HAMILTONIAN NORMAL CAYLEY GRAPHS

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Abstract

A variant of the Lovász Conjecture on hamiltonian paths states that every finite connected Cayley graph contains a hamiltonian cycle. Given a finite group $G$ and a connection set $S$, the Cayley graph $\text{Cay}(G, S)$ will be called normal if for every $g \in G$ we have that $g^{-1}Sg = S$. In this paper we present some conditions on the connection set of a normal Cayley graph which imply the existence of a hamiltonian cycle in the graph.

Keywords: Cayley graph, hamiltonian cycle, normal connection set.

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1. Introduction

Let $G$ be a finite group. A subset $S \subseteq G$ will be called symmetric if $S = S^{-1}$. Given a symmetric subset $S \subseteq G \setminus \{e\}$ (with $e$ the identity of $G$), the Cayley graph $\text{Cay}(G, S)$ is the graph with vertex set $G$ and a pair $\{\alpha, \beta\}$ is an edge of $\text{Cay}(G, S)$ if and only if there is $s \in S$ such that $\alpha = \beta s$ (since $S$ is symmetric, observe that $s^{-1} \in S$ and $\beta = \alpha s^{-1}$). A Cayley graph $\text{Cay}(G, S)$ will be called normal if for every $\alpha \in G$, $\alpha^{-1}S\alpha = S$. In the literature there is another definition of normal Cayley graph, which is different from the one used in this paper, that said that a Cayley graph on a group $G$ is normal if the right regular representation

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of the group $G$ is normal in the full automorphism group of the graph (see, for instance [15, 19]).

The problem of finding hamiltonian cycles in graphs is a difficult problem, and since 1969 has received a great attention by the Lovász Conjecture which states that every vertex-transitive graph has a hamiltonian path. A variant of the Lovász Conjecture on hamiltonian paths states that every finite connected Cayley graph contains a hamiltonian cycle (see, for instance [1, 3, 14, 18]). In particular, there are several works on the existence of hamiltonian cycles in Cayley graphs generated by two elements (see, for instance [6–10, 12, 20]).

In this paper we present the following results.

**Theorem 1.** Let $G$ be a finite non-abelian simple group such that $\langle \delta_1, \delta_2 \rangle = G$. If $Cay(G, S)$ is a normal Cayley graph with $\{\delta_1, \delta_2\} \subseteq S$, then $Cay(G, S)$ contains a hamiltonian cycle.

**Theorem 2.** Let $G$ be a finite group, $G = G_0 \triangleright G_1, \ldots, G_{l-1} \triangleright G_l$ be a composition series of $G$ and let $\{\delta_0, \ldots, \delta_{l+1}\} \subseteq G$ such that, for each $0 \leq i \leq l$, $G_i/G_{i+1} = \langle \delta_i G_{i+1}, \delta_{i+1} G_{i+1} \rangle$. If $Cay(G, S)$ is a normal Cayley graph with $\{\delta_0, \ldots, \delta_{l+1}\} \subseteq S$, then $Cay(G, S)$ contains a hamiltonian cycle.

Observe that the normal Cayley graphs with vertex set a group generated by two elements have girth 4. The results are obtained via a generalization of known methods for hamiltonicity of Cayley graphs of girth 4 (see [5, 11, 13, 14]). For general concepts, we may refer the reader to [2, 16].

2. Notation and Previous Results

In order to prove the main theorems, we need some definitions and previous results.

**Theorem 3** [17]. Let $G$ be a simple, non-abelian and finite group. $G$ can be generated by two elements.

In all this section let $G = \langle \delta_1, \delta_2 \rangle$ be a simple, non-abelian and finite group and $Cay(G, S)$ be a normal Cayley graph with connection set $S$ such that $\{\delta_1, \delta_2\} \subseteq S$. Let $G_0 = \langle \delta_1 \rangle$, and let

