GENERALIZED SUM LIST COLORINGS OF GRAPHS

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Abstract

A (graph) property $\mathcal{P}$ is a class of simple finite graphs closed under isomorphisms. In this paper we consider generalizations of sum list colorings of graphs with respect to properties $\mathcal{P}$.

If to each vertex $v$ of a graph $G$ a list $L(v)$ of colors is assigned, then in an $(L, \mathcal{P})$-coloring of $G$ every vertex obtains a color from its list and the subgraphs of $G$ induced by vertices of the same color are always in $\mathcal{P}$. The $\mathcal{P}$-sum choice number $\chi_{\mathcal{P}}^{sc}(G)$ of $G$ is the minimum of the sum of all list sizes such that, for any assignment $L$ of lists of colors with the given sizes, there is always an $(L, \mathcal{P})$-coloring of $G$.

We state some basic results on monotonicity, give upper bounds on the $\mathcal{P}$-sum choice number of arbitrary graphs for several properties, and determine the $\mathcal{P}$-sum choice number of specific classes of graphs, namely, of all complete graphs, stars, paths, cycles, and all graphs of order at most 4.

Keywords: sum list coloring, sum choice number, generalized sum list coloring, additive hereditary graph property.

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1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$, and for every vertex $v \in V$ let $L(v)$ be a set (list) of available colors. The graph $G$ is called $L$-colorable if there is a proper coloring $c$ of the vertices with $c(v) \in L(v)$ for all $v \in V$. A function $f$ from the vertex set $V$ of $G$ to the positive integers is called a choice function of $G$ if $G$ is $L$-colorable for every list assignment $L$ with $|L(v)| = f(v)$ for all $v \in V$. If the list length of all vertices coincide then this is the ordinary list colorability. The sum choice number of $G$ is called $\chi_{sc}(G)$ denotes the minimum of $\sum_{v \in V} f(v)$ over all choice functions $f$ of $G$. Since the considered colorings are proper, vertices of the same color induce an edgeless graph.

Sum list colorings were introduced by Isaak in 2002 [7]. Results on the sum choice number can be found, e.g., in [1, 2, 6–9, 11].

In this paper we examine a generalization of this concept. We consider vertex assignments $L$, choice functions $f$ of the vertices colored $i$ belongs to some specific given class of graphs (and not necessarily to the class of edgeless graphs).

A (graph) property $\mathcal{P}$ is a non-empty isomorphism-closed subclass of $I$, where $I$ denotes the class of all finite simple graphs (see [3]). We assume in the entire paper that $K_1 \in \mathcal{P}$ for the considered properties $\mathcal{P}$. A property $\mathcal{P}$ is called additive if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$ are disjoint where $G$ and $H$ are two graphs of $I$. A property $\mathcal{P}$ is called hereditary (induced hereditary) if $G \in \mathcal{P}$ and $H \subseteq G$ ($H \leq G$) implies $H \in \mathcal{P}$, where $H \subseteq G$ ($H \leq G$) means that $H$ is a subgraph (an induced subgraph) of $G$. Therefore, every hereditary property is also induced hereditary. Obviously, $K_1 \in \mathcal{P}$ for any (induced) hereditary property $\mathcal{P}$.

The graph $G$ is called $(L, \mathcal{P})$-colorable if there exists a mapping (coloring) $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ for each vertex $v \in V(G)$ and, for each $i \in \mathbb{N}$, the graph induced in $G$ by the vertices colored $i$ belongs to $\mathcal{P}$. Such a mapping is called an $(L, \mathcal{P})$-coloring or a $\mathcal{P}$-list coloring of $G$.

Let $f : V(G) \to \mathbb{N}$ be a function which assigns list sizes to the vertices of $G$. The graph $G$ is $(f, \mathcal{P})$-choosable and $f$ is a $\mathcal{P}$-choice function of $G$ if for every list assignment $L$ with list sizes specified by $f$, that is, $|L(v)| = f(v)$ for each $v \in V(G)$, the graph $G$ is $(L, \mathcal{P})$-colorable. The $\mathcal{P}$-sum choice number $\chi_{sc}^\mathcal{P}(G)$ of a graph $G$ is the minimum of the sum of list sizes in $f$ taken over all $\mathcal{P}$-choice functions $f$ of $G$. Thus

$$\chi_{sc}^\mathcal{P}(G) = \min \left\{ \sum_{v \in V(G)} f(v) : f \text{ is a } \mathcal{P}\text{-choice function of } G \right\}.$$ 

We use the following standard notation for specific graph properties.
\[ O = \{ G \in I : E(G) = \emptyset \}, \]
\[ O_k = \{ G \in I : \text{each component of } G \text{ has at most } k + 1 \text{ vertices} \}, \]
\[ S_k = \{ G \in I : \Delta(G) \leq k \}, \]
\[ D_k = \{ G \in I : \text{each subgraph of } G \text{ contains a vertex of degree } \leq k \}, \]
\[ O^k = \{ G \in I : \chi(G) \leq k \}, \]
\[ J_k = \{ G \in I : \chi'(G) \leq k \}, \]
\[ \mathcal{I}_k = \{ G \in I : G \text{ does not contain } K_{k+2} \}. \]

All these properties are additive induced hereditary properties. Note that \( O_k \subseteq S_k \subseteq D_k \subseteq O^{k+1} \subseteq \mathcal{I}_k \) for \( k \geq 1 \) (see [3]).

The completeness \( c(\mathcal{P}) \) of an induced hereditary property \( \mathcal{P} \) is defined as \( c(\mathcal{P}) = \max\{ k : K_{k+1} \in \mathcal{P} \} \): we write \( c(\mathcal{P}) = \infty \) if the maximum does not exist. For example, \( c(\mathcal{P}) = 0 \) if and only if \( \mathcal{P} \subseteq O \), \( c(I) = \infty \), and \( c(O_k) = c(S_k) = c(D_k) = c(O^{k+1}) = c(\mathcal{I}_k) = k \), as well as \( c(J_k) = k \) if \( k \) is odd and \( c(J_k) = k - 1 \) if \( k \) is even. Moreover, if \( \mathcal{P} \) is an additive hereditary property with \( c(\mathcal{P}) = k \), then \( O_k \subseteq \mathcal{P} \subseteq \mathcal{I}_k \) (see [3]).

