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FACIAL [*r,s,t*]-COLORINGS OF PLANE GRAPHS

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Abstract

Let G be a plane graph. Two edges are facially adjacent in G if they are consecutive edges on the boundary walk of a face of G. Given nonnegative integers r, s, and t, a facial $[r, s, t]$-coloring of a plane graph $G = (V, E)$ is a mapping $f : V \cup E \to \{1, \ldots, k\}$ such that $|f(v_1) - f(v_2)| \geq r$ for every two adjacent vertices v_1 and v_2, $|f(e_1) - f(e_2)| \geq s$ for every two facially adjacent edges e_1 and e_2, and $|f(v) - f(e)| \geq t$ for all pairs of incident vertices v and edges e. The facial $[r, s, t]$-chromatic number $\overline{\chi}_{r,s,t}(G)$ of G is defined to be the minimum k such that G admits a facial $[r, s, t]$-coloring with colors $1, \ldots, k$. In this paper we show that $\overline{\chi}_{r,s,t}(G) \leq 3r + 3s + t + 1$ for every plane graph G. For some triplets $[r, s, t]$ and for some families of plane graphs this bound is improved. Special attention is devoted to the cases when the parameters r, s, and t are small.

Keywords: plane graph, boundary walk, edge-coloring, vertex-coloring, total-coloring.

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1. Introduction and Notations

All graphs considered in this paper are finite, connected, and simple. We use standard graph theory terminology according to the book [3].

The concept of $[r,s,t]$-coloring is a generalization of the classical colorings: vertex-coloring, edge-coloring, and total-coloring. It was introduced by Kemnitz and Marangio [18]. Given nonnegative integers r, s, and t, an $[r,s,t]$-*coloring* of a graph $G=(V,E)$ is a mapping $f : V \cup E \to \{1,\dots,k\}$ such that $|f(v_1)-f(v_2)| \geq r$ for every two adjacent vertices v_1 and v_2, $|f(e_1)-f(e_2)| \geq s$ for every two adjacent edges e_1 and e_2, and $|f(v)-f(e)| \geq t$ for all pairs of incident vertices v and edges e. The $[r,s,t]$-*chromatic number* $\chi_{r,s,t}(G)$ of G is defined to be the minimum k such that G admits an $[r,s,t]$-coloring with colors $1,\dots,k$. Obviously, a $[1,0,0]$-coloring is a classical proper vertex-coloring, a $[0,1,0]$-coloring is a classical proper edge-coloring, and a $[1,1,1]$-coloring is a classical proper total-coloring. Results concerning $[r,s,t]$-colorings of graphs can be found in [7, 8, 16–20, 22, 25].

In this paper we deal with a relaxation of the $[r,s,t]$-coloring for plane graphs. A *plane graph* is a particular drawing of a planar graph in the Euclidean plane such that edges only meet at their ends. Let G be a plane graph with vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$. A closed walk $W = v_0, e_0, v_1, e_1, \dots, e_{n-1}, v_n$ in G is the *boundary walk* corresponding to a face α if all vertices and edges of W are incident with α, and for every $i \in \{1,\dots,n\}$, the edges e_{i-1} and e_i (subscripts modulo n) occur consecutively in the counterclockwise cyclic ordering of the edges incident to v_i in the embedding of G. Two edges of a graph are adjacent if they share a vertex. In plane graphs some adjacent edges have a stronger property: they are adjacent and in addition they are incident with the same face. We say that two edges are *facially adjacent* in a plane graph G if they are consecutive edges on the boundary walk of a face of G. Using this adjacency we define a facial $[r,s,t]$-coloring of plane graphs as follows.

Given nonnegative integers r, s, and t, a *facial* $[r,s,t]$-*coloring* of a plane graph $G=(V,E)$ is a mapping $f : V \cup E \to \{1,\dots,k\}$ such that $|f(v_1)-f(v_2)| \geq r$ for every two adjacent vertices v_1 and v_2, $|f(e_1)-f(e_2)| \geq s$ for every two facially adjacent edges e_1 and e_2, and $|f(v)-f(e)| \geq t$ for all pairs of incident vertices v and edges e. The *facial* $[r,s,t]$-*chromatic number* $\overline{\chi}_{r,s,t}(G)$ of G is defined to be the minimum k such that G admits a facial $[r,s,t]$-coloring with colors $1,\dots,k$.