$$
P = \{a_0 G_0 \cup a_1 G_0, \ldots, a_n G_0\}
$$

be the partition of $G$ in cosets induced by the subgroup $G_0$ (with $a_0$ the identity element of $G$). For each $0 \leq i \leq n$, $C(a_i G_0)$ will denote the subdigraph of $Cay(G, S)$ induced by the set of vertices $a_i G_0$. Given two isomorphic vertex disjoint subgraphs $H$ and $H'$ of $Cay(G, S)$, we will say that $H$ and $H'$ are attached if there is an isomorphism $\Psi$ between $H$ and $H'$ such that for every $x \in V(H)$, $\{x, \Psi(x)\}$ is an edge of $Cay(G, S)$.
Lemma 4. For every $0 \leq i, j \leq n$, $C(a_iG_0) \cong C(a_jG_0)$. Moreover, for every $0 \leq i \leq n$ and $\delta \in \{\delta_1, \delta_2\}$, $C(a_iG_0)$ and $C(\delta a_iG_0)$ are attached.

Proof. Given $a_i, a_j$ let $\Phi : a_iG_0 \to a_jG_0$ be defined, for each $g \in G_0$, as $\Phi(a_i)g = a_jg$. If $\Phi(a_i)g = \Phi(a_j)g_1$ then $a_jg = a_jg_1$, so $g = g_1$. Therefore $\Phi$ is injective and since all cosets have the same cardinality, $\Phi$ is bijective. If $a_iG_1$ and $a_iG_2$ are adjacent in $C(a_iG_0)$, then $g_i^{-1}a_i^{-1}a_ig_2 = g_i^{-1}g_2 \in S$. Therefore $\Phi(a_i^{-1}g_i) = g_i^{-1}a_i^{-1}a_ig_2 = g_i^{-1}g_2 \in S$ and then $\Phi(a_i)g_1$ and $\Phi(a_i)g_2$ are adjacent in $C(a_iG_0)$, and the first part of the lemma follows. For the second part, let $a \in a_iG_0$ and $\delta a \in a_iG_0$. Clearly the map $a \to \delta a$ define an isomorphism between $C(a_iG_0)$ and $C(\delta a_iG_0)$ and since $S$ is normal, $a^{-1}\delta a \in S$, therefore $\{a, \delta a\}$ is an edge in $Cay(G, S)$ (see Figure 1), and the lemma follows.

As a word on $\{\delta_1, \delta_2\}$ we will understand a product $s_1s_2\cdots s_{n-1}s_n$ of $\delta_1$ and $\delta_2$, where two consecutive elements in the product are not powers of the same elements, that is to say, if $s_i \in \{a\}$ then $s_{i+1}, s_{i-1} \notin \{a\}$. The length of a word $s_1s_2\cdots s_{n-1}s_n$ is $n$. Since $G = \langle \delta_1, \delta_2 \rangle$, it follows that for each $\alpha \in G$, $\alpha = s_1s_2\cdots s_{n-1}s_n$ for some word $s_1s_2\cdots s_{n-1}s_n$ on $\{\delta_1, \delta_2\}$. For each $\alpha \in G$, let $\ell(\alpha)$ be the minimum length of a word on $\{\delta_1, \delta_2\}$ such that $\alpha = s_1s_2\cdots s_{n-1}s_n$.

Let $H_0 = \{G_0\}$ and, for each $k \geq 1$, $H_k = \{a_iG_0 \in \mathcal{P} : \ell(a_i) = k\}$.

Given a coset $a_iG_0 \in \mathcal{P}$, let $L[a_iG_0] = \{a_iG_0\} \cup \{\delta a_iG_0 \in \mathcal{P} : \delta \in \{\delta_1, \delta_2\}, r \geq 1\}$. Observe that if $\ell(a_i) = k$, then for every $aG_0 \in L[a_iG_0] \backslash \{a_iG_0\}$, $\ell(a) = k+1$, and the number of cosets in the set $\{\delta a_iG_0 \in \mathcal{P} : \delta \in \{\delta_1, \delta_2\}, r \geq 1\}$ depends on the commutativity of the words $\delta a_1\delta_1^2, \delta_2 a_1\delta_2^2, \ldots, \delta^{n-1} a_1\delta_j^2$ for $j \geq 1$.

Lemma 5. If for some $k \geq 1$, $\delta^k a_iG_0 \in L[a_iG_0]$, then $\delta^{k-1} a_iG_0 \in L[a_iG_0]$.