The \( \mathcal{P} \)-sum choice number is a generalization of the usual sum choice number since \( \chi^O_{ac}(G) = \chi_{ac}(G) \) for all graphs \( G \). This concept was introduced in [4]. In [4,5] the \( \mathcal{P} \)-sum choice number for induced hereditary properties \( \mathcal{P} \) was studied, especially for \( \mathcal{P} = D_1 \), that is, for the class of acyclic graphs. In [10] lower and upper bounds on \( \chi^\mathcal{P}_{ac}(G) \) are given for arbitrary induced hereditary properties \( \mathcal{P} \) where \( G \) is the union of two graphs with exactly one vertex in common.

This paper is organized as follows. In Section 2 we collect some basic results, most of them from the literature. In Section 3 we present upper bounds on the \( \mathcal{P} \)-sum choice number for arbitrary graphs and specific additive induced hereditary properties \( \mathcal{P} \), namely \( D_k, \mathcal{I}_k, J_k, O_k, O^k, \) and \( S_k \). Moreover, Theorem 10 contains a general upper bound for all additive hereditary properties and Theorem 14 for all additive properties. In Section 4 we determine the \( \mathcal{P} \)-sum choice number of some known classes of graphs including complete graphs, stars, paths, cycles, and all graphs of order at most 4 for arbitrary additive induced hereditary properties \( \mathcal{P} \).

2. Preliminaries

In this section we state some basic results.

**Proposition 1.** Let \( \mathcal{P}, \mathcal{Q} \) be arbitrary properties. If \( \mathcal{P} \subseteq \mathcal{Q} \), then \( \chi^\mathcal{Q}_{ac}(G) \leq \chi^\mathcal{P}_{ac}(G) \).

**Proof.** Each \((L, \mathcal{P})\)-coloring of \( G \) is also an \((L, \mathcal{Q})\)-coloring of \( G \) since each graph in \( \mathcal{P} \) is contained in \( \mathcal{Q} \). This implies that each \( \mathcal{P} \)-choice function of \( G \) is a \( \mathcal{Q} \)-choice function of \( G \), hence \( \chi^\mathcal{Q}_{ac}(G) \leq \chi^\mathcal{P}_{ac}(G) \).
The following proposition collects some bounds that can be found in [4] or deduced from some results there.

**Proposition 2** [4]. Let $\mathcal{P}$ be a hereditary (an induced hereditary) property and $H \subseteq G$ ($H \leq G$). Then $\chi_{sc}^P(G) = \chi_{sc}^P(H) + \chi_{sc}^P(G - V(H)) \geq \chi_{sc}^P(H) + |V(G)| - |V(H)|$.

A direct implication is the following result.

**Corollary 3.** If $\mathcal{P}$ is an induced hereditary property and $V(G) = V_1 \cup \cdots \cup V_l$ is a partition of the vertex set of a graph $G$, then $\chi_{sc}^P(G) \geq \sum_{i=1}^{l} \chi_{sc}^P(G[V_i])$.

**Proof.** The result follows by iterative application of Proposition 2 on the induced subgraphs $G[V_1], \ldots, G[V_{l-1}]$. $\blacksquare$

For $\mathcal{P} = \mathcal{O}$ (that is, for the sum choice number) the lower bound can be improved to $\chi_{sc}^O(G) \geq \sum_{i=1}^{l} \chi_{sc}^O(G[V_i]) + l - c(G)$ where $c(G)$ is the number of components of $G$ (see [6,9]).

In the proof in [9], $l - c(G)$ edges that induce bridges (that is, blocks $K_2$) were added to the subgraph $G[V_1] \cup \cdots \cup G[V_l]$, and each bridge increases the sum choice number by $\chi_{sc}(K_2) - 2 = 3 - 2 = 1$. Since $\chi_{sc}(K_2) = 2$ for $\mathcal{P} \neq \mathcal{O}$, the larger subgraph does not increase the lower bound on the $\mathcal{P}$-sum choice number if $\mathcal{P} \neq \mathcal{O}$.

The following result is proved in [4] using some hypergraph method.

**Proposition 4** [4]. Let $\mathcal{P}$ be an additive induced hereditary property. If $G = F \cup H$ is the disjoint union of $F$ and $H$, respectively. Let $L$ be a list assignment of $G$ with sizes determined by $f$. An $(L|V(F), \mathcal{P})$-coloring of $F$ and an $(L|V(H), \mathcal{P})$-coloring of $H$ provide an $(L, \mathcal{P})$-coloring of $G = F \cup H$ since for each color $i$ the subgraphs of $F$ and of $H$ induced by vertices of color $i$ are disjoint and contained in $\mathcal{P}$, hence their union, that is, the corresponding induced subgraph of $G = F \cup H$, is also in $\mathcal{P}$ since $\mathcal{P}$ is additive. This implies $\chi_{sc}^P(G) \leq \chi_{sc}^P(F) + \chi_{sc}^P(H)$. Equality holds by Proposition 2. $\blacksquare$

This result implies that the $\mathcal{P}$-sum choice number of a graph is equal to the sum of the $\mathcal{P}$-sum choice numbers of its components for additive induced hereditary properties $\mathcal{P}$.

**Corollary 5.** Let $\mathcal{P}$ be an additive induced hereditary property. If $G$ has $c$ components $H_1, \ldots, H_c$, then $\chi_{sc}^P(G) = \chi_{sc}^P(H_1) + \cdots + \chi_{sc}^P(H_c)$. 

3. Upper Bounds for Specific Properties

In this section we present upper bounds on \( \chi_{sc}^P(G) \) for arbitrary graphs \( G = (V, E) \) and specific additive (induced) hereditary properties \( P \).

The greedy bound \( GB(G) = |V| + |E| \) is an upper bound on the sum choice number \( \chi_{sc}(G) = \chi_{sc}^O(G) \), and obviously \( \chi_{sc}^P(G) = |V| \) holds (each vertex obtains a list of size 1). Since \( O \subseteq P \subseteq I \) for any additive property \( P \), we have \( |V| \leq \chi_{sc}^P(G) \leq |V| + |E| \) by Proposition 1.

The next result states an upper bound on the \( D_1 \)-sum choice number proved in [4]. We present a simple direct proof.

