_{j=0}^{k} \left( \bigcup_{A \in \mathcal{H}_j} A \right)$$

contains a hamiltonian cycle $C$ such that for each $a_jG_0 \in \mathcal{H}_k$, there is $s_j$ such that $\left\{ a_i\delta_1^{s_j}, a_i\delta_1^{s_j+1} \right\}$ is an edge of $C$.

For $k = 1$, observe that $\mathcal{H}_0 = \{G_0\}$ and $\mathcal{H}_1 = \left\{\delta_2G_0, \delta_2^2G_0, \ldots, \delta_2^mG_0\right\}$. Thus, the subgraph $M$ of $Cay(G,S)$ induced by $\bigcup_{j=0}^{1} \left( \bigcup_{A \in \mathcal{H}_j} A \right)$ is the leaf of $Cay(G,S)$ induced by $L[G_0] = \left\{G_0, \delta_2G_0, \delta_2^2G_0, \ldots, \delta_2^mG_0\right\}$. Let $e, \delta_1 \in G_0$. By

Lemma 6 there is an $(e, \delta_1, M)$-complete path $P$. Therefore $C = P \circ \left(\delta_1, \delta_1^2, \ldots, \delta_1^{n-1}, e\right)$ is a hamiltonian cycle of $M$ such that for every $1 \leq j \leq m$, there is $s_j$ such that $\left\{\delta_2^j \delta_1^{s_j}, \delta_2^j \delta_1^{s_j+1}\right\} \subseteq \delta_2^j G_0$ is an edge of $C$.

Suppose that the statement is true for $1 \leq m \leq k$; let $Q$ be the subgraph of $Cay(G, S)$ induced by $\bigcup_{j=0}^{k+1} \left(\bigcup_{A \in \mathcal{H}_j} A\right)$ and let $Q'$ be the subgraph of $Cay(G, S)$ induced by $\bigcup_{j=0}^{k} \left(\bigcup_{A \in \mathcal{H}_j} A\right)$. By induction hypothesis, there is a hamiltonian cycle $C$ of $Q'$ such that for each $a_j G_0 \in \mathcal{H}_k$, there is $s_j$ such that $\left\{a_j \delta_1^{s_j}, a_j \delta_1^{s_j+1}\right\}$ is an edge of $C$.

For each $a_j G_0 \in \mathcal{H}_k$, by Lemma 6, there is an $\left(a_j \delta_1^{s_j}, a_j \delta_1^{s_j+1}, M\right)$-complete path $P$ with $M$ the leaf induced by $L[a_j G_0] = \left\{a_j G_0, \delta a_j G_0, \delta^2 a_j G_0, \ldots, \delta^m a_j G_0\right\}$. Therefore, by deleting from $C$ the edge $\left\{a_j \delta_1^{s_j}, a_j \delta_1^{s_j+1}\right\}$, and attach to $C \setminus \left\{a_j \delta_1^t, a_j \delta_1^{t+1}\right\}$ the path $P$ we obtain a hamiltonian cycle $C'$ of the subgraph of $Cay(G, S)$ induced by $V(Q') \cup V(M)$, and such that for each $1 \leq i \leq m$ there is $s_i$ such that $\left\{\delta^i a_j \delta_1^{s_i}, \delta^i a_j \delta_1^{s_i+1}\right\}$ is an edge of $C'$. Following this procedure for each coset in $\mathcal{H}_k$, since $\{L[a_i G_0] : a_i G_0 \in \mathcal{H}_k\}$ is a partition of $\mathcal{H}_k \cup \mathcal{H}_{k+1}$, we obtain a hamiltonian cycle $C$ of $Q$ such that for each $a_j G_0 \in \mathcal{H}_{k+1}$, there is $s_j$ such that $\left\{a_j \delta_1^{s_j}, a_j \delta_1^{s_j+1}\right\}$ is an edge of $C$. From here, the result follows. ■

***Proof of Theorem* 2.** We will prove the theorem by induction on the order of the group. For $|G| = 3$, we see that $G \cong Z_3$ and the only possible normal Cayley graphs are $Cay(Z_3, \{1\})$ and $Cay(G, \{1, 2\})$ which are both hamiltonian graphs.