Proof. Let us suppose that for some $k \geq 1$, $\delta^k a_iG_0 \in L[a_iG_0]$ and let $b \in G$ such that $\delta^{k-1} a_i = bG_0 \in \mathcal{P}$. Thus, $\delta^{k-1} a_i = b\delta_i^j$ and therefore $\delta^k a_i = \delta b\delta_i^j$ which implies that $\delta^k a_i \in bG_0 = \delta^k a_iG_0$. Hence $\delta^k a_i = \delta b$ and $\delta^{k-1} a_i = b$, and the result follows.
Observe that for each \( k \geq 0 \), the set \( \{ L[a_iG_0] : a_iG_0 \in \mathcal{H}_k \} \) is a partition of \( \mathcal{H}_k \cup \mathcal{H}_{k+1} \). Given a coset \( a_iG_0 \in \mathcal{P} \), the subgraph of \( \text{Cay}(G, S) \) induced by \( L[a_iG_0] \) will be called a leaf.

Given a leaf \( M \) induced by \( L[a_iG_0] = \{ a_iG_0, \delta a_iG_0, \ldots, \delta^m a_iG_0 \} \) (with \( \delta \in \{ \delta_1, \delta_2 \} \)), and a pair of elements \( a_i\delta_1^0, a_i\delta_1^{t+1} \in a_iG_0 \), a path of \( \text{Cay}(G, S) \) with vertex-set \( \{ a_i\delta_1^0, a_i\delta_1^{t+1} \} \cup \bigcup_{j=1}^m \delta^j a_iG_0 \) which starts at \( a_i\delta_1^0 \), ends at \( a_i\delta_1^{t+1} \) and such that for every \( 1 \leq j \leq m \), there is \( s_j \) such that \( \{ \delta^j a_i\delta_1^0, \delta^j a_i\delta_1^{t+1} \} \) is an edge of the path \( P \), will be called an \( (a_i\delta_1^0, a_i\delta_1^{t+1}, M) \)-complete path.

**Lemma 6.** Let \( M \) be a leaf of \( \text{Cay}(G, S) \) induced by 

\[
L[a_iG_0] = \{ a_iG_0, \delta a_iG_0, \ldots, \delta^m a_iG_0 \}
\]

(with \( \delta \in \{ \delta_1, \delta_2 \} \)). For every pair of elements \( a_i\delta_1^0, a_i\delta_1^{t+1} \in a_iG_0 \) there is an \( (a_i\delta_1^0, a_i\delta_1^{t+1}, M) \)-complete path.

**Proof.** From Lemma 4 we see that any two “consecutive” subgraphs of the leaf, \( C(\delta^i aG_0) \) and \( C(\delta^{i+1} aG_0) \), are attached, and again, by Lemma 4, each subgraph of the leaf is isomorphic to \( C(G_0) \), which is a cycle of the form \( (e, \delta_1, \delta_2, \ldots, \delta_t^i = e) \). Since \( G = (\delta_1, \delta_2) \), from here it follows that for each \( 1 \leq k \leq m \),

\[
(\delta^k a_1, \delta^k a_1\delta_1, \delta^k a_1\delta_1^2, \ldots, \delta^k a_1\delta_1^n) = \delta^k a_1
\]

is a hamiltonian cycle of \( C(\delta^k aG_0) \).

Let \( a_i\delta_1^0, a_i\delta_1^{t+1} \in a_iG_0 \) and \( \delta \in \{ \delta_1, \delta_2 \} \). To simplify the notation, let \( a_0 = a_i\delta_1^0, \beta_0 = a_i\delta_1^{t+1}, \epsilon_0 = a_i\delta_1^t \), and for each \( 1 \leq k \leq m \), let \( a_k = \delta^k a_1\delta_1^t \), \( \beta_k = \delta^k a_1\delta_1^{t+1} \) and \( \epsilon_k = \delta^k a_1\delta_1^1 \).