**Theorem 6** [4]. \( \chi_{sc}^{D_1}(G) \leq |E| + c(G) \), where \( c(G) \) is the number of components of \( G \).

**Proof.** If \( G \) is connected, then order the vertices \( V = \{v_1, \ldots, v_n\} \) in such a way that \( v_i, i \geq 2 \), is connected to \( v_j, 1 \leq j \leq i - 1 \). Set \( f(v_1) = 1 \) and
\[
\sum_{i=1}^n f(v_i) = |\{e \in E : e = v_iv_j, j < i\}| \geq 1 \text{ for } i \geq 2.
\]
Then \( \sum_{i=1}^n f(v_i) = |E| + 1 \). We prove that \( f \) is a \( D_1 \)-choice function of \( G \) by a greedy \( (L, P) \)-coloring of \( v_1, \ldots, v_n \) for an arbitrary list assignment \( L \) with list sizes defined by \( f \). Obviously, \( v_1 \) can be colored. Assume that \( v_1, \ldots, v_{i-1} \) are colored and consider \( v_i, i \geq 2 \). If all neighbors of \( v_i \) in \{ \( v_1, \ldots, v_{i-1} \) \} are colored distinctly, then \( v_i \) can be colored by any color from its list since neither adding an isolated vertex nor a pending edge to an acyclic graph does create a cycle. If at least two neighbors have the same color, then there is a color in \( L(v_i) \) not used in any neighbor of \( v_i \), which can be used to color \( v_i \), and we are done.

If \( G \) is not connected, then use Corollary 5 and apply the preceding result for all components of \( G \).

Note that this bound can be improved (see Theorem 7), but it is also tight in some specific cases. For example, it holds that \( \chi_{sc}^{D_1}(C_n) = n + 1 = |E(C_n)| + 1 \) (see Theorem 19).

The following result can be deduced from Corollary 9 in [4]. We give a direct proof instead without using hypergraph methods.

**Theorem 7.** Let \( v_1, \ldots, v_n \) be an arbitrary ordering of the vertices of \( G \) and \( G_i = G[\{v_1, \ldots, v_i\}], i \in \{1, \ldots, n\} \). Then
\[
\chi_{sc}^{D_i}(G) \leq n + \sum_{i=1}^n \left\lfloor \frac{d_{G_i}(v_i)}{k+1} \right\rfloor \leq |V| + \frac{|E|}{k+1}.
\]

**Proof.** Define \( f : V \rightarrow \mathbb{N} \) by \( f(v_i) = 1 + \left\lfloor \frac{d_{G_i}(v_i)}{k+1} \right\rfloor \), \( i \in \{1, \ldots, n\} \). Then
\[
\sum_{i=1}^n f(v_i) = n + \sum_{i=1}^n \left\lfloor \frac{d_{G_i}(v_i)}{k+1} \right\rfloor \leq n + \frac{1}{k+1} \sum_{i=1}^n d_{G_i}(v_i) = |V| + \frac{1}{k+1} |E|.
\]
We prove in the following that \( f \) is a \( D_k \)-choice function of \( G \). Let \( L \) be a list assignment with \( |L(v)| = f(v) \) for every \( v \in V \). Vertex \( v_1 \) can be colored with the color from its list. Assume that vertices \( v_1, \ldots, v_{i-1} \) are already colored in a partial \((L, D_k)\)-coloring of \( G \), and consider the next vertex \( v_i \). If at most \( k \) neighbors of \( v_i \) are colored by a color \( \alpha \in L(v_i) \), then \( v_i \) can also be colored by \( \alpha \) since the subgraph \( C_\alpha \) of \( G_i \) induced by vertices of color \( \alpha \) is \( k \)-degenerate. Each subgraph of \( C_\alpha \) without \( v_i \) has a vertex of degree at most \( k \) because of the assumed coloring, and if \( v_i \) is a vertex of the subgraph, then \( v_i \) is a vertex of degree at most \( k \). This means that at most \( \left\lfloor \frac{\delta_{G_i}(v_i)}{k+1} \right\rfloor \) colors cannot be used for \( v_i \), but \( L(v_i) \) has at least one color which is not forbidden. Therefore, the \((L, D_k)\)-coloring of \( G \) can be completed and \( f \) is a \( D_k \)-choice function of \( G \). \( \blacksquare \)

If \( D_k \subseteq \mathcal{P} \), then \( \chi_{\mathcal{P}}^{D_k}(G) \leq \chi_{\mathcal{P}}^{D_k}(G) \) by Proposition 1, thus the upper bound of Theorem 7 is also an upper bound on the \( \mathcal{P} \)-sum choice number of \( G \). Since \( D_k \subseteq O^{k+1} \subseteq I_k \) we obtain the following bounds.

Corollary 8. Let \( v_1, \ldots, v_n \) be an arbitrary ordering of the vertices of \( G \) and \( G_i = G[\{v_1, \ldots, v_i\}] \), \( i \in \{1, \ldots, n\} \). Then

\[
\chi_{\mathcal{P}}^{O_k}(G) \leq n + \sum_{i=1}^{n} \left\lfloor \frac{\delta_{G_i}(v_i)}{k} \right\rfloor \leq |V| + \frac{|E|}{k}.
\]

Note that for \( O^1 = O \) Corollary 8 gives the greedy bound of \( G \): \( \chi_{\mathcal{P}}^{O_k}(G) = \chi_{\mathcal{P}}^{O_k}(G) \leq GB(G) = |V| + |E| \).

Corollary 9. Let \( v_1, \ldots, v_n \) be an arbitrary ordering of the vertices of \( G \) and \( G_i = G[\{v_1, \ldots, v_i\}] \), \( i \in \{1, \ldots, n\} \). Then

\[
\chi_{\mathcal{P}}^{I_k}(G) \leq n + \sum_{i=1}^{n} \left\lfloor \frac{\delta_{G_i}(v_i)}{k+1} \right\rfloor \leq |V| + \frac{|E|}{k+1}.
\]

Let us mention that it is possible to generalize these results and prove that \( f : V \to \mathbb{N} \) with \( f(v_i) = 1 + \left\lfloor \frac{\delta_{G_i}(v_i)}{d(P, G)} \right\rfloor \) is a \( \mathcal{P} \)-choice function of \( G \) for an appropriate divisor \( d(P, G) \). Note that \( d(P, G) = 1 \) leads to a choice function \( f \) with sum of list sizes equal to the greedy bound \( GB(G) \) which is indeed an upper bound on the \( \mathcal{P} \)-choice number of \( G \). In Corollary 9 in [4] a divisor \( d(P, G) = \delta(P) \) was used, that is, the smallest minimum degree of a minimal forbidden graph of \( \mathcal{P} \) (which is a graph not contained in \( \mathcal{P} \) whose proper induced subgraphs are all in \( \mathcal{P} \)). Obviously, it suffices to consider just subgraphs of \( G \).