Obviously, a facial $[1,0,0]$-coloring is a classical proper vertex-coloring. Since plane graphs are 4-colorable, see [1, 2], we have $\overline{\chi}_{1,0,0}(G) \leq 4$ for every plane graph G. For some families of plane graphs this bound can be lowered to three. For instance, the theorem by Grötzsch [12] says that every triangle-free plane graph admits a proper vertex-coloring with at most three colors. Clearly, $\overline{\chi}_{1,0,0}(G) = 2$ if and only if G is bipartite.

A facial $[0,1,0]$-coloring is a classical facial edge-coloring (i.e., facially adjacent edges have distinct colors) which was first studied for the family of cubic bridgeless plane graphs and for the family of plane triangulations. Already Tait [24] observed that the Four Color Problem is equivalent to the problem of facial 3-edge-coloring of any plane triangulation and to the problem of facial 3-edge-coloring of cubic bridgeless plane graphs. It is known that every plane graph admits a facial edge-coloring with at most four colors, see [9]. The bound four is tight which can be easily seen on the graph of a wheel on six vertices.

A challenging open problem in this direction is the following.

Problem 1 [5]**.** Characterize all plane graphs that admit a facial edge-coloring with three colors.

If we color all vertices of a plane graph G with 1 and all edges with 2, then we obtain a facial $[0, 0, 1]$-coloring of G. Therefore, $\overline{\chi}_{0,0,1}(G) \leq 2$. The facts $\overline{\chi}_{1,0,0}(G) \leq 4$ and $\overline{\chi}_{0,1,0}(G) \leq 4$ imply that $\overline{\chi}_{1,1,0}(G) \leq 4$ (we use the same set of colors for the vertices and edges). The facial $[1, 0, 1]$-chromatic number is at most four for every plane graph. To see this, color the vertices with at most four colors so that adjacent vertices receive different colors. Then color each edge with a color which does not appear on its endvertices.

We say that a color c appears at a vertex v in an edge-colored graph G if v is incident with an edge of color c. Czap and Šugerek [6] proved that every plane graph admits a facial edge-coloring with at most four colors such that at most three colors appear at each vertex. Therefore, if we color each vertex with a color which does not appear on the incident edges we obtain a facial $[0, 1, 1]$-coloring, i.e., $\overline{\chi}_{0,1,1}(G) \leq 4$ for every plane graph G.

Facial $[1, 1, 1]$-colorings were introduced by Fabrici, Jendrol', and Voigt [11] under the name facial total-coloring. They proved that $\overline{\chi}_{1,1,1}(G) \leq 6$ for every plane graph G and they presented an outerplane graph H such that $\overline{\chi}_{1,1,1}(H) = 5$.

The rest of the paper is organized as follows. In Section 2 some general upper bounds are proved. In Section 3 situations with $\min\{r, s, t\} = 0$ are considered. Sections 4 and 5 are devoted to study facial $[r, s, t]$-colorings when $\max\{r, s, t\} \leq 2$. Namely, in Section 4, general plane graphs and, in Section 5, trees are investigated.

2. General Upper Bounds

Schiermeyer and Villà [22] determined the $[r, s, t]$-chromatic number for paths for all possible triplets $[r, s, t]$. $[r, s, t]$-chromatic numbers for cycles were determined by Villà [25]. Observe, that for every subcubic plane graph G it holds $\overline{\chi}_{r,s,t}(G) = \chi_{r,s,t}(G)$ (since every two adjacent edges are also facially adjacent).