**Case 1.** \( n \) is even (see Figure 2).

![Figure 2](image_url)

Let

\[
P = (a_0, a_1, \ldots, a_m = \delta^m a_1\delta_1^t, \delta^m a_1\delta_1^{t-1}, \ldots, (\delta^m a_1\delta_1^{t+1} = \beta_m), (\beta_{m-1} = \delta^{m-1} a_1\delta_1^t, \delta^{m-1} a_1\delta_1^{t+2}, \ldots, (\delta^{m-1} a_1\delta_1^{t-1} = \epsilon_{m-1}), (\epsilon_{m-2} = \delta^{m-2} a_1\delta_1^t, \delta^{m-2} a_1\delta_1^{t+2}, \ldots, (\delta^{m-2} a_1\delta_1^{t+1} = \beta_{m-2}), \ldots, (\epsilon_1 = \delta^1 a_1\delta_1^t, \delta^1 a_1\delta_1^{t-2}, \ldots, (\delta a_1\delta_1^{t+1} = \beta_1), \beta_0).\]
Case 2. $n$ is odd (see Figure 3).

Let

$$P = (\alpha_0, \alpha_1, \ldots, \alpha_m = \delta^mS_1\delta_1^{-1}, \delta^mS_1\delta_1^{-1+1}, \ldots, (\delta^mS_1\delta_1^{-1} = \epsilon_m),$$

$$(\epsilon_{m-1} = \delta^{m-1}S_1\delta_1^{-1}), \delta^{m-1}S_1\delta_1^{-1+2}, \ldots, (\delta^{m-1}S_1\delta_1^{-1+1} = \beta_{m-1}),$$

$$(\beta_{m-2} = \delta^{m-2}S_1\delta_1^{-1+1}), \delta^{m-2}S_1\delta_1^{-1+2}, \ldots, (\delta^{m-2}S_1\delta_1^{-1+1} = \epsilon_{m-2}), \ldots,$$

$$(\epsilon_1 = \delta a_i^t\delta_1^{-1}), \delta a_i^t\delta_1^{-2}, \ldots, (\delta a_i^t\delta_1^{-1+1} = \beta_1), \beta_0).$$

From here the result follows.

3. The Proofs of the Main Results

Proof of Theorem 1. Let $G = \langle \delta_1, \delta_2 \rangle$ be a non-abelian simple group and $Cay(G, S)$ be a normal Cayley graph with $\{\delta_1, \delta_2\} \subseteq S$. Let $G_0 = \langle \delta_1 \rangle$, and let $P = \{G_0, a_1G_0, \ldots, a_nG_0\}$ be the partition of $G$ in cosets induced by the subgroup $G_0$.

Let $H_0 = \{G_0\}$ and, for each $k \geq 1$, let $H_k = \{a_iG_0 \in P : \ell(a_i) = k\}$. Since $G$ is finite, it follows that for some $p \geq 1$, $G = \bigcup_{j=0}^{p} \left( \bigcup_{A \in H_j} A \right)$.

We will prove the result by showing, by induction on $k$, that for every $k \geq 1$ the subgraph of $Cay(G, S)$ induced by

$$\bigcup_{j=0}^{k} \left( \bigcup_{A \in H_j} A \right)$$

contains a hamiltonian cycle $C$ such that for each $a_jG_0 \in H_k$, there is $s_j$ such that $\{a_j^s, a_i^t\}$ is an edge of $C$.

For $k = 1$, observe that $H_0 = \{G_0\}$ and $H_1 = \{\delta_2G_0, \delta_2G_0, \ldots, \delta_2^mG_0\}$. Thus, the subgraph $M$ of $Cay(G, S)$ induced by $\bigcup_{j=0}^{1} \left( \bigcup_{A \in H_j} A \right)$ is the leaf of $Cay(G, S)$ induced by $L[G_0] = \{G_0, \delta_2G_0, \delta_2G_0, \ldots, \delta_2^mG_0\}$. Let $e, \delta_1 \in G_0$. By
Lemma 6 there is an \((e, \delta, M)-complete\) path \(P\). Therefore \(C = P \circ (\delta_1, \delta_2, \ldots, \delta_m)\) is a Hamiltonian cycle of \(M\) such that for every \(1 \leq j \leq m\), there is \(s_j\) such that \(\{\delta_j^{s_j}, \delta_j^{s_j+1}\} \subseteq \delta_jG_0\) is an edge of \(C\).