The following result provides a general upper bound on \( \chi_{\mathcal{P}}^{\mathcal{P}}(G) \).
Theorem 10. Let $\mathcal{P}$ be an additive hereditary property, $v_1, \ldots, v_n$ be an arbitrary ordering of the vertices of $G$, $G_i = G[\{v_1, \ldots, v_i\}]$ for $i \in \{1, \ldots, n\}$, and $k = c(\mathcal{P})$. Then

$$
\chi_{sc}^{\mathcal{P}}(G) \leq n + \sum_{i=1}^{n} \min \left\{ d_{G_i}(v_i), \left\lceil \frac{i-1}{k+1} \right\rceil \right\}.
$$

**Proof.** Define $f : V(G) \to \mathbb{N}$ by $f(v_i) = 1 + \min \left\{ d_{G_i}(v_i), \left\lceil \frac{i-1}{k+1} \right\rceil \right\}$, $i \in \{1, \ldots, n\}$. Then $\sum_{i=1}^{n} f(v_i) = n + \sum_{i=1}^{n} \min \left\{ d_{G_i}(v_i), \left\lceil \frac{i-1}{k+1} \right\rceil \right\}$ as stated.

Let $L$ be a list assignment with list sizes defined by $f$. Vertex $v_1$ can be colored with the color from its list of size $f(v_1) = 1$. Assume that vertices $v_1, \ldots, v_{i-1}$ are already colored in a partial $(L, \mathcal{P})$-coloring of $G$, and consider next the vertex $v_i$.

If $f(v_i) = 1 + d_{G_i}(v_i)$, then we can color $v_i$ by a color distinct from the colors of all of its already colored $d_{G_i}(v_i)$ neighbors. Thus $v_i$ belongs to none of the so far existing components induced by vertices of the same color. Let now $f(v_i) = 1 + \left\lceil \frac{i-1}{k+1} \right\rceil$. If at most $k$ vertices in $v_1, \ldots, v_{i-1}$ are colored with a color $\alpha \in L(v_i)$, then $v_i$ can also be colored by $\alpha$ since all subgraphs of $K_{k+1}$ are contained in the hereditary property $\mathcal{P}$. Hence at most $\left\lceil \frac{i-1}{k+1} \right\rceil$ colors are forbidden for vertex $v_i$, but $L(v_i)$ contains at least one additional color which can be used to color $v_i$.

This implies that the $(L, \mathcal{P})$-coloring of $G$ can be inductively completed, and therefore $f$ is a $\mathcal{P}$-choice function of $G$. 

Note that for $\mathcal{P} \supseteq \mathcal{D}_k$ the degree $d_{G_i}(v_i)$ can be replaced by $\left\lceil \frac{d_{G_i}(v_i)}{k+1} \right\rceil$. Since $d_{G_i}(v_i) \leq i - 1$, we obtain the upper bound of Theorem 7.

**Corollary 11.** Let $\mathcal{P}$ be an additive hereditary property, $G$ be a graph with $n$ vertices, and $k = c(\mathcal{P})$. Then

$$
\chi_{sc}^{\mathcal{P}}(G) \leq \chi_{\mathcal{O}_k}(G) \leq n + \sum_{i=1}^{n} \left\lceil \frac{i-1}{k+1} \right\rceil.
$$

**Proof.** The first inequality follows from Proposition 1 since $\mathcal{O}_k \subseteq \mathcal{P}$, the second by Theorem 10 since $c(\mathcal{O}_k) = k$.

For complete graphs equality holds in Theorem 10 and Corollary 11 (see Theorem 15).

The square $G^2$ of a graph $G$ is the graph with $V(G^2) = V(G)$ and $uv \in E(G^2)$ if and only if the distance between $u$ and $v$ in $G$ is at most 2.
Theorem 12. Let \( v_1, \ldots, v_n \) be an arbitrary ordering of the vertices of \( G \) and \( G_i = G[\{v_1, \ldots, v_i\}] \) for \( i \in \{1, \ldots, n\} \). Then

\[
\chi_{sc}^k(G) \leq n + \sum_{i=1}^{n} \left\lfloor \frac{d_{G_i^2(v_i)}}{k+1} \right\rfloor \leq |V(G)| + \left| E(G^2) \right| \frac{k}{k+1}.
\]

Proof. Define \( f : V(G) \to \mathbb{N} \) by \( f(v_i) = 1 + \left\lfloor \frac{d_{G_i^2(v_i)}}{k+1} \right\rfloor, i \in \{1, \ldots, n\} \). Then

\[
\sum_{i=1}^{n} f(v_i) = n + \sum_{i=1}^{n} \left\lfloor \frac{d_{G_i^2(v_i)}}{k+1} \right\rfloor \leq n + \sum_{i=1}^{n} \frac{d_{G_i^2(v_i)}}{k+1} = |V(G)| + \frac{1}{k+1} \left| E(G^2) \right|.
\]

The first inequality follows from \( G_i = G[\{v_1, \ldots, v_i\}] \) which implies that two vertices at distance at most 2 in \( G_i \) have also distance at most 2 in \( G \), that is, \( G_i^2 \subseteq G^2[\{v_1, \ldots, v_i\}] \).

We prove in the following that \( f \) is an \( S_k \)-choice function of \( G \). Let \( L \) be a list assignment with list sizes defined by \( f \). Vertex \( v_i \) can be colored with the color from its list. Assume that \( v_1, \ldots, v_{i-1} \) are already colored in a partial \( (L, S_k) \)-coloring of \( G \) and consider the next vertex \( v_i, i \in \{2, \ldots, n\} \). A color \( \alpha \in L(v_i) \) is forbidden for \( v_i \) if either \( v_i \) is adjacent to at least \( k+1 \) vertices of color \( \alpha \) in \( G_i \), or if \( v_i \) is adjacent to a vertex \( v_j \) of color \( \alpha, j < i \), which is adjacent to at least \( k \) vertices of color \( \alpha \). In either case, at least \( k+1 \) vertices of \( N_{G_i^2}(v_i) \) must be already colored with \( \alpha \) in order to forbid this color for \( v_i \). This implies that at most \( \left\lfloor \frac{1}{k+1} d_{G_i^2(v_i)} \right\rfloor \) different colors are forbidden, hence \( v_i \) can be colored with a color from \( L(v_i) \), and the \( (L, S_k) \)-coloring of \( G \) can be inductively completed. \( \blacksquare \)

If \( S_k \subseteq \mathcal{P} \), then \( \chi_{sc}^P(G) \leq \chi_{sc}^{S_k}(G) \) by Proposition 1, thus the upper bound of Theorem 12 is also an upper bound on the \( \mathcal{P} \)-sum choice number of \( G \). This improves the upper bound on \( \chi_{sc}^P(G) \) from Corollary 11 if \( S_k \subseteq \mathcal{P} \) since \( d_{G_i^2(v_i)} \leq i-1 \). For example, the Theorem of Vizing states that \( \chi'(G) \leq \Delta(G) + 1 \), that is, \( \Delta(G) \leq k-1 \) implies \( \chi'(G) \leq k \). Therefore, \( \chi_{sc}^k(G) \leq \chi_{sc}^{S_k-1}(G) \) by Proposition 1. From Theorem 12 we obtain the following bounds.