Lemma 2. *Let G be a plane graph. If $\tilde{r} \leq r$, $\tilde{s} \leq s$, and $\tilde{t} \leq t$, then $\overline{\chi}_{\tilde{r},\tilde{s},\tilde{t}}(G) \leq \overline{\chi}_{r,s,t}(G)$.*

Proof. A facial $[r,s,t]$-coloring of G is, by the definition, also a facial $[\tilde{r},\tilde{s},\tilde{t}]$-coloring of G if $\tilde{r} \le r$, $\tilde{s} \le s$, and $\tilde{t} \le t$. ■

The following multiplication property implies that it is sufficient to consider facial $[r,s,t]$-colorings such that the greatest common divisor of r, s, and t is 1.

Lemma 3. *Let G be a plane graph and let ℓ be a nonnegative integer. Then*

$$\overline{\chi}_{\ell r,\ell s,\ell t}(G) = \ell \cdot \overline{\chi}_{r,s,t}(G) - \ell + 1.$$

Proof. If $0 \le \ell \le 1$, then the lemma trivially holds. So assume $\ell \ge 2$.

Let f be a facial $[r,s,t]$-coloring of G with colors $1,\ldots,\overline{\chi}_{r,s,t}(G)$. If we multiply all assigned colors by ℓ, then we obtain a facial $[\ell r,\ell s,\ell t]$-coloring of G. Observe that the smallest color in this coloring is not smaller than ℓ. So if we decrease each color by $\ell - 1$, then the obtained coloring is also a facial $[\ell r,\ell s,\ell t]$-coloring. Consequently, $\overline{\chi}_{\ell r,\ell s,\ell t}(G) \le \ell \cdot \overline{\chi}_{r,s,t}(G) - (\ell - 1)$.

Now we show the opposite inequality. Suppose to the contrary that G has a facial $[\ell r,\ell s,\ell t]$-coloring c with colors $1,\ldots,\ell \cdot \overline{\chi}_{r,s,t}(G) - \ell$. We define a coloring c' of G by $c'(x) = \left\lceil \frac{c(x)}{\ell} \right\rceil$, where $x \in V \cup E$.

Let $y, z \in V \cup E$ be two facially adjacent or facially incident elements of G. Without loss of generality we can assume that $c(y) > c(z)$. Then $c(y) - c(z) \ge \ell p$, for some $p \in \{r,s,t\}$. The fact $c(y) - c(z) \ge \ell p$ implies $\left\lceil \frac{c(y)}{\ell} \right\rceil \ge \frac{c(y)}{\ell} \ge \frac{c(z)}{\ell} + p$. Since $\left\lceil \frac{c(z)}{\ell} \right\rceil < \frac{c(z)}{\ell} + 1$ we have $-\left\lceil \frac{c(z)}{\ell} \right\rceil > -\dfrac{c(z)}{\ell} - 1$. Consequently,

$$\left\lceil \frac{c(y)}{\ell} \right\rceil - \left\lceil \frac{c(z)}{\ell} \right\rceil > p - 1.$$

Therefore, c' is a facial $[r,s,t]$-coloring of G with colors $1,\ldots,\overline{\chi}_{r,s,t}(G) - 1$ which contradicts the definition of the facial $[r,s,t]$-chromatic number. ■

Corollary 4. *Let G be a plane graph and let ℓ be a nonnegative integer. Then*

$$\overline{\chi}_{\ell,\ell,\ell}(G) = \ell \cdot \overline{\chi}_{1,1,1}(G) - \ell + 1.$$

Corollary 5. *Let G be a plane graph. If $M = \max\{r,s,t\}$, then*

$$\overline{\chi}_{r,s,t}(G) \le 5M + 1.$$

Proof. From Lemma 2 it follows that $\overline{\chi}_{r,s,t}(G) \le \overline{\chi}_{M,M,M}(G)$. By Corollary 4 we have $\overline{\chi}_{M,M,M}(G) = M \cdot \overline{\chi}_{1,1,1}(G) - M + 1$. Fabrici *et al.* [11] proved that $\overline{\chi}_{1,1,1}(G) \le 6$, consequently $\overline{\chi}_{r,s,t}(G) \le 5M + 1$. ■

In the rest of the paper we will use the following notations $\overline{\chi}(G) := \overline{\chi}_{1,0,0}(G)$ and $\overline{\chi}'(G) := \overline{\chi}_{0,1,0}(G)$.