Let $G$ be a finite group of order greater than 3, $G = G_0 \trianglerighteq G_1, \ldots, G_{l-1} \trianglerighteq G_l$ be a composition series of $G$, and let $\{\delta_0, \ldots, \delta_{l+1}\} \subseteq G$ such that, for each $0 \leq i \leq l$, $G_i/G_{i+1} = \langle \delta_i G_{i+1}, \delta_{i+1} G_{i+1} \rangle$. Let $Cay(G, S)$ be a normal Cayley graph with $\{\delta_0, \ldots, \delta_{l+1}\} \subseteq S$.

Let $S/G_1 = \{sG_1 : s \in S\}$ and consider the Cayley graph $Cay(G/G_1, S/G_1)$. If $G/G_1$ is an abelian group, it is known that $Cay(G/G_1, S/G_1)$ contains a hamiltonian cycle (see [4]). If $Cay(G/G_1, S/G_1)$ is not an abelian group, consider the following.

**Claim 1.** *$Cay(G/G_1, S/G_1)$ is a normal Cayley graph.*

***Proof.*** Let $g \in G$ and $s \in S$. Since $G_1$ is a normal subgroup it follows that $g^{-1} G_1 s G_1 g G_1 = g^{-1} s g G_1$ and since $S$ is a normal connection set, $g^{-1} s g = s_1 \in S$. Therefore $g^{-1} s g G_1 = s_1 G_1 \in S/G_1$ and the claim follows. □

Thus, by Claim 1, $Cay(G/G_1, S/G_1)$ is a normal Cayley graph; $G/G_1 = \langle \delta_0 G_1, \delta_1 G_1 \rangle$ is a simple non-abelian group and, by hypothesis, $\{\delta_0, \delta_1\} \subseteq S$ which implies that $\{\delta_0 G_1, \delta_1 G_1\} \subseteq S/G_1$. Therefore, from Theorem 1 it follows that there is a hamiltonian cycle in $Cay(G/G_1, S/G_1)$.

Let $\mathcal{C} = (G_1, g_1G_1, \ldots, g_nG_1, G_1)$, with $n = |G/G_1|$, be a hamiltonian cycle in $Cay(G/G_1, S/G_1)$ (see Figure 4).

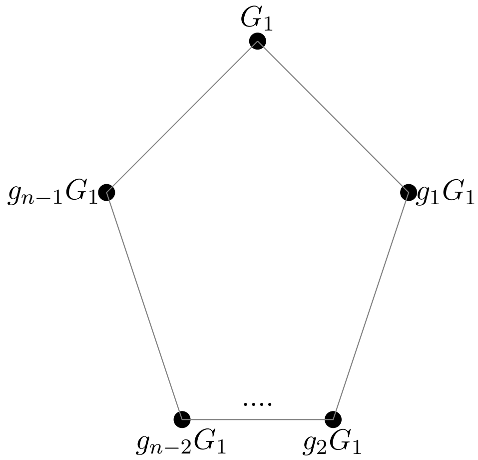


Figure 4

On the other hand, let $S\ |_{G_1} = S \cap G_1$ and consider the Cayley graph $Cay(G_1, S\ |_{G_1})$.

**Claim 2.** *$Cay(G_1, S\ |_{G_1})$ is a normal Cayley graph.*

***Proof.*** Since $S$ is a normal connection set and $G_1$ is a normal subgroup of $G$ we see that $g^{-1}Sg = S$ and $g^{-1}G_1g = G_1$, so $g^{-1}(S\ |_{G_1})g = g^{-1}(S \cap G_1)g = S \cap G_1 = S\ |_{G_1}$. □