Corollary 13. Let \( v_1, \ldots, v_n \) be an arbitrary ordering of the vertices of \( G \), \( G_i = G[\{v_1, \ldots, v_i\}] \) for \( i \in \{1, \ldots, n\} \), and \( k \geq 1 \). Then

\[
\chi_{sc}^{J_k}(G) \leq n + \sum_{i=1}^{n} \left\lfloor \frac{d_{G_i^2(v_i)}}{k} \right\rfloor \leq |V(G)| + \frac{|E(G^2)|}{k}.
\]
In [9] an upper bound on the sum choice number of $G$ was proved that depends on a partition $V(G) = V_1 \cup \cdots \cup V_l$ of the vertex set of $G$ and the sum choice numbers of the induced subgraphs $G[V_i], i = 1, \ldots, l$. The bound can be generalized as follows.

**Theorem 14.** If $\mathcal{P}$ is an additive property and $V(G) = V_1 \cup \cdots \cup V_l$ is a partition of $V(G)$, then

$$\chi_{sc}^P(G) \leq \sum_{i=1}^{l} \chi_{sc}^P(G[V_i]) + |E(G)| - \sum_{i=1}^{l} |E(G[V_i])|. \tag{1}$$

**Proof.** For $i \in \{1, \ldots, l\}$ let $f_i : V_i \to \mathbb{N}$ be a $\mathcal{P}$-choice function of $G[V_i]$ with $\sum_{v \in V_i} f_i(v) = \chi_{sc}^P(G[V_i])$. Define $f : V(G) \to \mathbb{N}$ as follows:

$$f(v) = f_i(v) + |N(v) \cap (V_1 \cup \cdots \cup V_{i-1})| \quad \text{for } v \in V_i, i \in \{1, \ldots, l\}. \tag{2}$$

Consider an arbitrary list assignment $L$ with $|L(v)| = f(v)$ for each vertex $v \in V(G)$. Color at first the vertices of $V_1$ which is possible since $f|_{V_1} = f_1$ and $f_1$ is a $\mathcal{P}$-choice function of $G[V_1]$. Assume that all vertices of $V_1 \cup \cdots \cup V_{i-1}$ are colored by a partial $(L, \mathcal{P})$-coloring $\varphi$ of $G$ and consider next the set $V_i, i \in \{2, \ldots, l\}$.

A vertex $v \in V_i$ will be colored distinctly from the previously colored neighbors, that is, only the colors of $L_i(v) = L(v) \setminus \{\varphi(w) : w \in N(v) \cap (V_1 \cup \cdots \cup V_{i-1})\}$ will be used. Since $|L_i(v)| \geq f_i(v)$ for all $v \in V_i$ and $f_i$ is a $\mathcal{P}$-choice function of $G[V_i]$, each vertex $v \in V_i$ can be colored with a color from $L_i(v) \subseteq L(v)$. The coloring is a partial $(L, \mathcal{P})$-coloring of $G$ since $\mathcal{P}$ is additive.

This implies that $f$ is a $\mathcal{P}$-choice function of $G$ with

$$\sum_{v \in V(G)} f(v) = \sum_{i=1}^{l} \chi_{sc}^P(G[V_i]) + |E(G) \setminus E(G[V_1] \cup \cdots \cup G[V_l])|$$

$$= \sum_{i=1}^{l} \chi_{sc}^P(G[V_i]) + |E(G)| - \sum_{i=1}^{l} |E(G[V_i])|. \tag{3}$$

4. **Specific Graph Classes**

In this section we determine the $\mathcal{P}$-sum choice number of some well-known classes of graphs for arbitrary additive induced hereditary properties $\mathcal{P}$. We begin with complete graphs whose $\mathcal{P}$-sum choice numbers only depend on the complete graphs contained in $\mathcal{P}$, that is, on the completeness $c(\mathcal{P})$ of $\mathcal{P}$. The proof is similar to the proof for the determination of the sum choice number $\chi_{sc}(K_n)$ in [8]. In fact, the following theorem is a generalization of this result.
Theorem 15. Let \( b(n, k) = \sum_{i=1}^{n} \left( 1 + \left\lfloor \frac{i-1}{k+1} \right\rfloor \right) \) for \( n \in \mathbb{N}, k \in \mathbb{N}_0 \) and \( \mathcal{P} \) be an induced hereditary property. If \( c(\mathcal{P}) = k \), then \( \chi_{sc}(K_n) = b(n, k) \).

**Proof.** The proof of Theorem 10 implies \( \chi_{sc}(K_n) \leq b(n, k) \) if \( G = K_n \) since \( d_{G_i}(v_i) = i - 1 \). We only need to require that \( \mathcal{P} \) is induced hereditary, since the subgraphs of \( G \) induced by vertices of the same color are also complete and thus connected induced subgraphs.