Suppose that the statement is true for \(1 \leq m \leq k\); let \(Q\) be the subgraph of \(\text{Cay}(G, S)\) induced by \(\bigcup_{j=0}^{k+1} \left( \bigcup_{A \in \mathcal{H}_j} A \right)\) and let \(Q'\) be the subgraph of \(\text{Cay}(G, S)\) induced by \(\bigcup_{j=0}^{k} \left( \bigcup_{A \in \mathcal{H}_j} A \right)\). By induction hypothesis, there is a Hamiltonian cycle \(C\) of \(Q'\) such that for each \(a_jG_0 \in \mathcal{H}_k\), there is \(s_j\) such that \(\{a_j^{s_j}, a_j^{s_j+1}\}\) is an edge of \(C\).

For each \(a_jG_0 \in \mathcal{H}_k\) by Lemma 6, there is an \((a_j^{s_j}, a_j^{s_j+1}, M)-complete\) path \(P\) with \(M\) the leaf induced by \(L[a_jG_0] = \{a_jG_0, \delta a_jG_0, \delta^2 a_jG_0, \ldots, \delta^m a_jG_0\}\). Therefore, by deleting from \(C\) the edge \(\{a_j^{s_j}, a_j^{s_j+1}\}\), and attaching to \(C\) an edge \(a_j^{s_j}, a_j^{s_j+1}\), we obtain a Hamiltonian cycle \(C'\) of the subgraph of \(\text{Cay}(G, S)\) induced by \(V(Q') \cup V(M)\), and such that for each \(1 \leq i \leq m\) there is \(s_i\) such that \(\{\delta^{s_i}, \delta a_j^{s_j+1}\}\) is an edge of \(C'\). Following this procedure for each coset in \(\mathcal{H}_k\), since \(\{L[a_jG_0] : a_jG_0 \in \mathcal{H}_k\}\) is a partition of \(\mathcal{H}_k \cup \mathcal{H}_{k+1}\), we obtain a Hamiltonian cycle \(C\) of \(Q\) such that for each \(a_jG_0 \in \mathcal{H}_{k+1}\), there is \(s_j\) such that \(\{a_j^{s_j}, a_j^{s_j+1}\}\) is an edge of \(C\). From here, the result follows. 

**Proof of Theorem 2.** We will prove the theorem by induction on the order of the group. For \(|G| = 3\), we see that \(G \cong \mathbb{Z}_3\) and the only possible normal Cayley graphs are \(\text{Cay}(\mathbb{Z}_3, \{1\})\) and \(\text{Cay}(G, \{1, 2\})\) which are both Hamiltonian graphs.

Let \(G\) be a finite group of order greater than \(3\), \(G = G_0 \triangleright G_1, \ldots, G_{l-1} \triangleright G_1\) be a composition series of \(G\), and let \(\{\delta_0, \ldots, \delta_{l-1}\} \subseteq G\) such that, for each \(0 \leq i \leq l\), \(G_i/G_{i-1} = (\delta_iG_{i+1}, \delta_{i+1}G_{i+1})\). Let \(\text{Cay}(G, S)\) be a normal Cayley graph with \(\{\delta_0, \ldots, \delta_{l+1}\} \subseteq S\).

Let \(S/G_1 = \{sG_1 : s \in S\}\) and consider the Cayley graph \(\text{Cay}(G/G_1, S/G_1)\). If \(G/G_1\) is an abelian group, it is known that \(\text{Cay}(G/G_1, S/G_1)\) contains a Hamiltonian cycle (see [4]). If \(\text{Cay}(G/G_1, S/G_1)\) is not an abelian group, consider the following.