**Claim 3.** $\{\delta_1, \ldots, \delta_{l+1}\} \subseteq S\ |_{G_1}$.

***Proof.*** Since for each $i \in \{0, \ldots, l+1\}$ we have $G_i/G_{i+1} = \langle \delta_i G_{i+1}, \delta_{i+1}G_{i+1} \rangle$ it follows that $\delta_i \in G_i \subset G_1$ and $\delta_i \in G_1$ for all $1 \leq i \leq l+1$. □

Clearly $|G_1| < |G_0|$, and since $G_1 \trianglerighteq G_2, \ldots, G_{l-1} \trianglerighteq G_l$ is a composition series of $G_1$, by Claims 2 and 3 and by induction hypothesis we see that there is a hamiltonian cycle $\mathcal{C}'$ in $C(G_1, S\ |_{G_1})$. Let $\mathcal{C}' = (1, n_1, n_2, \ldots, n_{i-1}, 1)$ with $i = |G_1|$.

Let $g_lG_1$ and $g_{l+1}G_1$ be two consecutive vertices of the hamiltonian cycle $\mathcal{C}$ of $Cay(G/G_1, S/G_1)$. By definition $(g_lG_1)^{-1}g_{l+1}G_1 \in S/G_1$ which implies that $G_1g_l^{-1}g_{l+1}G_1 = s_1G_1$ with $s_1 \in S$. Thus $g_l^{-1}g_{l+1} = s_1n_{l_1}$ with $n_{l_1} \in G_1$ and then $g_l^{-1}g_{l+1}n_{l_1}^{-1} = s_1 \in S$. Therefore, for every $n_j \in G_1$, we see that

$$n_j^{-1}g_l^{-1}g_{l+1}n_{l_1}^{-1}n_j = n_j^{-1}s_1n_j \in S,$$

which implies that for every $n_j \in G_1$, $g_ln_j$ is adjacent to $g_{l+1}n_{l_1}^{-1}n_j$ in $Cay(G, S)$.

Observe that the map $g_l n_j \to g_{l+1} n_{l_1}^{-1} n_j$ defines a bijection between $g_l G_1$ and $g_{l+1} G_1$, and that, given $\alpha, \beta \in G_1$, we see that $(g_l \alpha)^{-1} g_l \beta = \alpha^{-1}\beta \in S$ if and only if

$$\left(g_{l+1} n_{l_1}^{-1} \alpha\right)^{-1} \left(g_{l+1} n_{l_1}^{-1} \beta\right) = \alpha^{-1} n_{l_1} g_{l+1}^{-1} g_{l+1} n_{l_1}^{-1} \beta = \alpha^{-1}\beta \in S,$$

which implies that the subgraphs of $Cay(G, S)$ induced by $g_l G_1$ and $g_{l+1} G_1$ are attached (see Figure 5).

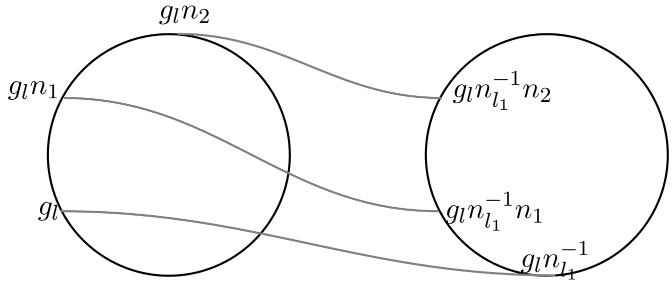


Figure 5

From here, and by an analogous argument than in the proof for the case $k = 1$ in Theorem 1, we obtain a hamiltonian cycle in $Cay(G, S)$ (see Figure 6). ■

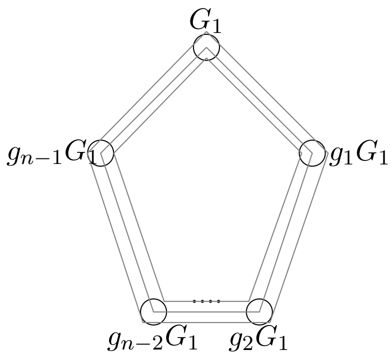


Figure 6

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