Consider an arbitrary \( \mathcal{P} \)-choice function \( f \) of \( K_n \) and denote the vertices of \( K_n \) in increasing order with respect to \( f \), \( f(v_1) \leq \cdots \leq f(v_n) \). Assume that there is a vertex \( v_j, 1 \leq j \leq n \), with \( f(v_j) < 1 + \left\lfloor \frac{j-1}{k+1} \right\rfloor \). Since \( f(v_j) \geq 1, j - 1 \geq k + 1 \). Let \( L \) be the list assignment with initial lists, \( L(v_i) = \{1, \ldots, f(v_i)\} \) for each \( i \in \{1, \ldots, n\} \). Then in any \((L, \mathcal{P})\)-coloring the vertices in \( V' = \{v_1, \ldots, v_j\} \) will be colored by at most \( q = \left\lfloor \frac{j-1}{k+1} \right\rfloor \geq 1 \) colors \( 1, \ldots, q \). By the pigeonhole principle, there is a color \( \alpha \in \{1, \ldots, q\} \) used in at least \( \left\lceil \frac{j}{q} \right\rceil \) vertices of \( V' \). Let \( r \) be the integer \( 0 \leq r \leq k \) with \( j - 1 = q(k + 1) + r \). Then \( \left\lceil \frac{j}{q} \right\rceil = \left\lceil \frac{q(k+1)+r+1}{q} \right\rceil = k + 1 + \left\lfloor \frac{r+1}{q} \right\rfloor > k + 1 \), that is, the graph induced by the vertices of color \( \alpha \) is a complete graph with more than \( k + 1 \) vertices, a contradiction to \( c(\mathcal{P}) = k \). It follows that \( f(v_i) \geq 1 + \left\lfloor \frac{i-1}{k+1} \right\rfloor \) for every \( i \in \{1, \ldots, n\} \) and therefore \( \chi_{sc}(K_n) \geq b(n, k) \). \( \blacksquare \)

In the following proposition we compute \( b(n, k) \).

**Proposition 16.** For \( n \in \mathbb{N}, k \in \mathbb{N}_0 \) let \( n = q(k + 1) + r \) with \( q, r \in \mathbb{N}_0, r \leq k \). Then \( b(n, k) = \frac{1}{2} (q+1)(n+r) = \frac{1}{2} \left( \left\lceil \frac{n}{k+1} \right\rceil + 1 \right) \left( 2n - \left\lceil \frac{n}{k+1} \right\rceil (k+1) \right) \).

**Proof.** Let \( n = q(k + 1) + r \) with \( 0 \leq r \leq k \). Then

\[
b(n, k) = \sum_{i=1}^{n} \left( 1 + \left\lfloor \frac{i-1}{k+1} \right\rfloor \right) = n + \sum_{i=1}^{q(k+1)+1} \left\lfloor \frac{i-1}{k+1} \right\rfloor + \sum_{i=q(k+1)+1}^{q(k+1)+r} \left\lfloor \frac{i-1}{k+1} \right\rfloor
\]

\[
= n + (k+1) \sum_{j=0}^{q-1} j + rq = n + \frac{1}{2} (k+1) (q-1) q + rq
\]

\[
= \frac{1}{2} (k+1) q(q+1) + r(q+1) = \frac{1}{2} (q+1)(n+r)
\]

\[
= \frac{1}{2} \left( \left\lceil \frac{n}{k+1} \right\rceil + 1 \right) \left( 2n - \left\lceil \frac{n}{k+1} \right\rceil (k+1) \right).
\] \( \blacksquare \)

For example, if \( \mathcal{P} = \mathcal{O} \), then \( c(\mathcal{O}) = k = 0 \) and therefore \( \chi_{sc}(K_n) = \chi_{sc}(K_n) = \frac{1}{2} (n+1) n = n + \left( \frac{n}{2} \right) = |V(K_n)| + |E(K_n)| \) (see [8]). For properties \( \mathcal{P} \) with \( c(\mathcal{P}) = k = 1 \) we obtain \( \chi_{sc}(K_n) = \frac{1}{2} \left( \left\lceil \frac{n}{2} \right\rceil + 1 \right) \left( 2n - 2 \left\lceil \frac{n}{2} \right\rceil \right) \), that is,
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χ_psc(K_n) = \frac{1}{4}n(n+2) if n even and χ_psc(K_n) = \frac{1}{4}(n+1)^2 if n odd. This generalizes Theorem 31 of [4] on D_psc(K_n).

In the next theorems stars, paths, and cycles are considered. Their P-sum choice number again only depends on the connected induced subgraphs contained in P.

**Theorem 17.** Let P be an additive induced hereditary property. If m ∈ N and s = max{k : k ≤ m and K_{1,k} ∈ P} for P ≠ O and s = 0 for P = O, then χ_psc(K_{1,m}) = m + 1 + ⌊\frac{m}{s+1}\rfloor.

**Proof.** Let V = {z, v_1, \ldots, v_m} be the vertex set of K_{1,m} such that z has degree m. Define f : V → N by f(v_i) = 1 for i ∈ {1, \ldots, m} and f(z) = 1 + ⌊\frac{m}{s+1}\rfloor. We prove that f is a P-choice function of K_{1,m}. Consider an arbitrary list assignment L with list sizes defined by f. Each vertex v_i must be colored with the color from its own list L(v_i) for each i ∈ {1, \ldots, m} which is possible since P is additive. Each color which is used to color at most s vertices v_i can be used to color z since K_{1,s} ∈ P and P is induced hereditary which implies that also all substars are in P. Therefore, at most ⌊\frac{m}{s+1}\rfloor colors are forbidden for z which implies that z can be colored with a color from its list. Therefore, f is a P-choice function of K_{1,m} and χ_psc(K_{1,m}) ≤ \sum_{v ∈ V} f(v) = m + 1 + ⌊\frac{m}{s+1}\rfloor.

Consider now an arbitrary P-choice function f of K_{1,m} and assume without loss of generality that f(v_1) = \cdots = f(v_a) = 1 and f(v_{a+1}), \ldots, f(v_m) ≥ 2, a ≥ 0. Consider an arbitrary list assignment L with list sizes defined by f. As above, v_i must be colored with the color from L(v_i) for i ∈ {1, \ldots, a}. It must hold that f(z) ≥ 1 + ⌊\frac{a}{s+1}\rfloor, which allows z to be colored with a color β ∈ L(z). Lastly, v_{a+1}, \ldots, v_m can always be colored by a color different from β since their list size is at least 2. It follows that \sum_{v ∈ V} f(v) ≥ a + 2(m - a) + 1 + \frac{a}{s+1}\rfloor = m + 1 + \frac{(m-a)(s+1)+a}{s+1} ≥ m + 1 + \frac{m}{s+1}\rfloor since s ≥ 0. Therefore, χ_psc(K_{1,m}) ≥ m + 1 + \frac{m}{s+1}\rfloor. □

For example, if P = O, then χ_psc(K_{1,m}) = χ_{sc}(K_{1,m}) = 2m + 1.

**Theorem 18.** Let P be an additive induced hereditary property. If n ∈ N and p = max{k : k ≤ n and P_k ∈ P}, then χ_psc(P_n) = n + \left\lceil\frac{n-1}{p}\right\rceil.