Lemma 6. *If G is a plane graph, then*

$$\overline{\chi}_{r,s,t}(G) \leq r \cdot (\overline{\chi}(G) - 1) + s \cdot (\overline{\chi}'(G) - 1) + t + 1.$$

Proof. A proper vertex-coloring of G with colors $1, 1+r, \ldots, 1+(\overline{\chi}(G)-1)\cdot r$ and a facial edge-coloring of G with colors $1+(\overline{\chi}(G)-1)\cdot r+t$, $1+(\overline{\chi}(G)-1)\cdot r+t+s$, $\ldots, 1+(\overline{\chi}(G)-1)\cdot r+t+(\overline{\chi}'(G)-1)\cdot s$ induce a facial $[r,s,t]$-coloring of G. ∎

Theorem 7. *If G is a plane graph, then*

$$\overline{\chi}_{r,s,t}(G) \leq \begin{cases} 3r+3s+t+1 & \text{in general,} \\ 2r+3s+t+1 & \text{if } G \text{ is triangle-free,} \\ 3r+2s+t+1 & \text{if } G \text{ is a triangulation,} \\ r+3s+t+1 & \text{if } G \text{ is bipartite,} \\ 3\cdot\max\{r,s\}+t+1 & \text{if } \max\{r,s\} \geq 2t, \\ 2r+3s+\max\{r,t\}+1 & \text{if } s \geq t. \end{cases}$$

Proof. In the first four cases the bounds follow from Lemma 6 and the known facts about the values of the chromatic number $\overline{\chi}(G)$ and the facial chromatic index $\overline{\chi}'(G)$ of G.

In the fifth case, if $r \geq s$, then a proper vertex-coloring of G with colors $1, 1+r, \ldots, 1+(\overline{\chi}(G)-1)\cdot r$ and a facial edge-coloring with colors $1+t$, $1+t+r, \ldots, 1+t+(\overline{\chi}'(G)-1)\cdot r$ induce a required facial $[r,s,t]$-coloring of G. If $r < s$, then we use a proper vertex-coloring with colors $1+t, 1+t+s, \ldots, 1+t+(\overline{\chi}(G)-1)\cdot s$ and a facial edge-coloring with colors 1, $1+s, \ldots, 1+(\overline{\chi}'(G)-1)\cdot s$.

Finally, assume that $s \geq t$. Czap and Šugerek [6] proved that every plane graph G admits a facial edge-coloring with at most four colors such that at most three colors appear at each vertex of G. Let c be such a facial edge-coloring with colors $1, 1+s, 1+2s, 1+3s$. Since plane graphs are 4-colorable, the vertex set of G can be divided into four independent sets V_1, V_2, V_3, and V_4. First we color each vertex of V_4 with a color from the set $\{1, 1+s, 1+2s, 1+3s\}$ which does not appear on the incident edges under the coloring c, thereafter we color all vertices in V_i, $i=1,2,3$, with color $1+3s+r(i-1)+\max\{r,t\}$. ∎

3. Facial $[r,s,t]$-Colorings with $\min\{r,s,t\}=0$

Theorem 8. *If G is a plane graph with at least one edge, then*

(a) $\overline{\chi}_{r,0,0}(G) = r\cdot\overline{\chi}(G) - r + 1$,

(b) $\overline{\chi}_{0,s,0}(G) = s\cdot\overline{\chi}'(G) - s + 1$,

(c) $\overline{\chi}_{0,0,t}(G) = t+1$,

(d) $\overline{\chi}_{r,s,0}(G) = \max\{\overline{\chi}_{r,0,0}(G), \overline{\chi}_{0,s,0}(G)\}$.