**Claim 1.** \(\text{Cay}(G/G_1, S/G_1)\) is a normal Cayley graph.

**Proof.** Let \(g \in G\) and \(s \in S\). Since \(G_1\) is a normal subgroup it follows that \(g^{-1}G_1sgG_1 = g^{-1}sgG_1\) and since \(S\) is a normal connection set, \(g^{-1}sg = s_1 \in S\). Therefore \(g^{-1}sgG_1 = s_1G_1 \subseteq S/G_1\) and the claim follows. \(\square\)

Thus, by Claim 1, \(\text{Cay}(G/G_1, S/G_1)\) is a normal Cayley graph; \(G/G_1 = \langle \delta_0 G_1, \delta_1 G_1 \rangle\) is a simple non-abelian group and, by hypothesis, \(\{\delta_0, \delta_1\} \subseteq S\) which implies that \(\{\delta_0 G_1, \delta_1 G_1\} \subseteq S/G_1\). Therefore, from Theorem 1 it follows that there is a Hamiltonian cycle in \(\text{Cay}(G/G_1, S/G_1)\).
Let $C = (G_1, g_1G_1, \ldots, g_nG_1, G_1)$, with $n = |G/G_1|$, be a hamiltonian cycle in $Cay(G/G_1, S/G_1)$ (see Figure 4).

On the other hand, let $S |_{G_1} = S \cap G_1$ and consider the Cayley graph $Cay(G_1, S |_{G_1})$.

**Claim 2.** $Cay(G_1, S |_{G_1})$ is a normal Cayley graph.

**Proof.** Since $S$ is a normal connection set and $G_1$ is a normal subgroup of $G$ we see that $g^{-1}Sg = S$ and $g^{-1}G_1g = G_1$, so $g^{-1}(S |_{G_1})g = g^{-1}(S \cap G_1)g = S \cap G_1 = S |_{G_1}$. □

**Claim 3.** $\{\delta_1, \ldots, \delta_{i+1}\} \subseteq S |_{G_1}$.

**Proof.** Since for each $i \in \{0, \ldots, l+1\}$ we have $G_i/G_{i+1} = \langle \delta_iG_{i+1}, \delta_{i+1}G_{i+1} \rangle$ it follows that $\delta_i \in G_i \subset G_1$ and $\delta_i \in G_1$ for all $1 \leq i \leq l+1$. □

Clearly $|G_1| < |G_0|$, and since $G_1 \supset G_2, \ldots, G_{i-1} \supset G_1$ is a composition series of $G_1$, by Claims 2 and 3 and by induction hypothesis we see that there is a hamiltonian cycle $C'$ in $C(G_1, S |_{G_1})$. Let $C' = (1, n_1, n_2, \ldots, n_{l-1}, 1)$ with $i = |G_1|$. Let $g_lG_1$ and $g_{l+1}G_1$ be two consecutive vertices of the hamiltonian cycle $C$ of $Cay(G/G_1, S/G_1)$. By definition $(g_lG_1)^{-1}g_{l+1}G_1 \in S/G_1$ which implies that $G_1g_l^{-1}g_{l+1}G_1 = S_1G_1$ with $s_1 \in S$. Thus $g_l^{-1}g_{l+1} = s_1n_l$ with $n_l \in G_1$ and then $g_l^{-1}g_{l+1}n_l^{-1} = s_1 \in S$. Therefore, for every $n_j \in G_1$, we see that

$$n_j^{-1}g_l^{-1}g_{l+1}n_l^{-1}n_j = n_j^{-1}s_1n_j \in S,$$

which implies that for every $n_j \in G_1$, $gn_j$ is adjacent to $g_{l+1}n_l^{-1}n_j$ in $Cay(G, S)$. 
Observe that the map $g_{ln_j} \rightarrow g_{l+1n^{-1}_j}$ defines a bijection between $g_lG_1$ and $g_{l+1}G_1$, and that, given $\alpha, \beta \in G_1$, we see that $(g_l\alpha)^{-1}g_l\beta = \alpha^{-1}\beta \in S$ if and only if

$$
\left(g_{l+1n^{-1}_l}\alpha\right)^{-1}\left(g_{l+1n^{-1}_l}\beta\right) = \alpha^{-1}n_l^{-1}g_{l+1}n_l^{-1}\beta = \alpha^{-1}\beta \in S,
$$

which implies that the subgraphs of $Cay(G, S)$ induced by $g_lG_1$ and $g_{l+1}G_1$ are attached (see Figure 5).

From here, and by an analogous argument than in the proof for the case $k = 1$ in Theorem 1, we obtain a hamiltonian cycle in $Cay(G, S)$ (see Figure 6).

**References**


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