**Proof.** Let P_n = (v_1, \ldots, v_n) and define f : V(P_n) → N by f(v_i) = 1 if i = 1 or if p ≥ i - 1, and f(v_i) = 2 otherwise. We prove that f is a P-choice function of P_n. Consider an arbitrary list assignment L with |L(v_i)| = f(v_i) for every i ∈ {1, \ldots, n}. We color the vertices in order, beginning with v_1. If f(v_1) = 1, then v_1 must be colored with the single color in its list. If f(v_1) = 2, then v_1 will be colored with a color different from the color of v_{n-1}. This implies that each graph induced by vertices of the same color consists of paths of order at most p.
and therefore belongs to $\mathcal{P}$ since $\mathcal{P}$ is additive and induced hereditary. Hence the coloring is an $(L, \mathcal{P})$-coloring, and $f$ is a $\mathcal{P}$-choice function which implies that $
chi_{sc}(P_n) \leq n + \left\lfloor \frac{n-1}{p} \right\rfloor$.

Assume that there is a $\mathcal{P}$-choice function $f$ of $P_n$ with $\sum_{v \in V(P_n)} f(v) = n - 1 + \left\lfloor \frac{n-1}{p} \right\rfloor$. Since $\nchi_{sc}(P_n) \geq n$, $p \leq n - 1$, which implies $P_{p+1} \notin \mathcal{P}$. There are less than $\left\lfloor \frac{n-1}{p} \right\rfloor$ vertices with list size at least 2, all other vertices have list size 1. Therefore, we either find $p + 1$ consecutive vertices with list size 1, or $(a + 2)p + 1$ consecutive vertices $v_j, \ldots, v_{j+(a+2)p}$, $a \geq 0$, with the following list sizes: $f(v_i) = 2$ if $i = j + lp$, $l = 1, \ldots, a + 1$, and $f(v_i) = 1$ otherwise. Every sequence of consecutive vertices of list size 1 is assigned the same list, alternating between \{1\} and \{2\}, all other vertices have initial lists $L(v) = \{1, \ldots, f(v)\}$. These lists force that any list coloring has $p + 1$ consecutive vertices of the same color, which is a contradiction to $P_{p+1} \notin \mathcal{P}$. Therefore, $\nchi_{sc}(P_n) \geq n + \left\lfloor \frac{n-1}{p} \right\rfloor$.

For example, if $\mathcal{P} = \mathcal{O}$, then $p = 1$ and $\nchi^{\mathcal{O}}_{sc}(P_n) = \nchi_{sc}(P_n) = 2n - 1$.

**Theorem 19.** Let $\mathcal{P}$ be an additive induced hereditary property. If $n \in \mathbb{N}$ and $p = \max\{k : k \leq n \text{ and } P_k \in \mathcal{P}\}$, then $\nchi_{sc}(C_n) = n$ if $C_n \in \mathcal{P}$, $\nchi_{sc}(C_n) = n + 1$ if $C_n \notin \mathcal{P}$, $p = n - 1$, and $\nchi_{sc}(C_n) = n + 1 + \left\lfloor \frac{n-1}{p} \right\rfloor$ otherwise.

**Proof.** The result is obvious if $C_n \in \mathcal{P}$, therefore assume in the following that $C_n \notin \mathcal{P}$ which implies $\nchi_{sc}(C_n) \geq n + 1$.

Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of $C_n$, and $V = \{v_1, \ldots, v_n\}$.

If $p = n - 1$, then define $f : V \to \mathbb{N}$ by $f(v_i) = 1$ if $i = 1$ or if $p \not| (i - 1)$, and $f(v_i) = 2$ otherwise (see the proof of Theorem 18) and $f' : V \to \mathbb{N}$ by $f'(v_i) = f(v_i)$ if $1 \leq i \leq n - 1$ and $f'(v_n) = f(v_n) + 1$. Consider an arbitrary list assignment $L$ with $|L(v_i)| = f'(v_i)$ for every $i \in \{1, \ldots, n\}$. Color the vertices $v_1, \ldots, v_n$ in order as in the proof of Theorem 18, but additionally remove the color of $v_1$ from $L(v_n)$ which is possible since its list size was increased by 1, thus forcing $v_n$ to be colored differently from $v_1$. This implies again that the coloring is an $(L, \mathcal{P})$-coloring and $f'$ is a $\mathcal{P}$-choice function of $C_n$. Therefore, $\nchi_{sc}(C_n) \leq n + 1 + \left\lfloor \frac{n-1}{p} \right\rfloor$. Note that because of the lower bound $n + 1$, equality holds for $p \geq n$. Since $p \neq n - 1$, let $p \leq n - 2$ in the following, which implies $P_{p+1} \notin \mathcal{P}$. 

Assume that there is a \( \mathcal{P} \)-choice function \( f \) of \( C_n \) with \( \sum_{v \in V} f(v) = n + \left\lfloor \frac{n-1}{p} \right\rfloor \). If there is a vertex \( v_i \) with \( f(v_i) \geq 3 \), then \( v_i \) can always be colored with a color different from the colors of its neighbors. This implies that \( \sum_{v \in V} f(v) \geq f(v_i) + \chi_{sc}(P_{n-1}) \geq 3 + n - 1 + \left\lfloor \frac{n-2}{p} \right\rfloor \geq n + 1 + \left\lfloor \frac{n-1}{p} \right\rfloor \) by Theorem 18, a contradiction to the assumption. Hence there are exactly \( a = \left\lfloor \frac{n-1}{p} \right\rfloor \) vertices with list size 2 and \( n-a \) vertices with list size 1. Set \( n = ap + r \) with \( 1 \leq r \leq p \). Since \( n > ap \), by the pigeonhole principle, we either find \( p + 1 \) consecutive vertices with list size 1 (e.g., if \( a = 1 \)) which leads to a list assignment with a monochromatic \( P_{p+1} \) which is not in \( \mathcal{P} \), a contradiction, or we find \( p \) consecutive vertices of list size 1 bounded by two vertices of list size 2. In this case, by removing the \( p \) vertices of list size 1 and reducing the list size of the end-vertices of the resulting \( P_{n-p} \) by 1 we obtain a \( \mathcal{P} \)-choice function of \( P_{n-p} \) which implies \( \sum_{v \in V} f(v) \geq p + 2 + \chi_{sc}(P_{n-p}) = p + 2 + n - p + \left\lfloor \frac{n-p-1}{p} \right\rfloor = n + 1 + \left\lfloor \frac{n-1}{p} \right\rfloor \) by Theorem 18, a contradiction to the initial assumption.