Proof. (a) and (b) follow from Lemma 3. Obviously, $\overline{\chi}_{0,0,t}(G) \geq t+1$ since $|f(v) - f(e)| \geq t$ for all pairs of incident vertices v and edges e. On the other hand if we color all vertices with 1 and all edges with $t+1$, then we obtain a facial $[0,0,t]$-coloring. The case (d) follows from Lemma 2 and the fact that vertices and edges can be colored independently. ∎

Theorem 9. *If G is a plane graph with $\overline{\chi}(G) = 2$, then*

$$\overline{\chi}_{r,0,t}(G) = \begin{cases} r+1 & \text{for} \quad r \geq 2t, \\ 2t+1 & \text{for} \quad t < r < 2t, \\ r+t+1 & \text{for} \quad r \leq t. \end{cases}$$

Proof. Since $\overline{\chi}(G) = 2$ (i.e., G is bipartite), the vertex set of G can be divided into two independent sets, say V_1 and V_2. By Lemma 2 and Theorem 8 we have $\overline{\chi}_{r,0,t}(G) \geq \overline{\chi}_{r,0,0}(G) = r+1$.

If $r \geq 2t$, then we color the vertices in V_1 and V_2 with 1 and $r+1$, respectively, and color all edges with color $t+1$.

Now assume that $r \leq t$. Consider an edge $e = uv$ and a facial $[r,0,t]$-coloring c of G. W.l.o.g. we can assume that $c(u) \leq c(v)$. If $c(e) \leq c(u)$ or $c(v) \leq c(e)$, then at least $r+t+1$ colors are needed. If $c(u) \leq c(e) \leq c(v)$, then at least $2t+1$ colors are needed. Since $r \leq t$ we have $2t+1 \geq r+t+1$. This means that, in this case, any facial $[r,0,t]$-coloring of G uses at least $r+t+1$ colors. On the other hand, Lemma 6 implies that $\overline{\chi}_{r,0,t}(G) \leq r+t+1$.

Finally, in the case $t < r < 2t$ we color the vertices in V_1 and V_2 with colors 1 and $2t+1$, respectively, and color the edges with color $t+1$. Thus, $\overline{\chi}_{r,0,t}(G) \leq 2t+1$. Similarly as in the previous case we can show that any facial $[r,0,t]$-coloring of G uses at least $2t+1$ colors. ∎

Theorem 10. *If G is a plane graph with $\overline{\chi}(G) \geq 3$, then*

$$\begin{aligned} \overline{\chi}_{r,0,t}(G) &= r \cdot \overline{\chi}(G) - r + 1 && \text{for } r \geq t, \\ \max\{r \cdot \overline{\chi}(G) - r + 1, t+1\} \leq \overline{\chi}_{r,0,t}(G) &\leq r \cdot \overline{\chi}(G) - r + t + 1 && \text{for } r < t. \end{aligned}$$

Proof. From Lemma 2 and Theorem 8 follows that $\overline{\chi}_{r,0,t}(G) \geq \overline{\chi}_{r,0,0}(G) = r \cdot \overline{\chi}(G) - r + 1$.

First assume that $r \geq t$. Let c be a proper vertex-coloring of G with colors $1, 1+r, \ldots, 1+(\overline{\chi}(G)-1) \cdot r$. If we color each edge uv of G with a color from the set $\{1, 1+r, 1+2r\} \setminus \{c(u), c(v)\}$ we obtain a facial $[r,0,t]$-coloring of G, hence $\overline{\chi}_{r,0,t}(G) \leq r \cdot \overline{\chi}(G) - r + 1$.

If $r < t$, then the lower bound follows from Theorem 8 and the upper bound follows from Lemma 6. ∎

In the following we consider facial $[0, s, t]$-colorings.

Theorem 11. *If G is a plane graph with $\overline{\chi}'(G) \geq 2$, then*

$$\overline{\chi}_{0,s,t}(G) = s \cdot \overline{\chi}'(G) - s + 1 \quad \textit{for} \quad s \geq 2t.$$

Proof. By Lemma 2 and Theorem 8 we have $\overline{\chi}_{0,s,t}(G) \geq \overline{\chi}_{0,s,0}(G) = s \cdot \overline{\chi}'(G) - s + 1$.