For example, if \( \mathcal{P} = \mathcal{O} \), then \( p = 1 \) and \( \chi_{sc}(C_n) = \chi_{sc}(C_n) = 2n \).

The results of this section allow the computation of the \( \mathcal{P} \)-sum choice number of all graphs of order at most 4 with the exception of the graph isomorphic to a paw \( K_{1,3} + e \) (a claw \( K_{1,3} \) with an additional edge) and of \( K_{1,1,2} \). Their \( \mathcal{P} \)-sum choice numbers will be determined in the next propositions.

**Proposition 20.** Let \( \mathcal{P} \) be an additive induced hereditary property. If \( G \cong K_{1,3} + e \), then \( \chi_{sc}(G) = 8 \), \( \chi_{sc}(G) = 4 \) if \( G \in \mathcal{P} \), and \( \chi_{sc}(G) = 5 \) if \( G \notin \mathcal{P} \).

**Proof.** If \( \mathcal{P} = \mathcal{O} \), then \( \chi_{sc}(G) = \chi_{sc}(G) = GB(G) = 8 \) [1]. If \( G \in \mathcal{P} \), then obviously \( \chi_{p}(G) = |V(G)| = 4 \). Therefore, let \( G \notin \mathcal{P} \). Since \( G \notin \mathcal{P} \), \( \chi_{sc}(G) \geq |V(G)|+1 = 5 \). Denote the vertices of \( G \) such that \( d_G(v_1) = 3 \), \( d_G(v_2) = 1 \), \( d_G(v_1) = d_G(v_2) = 2 \). Define \( f : V(G) \to \mathbb{N} \) by \( f(z) = 2 \) and \( f(v) = 1 \) for \( v \neq z \). Consider an arbitrary list assignment \( L \) with sizes defined by \( f \). In an \( (L, \mathcal{P}) \)-coloring of \( G \), all vertices except \( z \) must obtain the color of their list. If \( v_1 \) and \( v_2 \) are colored by the same color \( \alpha \), then color \( z \) differently from \( \alpha \). Otherwise, if the colors of \( v_1 \) and \( v_2 \) are not equal, then color \( z \) differently from the color of \( w \). In any case, at most two adjacent vertices share the same color, that is, \( f \) is a \( \mathcal{P} \)-choice function of \( G \) and \( \chi_{sc}(G) \leq 5 \).

**Proposition 21.** Let \( \mathcal{P} \) be an additive induced hereditary property. Then it holds \( \chi_{sc}(K_{1,1,2}) = 9 \), \( \chi_{sc}(K_{1,1,2}) = 6 \), \( \chi_{sc}(K_{1,1,2}) = 4 \) if \( K_{1,1,2} \in \mathcal{P} \), and \( \chi_{sc}(K_{1,1,2}) = 5 \) in the remaining cases.

**Proof.** If \( \mathcal{P} = \mathcal{O} \), then \( \chi_{sc}(K_{1,1,2}) = \chi_{sc}(K_{1,1,2}) = GB(K_{1,1,2}) = 9 \) [1]. If \( K_{1,1,2} \in \mathcal{P} \), then obviously \( \chi_{sc}(K_{1,1,2}) = |V(K_{1,1,2})| = 4 \).
If $P = O_1$, then each subgraph $P_3, C_3$ needs a list of size 2 to avoid a monochromatic $P_3$. Hence $\chi^{O_1}_{sc}(K_{1,1,2}) \geq 2 \cdot 2 + 2 \cdot 1 = 6$. Denote the vertices of $K_{1,1,2}$ such that $d_{K_{1,1,2}}(v_i) = 2$ and $d_{K_{1,1,2}}(w_i) = 3$, $i = 1, 2$. Set $f(v_i) = 2$ and $f(w_i) = 1$, $i = 1, 2$. Consider a list assignment $L$ with list sizes defined by $f$. Vertices $w_1$ and $w_2$ must be colored with the color from their lists. If $w_1$ is colored by $\alpha$ and $w_2$ by $\beta$ ($\alpha = \beta$ is allowed), then color $v_1$ by a color $\neq \alpha$ and $v_2$ by a color $\neq \beta$. It follows that $\chi^{O_1}_{sc}(K_{1,1,2}) \leq 6$.

In the remaining cases it holds that $K_{1,1,2} \notin \mathcal{P}$, $\mathcal{P} \neq \mathcal{O}$, and $\mathcal{P} \neq O_1$. Since $K_{1,1,2} \notin \mathcal{P}$, $\chi^{\mathcal{P}}_{sc}(K_{1,1,2}) \geq |V(G)| + 1 = 5$. Set $f(w_1) = 2$ for a vertex $w_1$ of degree 3 and $f(v) = 1$ for $v \neq w_1$. In any list assignment $L$ with list sized defined by $f$, the colors of the path $P_3 = (v_1, w_2, v_2)$ are fixed. Color then $w_1$ by a color different from the color of $w_2$. This implies that at most three vertices that induce a $P_3$ are colored by the same color, and $P_3 \in \mathcal{P}$. Therefore, $\chi^{\mathcal{P}}_{sc}(K_{1,1,2}) \leq 5$.

5. Concluding Remarks

In Section 3 we determined general upper bounds on the $\mathcal{P}$-sum choice number of arbitrary graphs for some of the most common properties $\mathcal{P}$, namely $O_k$, $S_k$, $D_k$, $O^k$, $J_k$, and $I_k$. It would be interesting to obtain reasonable lower bounds on the $\mathcal{P}$-sum choice number of arbitrary graphs for the same properties.

In Section 4 we determined the $\mathcal{P}$-sum choice number of complete graphs, stars, paths, cycles, and all graphs of order at most 4 for arbitrary additive induced hereditary properties $\mathcal{P}$. By the same methods and extensive case analysis we also determined the $\mathcal{P}$-sum choice number of all graphs of order 5 for arbitrary additive hereditary properties $\mathcal{P}$.

As mentioned above, we determined the $\mathcal{P}$-sum choice number of stars $K_{1,m}$. It would be an interesting task to study the $\mathcal{P}$-sum choice number of arbitrary complete bipartite graphs $K_{l,m}$. Partial results for $\mathcal{P} = D_1$ can be found in [4] and for $\mathcal{P} = O$ in [1, 6], for example.

References