Every facial edge-coloring with colors $1, s+1, \ldots, s \cdot (\overline{\chi}'(G) - 1) + 1$ can be extended to a facial $[0, s, t]$-coloring so that we color all vertices with color $t + 1$, therefore $\overline{\chi}_{0,s,t}(G) \leq s \cdot (\overline{\chi}'(G) - 1) + 1$. ∎

Theorem 12. *If G is a plane graph with $\overline{\chi}'(G) = 2$, then*

$$\overline{\chi}_{0,s,t}(G) = \begin{cases} 2t + 1 & \textit{for} \quad t < s < 2t, \\ s + t + 1 & \textit{for} \quad s \leq t. \end{cases}$$

Proof. Assume that $s \leq t$. Consider two facially adjacent edges $e_1 = uv$, $e_2 = vz$ and a facial $[0, s, t]$-coloring c of G. W.l.o.g. we can assume that $c(e_1) \leq c(e_2)$. If $c(v) \leq c(e_1)$ or $c(v) \geq c(e_2)$, then at least $s + t + 1$ colors are needed. If $c(e_1) \leq c(v) \leq c(e_2)$, then at least $2t + 1$ colors are needed. Since $s \leq t$ we have $2t + 1 \geq t + s + 1$. This means that, in this case, any facial $[0, s, t]$-coloring of G uses at least $s + t + 1$ colors. On the other hand, Lemma 6 implies that $\overline{\chi}_{0,s,t}(G) \leq s + t + 1$.

In the case $t < s < 2t$ we color the edges with colors 1 and $2t + 1$ so that facially adjacent edges receive distinct colors and color all vertices with color $t+1$. Thus, $\overline{\chi}_{0,s,t}(G) \leq 2t + 1$. Similarly as in the previous case we can show that any facial $[0, s, t]$-coloring of G uses at least $2t + 1$ colors. ∎

Each vertex v of a graph G having an odd (respectively, even) degree is an *odd-vertex* (respectively, *even-vertex*) of G.

Theorem 13. *If G is a plane graph with $\overline{\chi}'(G) = 3$, then*

$$2s + 1 \leq \overline{\chi}_{0,s,t}(G) \leq 4t + 1 \quad \textit{for} \quad t \leq s \leq 2t.$$

Moreover, if G contains an odd-vertex of degree at least three, then

$$\overline{\chi}_{0,s,t}(G) = 2s + t + 1 \quad \textit{for} \quad s \leq t.$$

Proof. By Lemma 2 and Theorem 8 we have $\overline{\chi}_{0,s,t}(G) \geq \overline{\chi}_{0,s,0}(G) = 2s + 1$.

If $t \leq s \leq 2t$, then every facial edge-coloring with colors $1, 2t + 1, 4t + 1$ can be extended to a facial $[0, s, t]$-coloring so that we color all vertices with color $t + 1$. Therefore, $\overline{\chi}_{0,s,t}(G) \leq 4t + 1$.

Now assume that G contains an odd-vertex of degree at least three and $s \leq t$. Let c be a facial $[0, s, t]$-coloring of G. Since G contains an odd-vertex and $\overline{\chi}'(G) = 3$, there are three adjacent edges e_1, e_2, e_3 which have distinct colors. W.l.o.g. we can assume that $c(e_1) \leq c(e_2) \leq c(e_3)$. Let v be its common endvertex. If $c(v) \leq c(e_1)$ or $c(v) \geq c(e_3)$, then at least $2s + t + 1$ colors are needed. Otherwise, at least $s + 2t + 1$ colors are needed. Since $s \leq t$ we have $s + 2t + 1 \geq 2s + t + 1$. This means that, in this case, any facial $[0, s, t]$-coloring of G uses at least $2s + t + 1$ colors. On the other hand, Lemma 6 implies that $\overline{\chi}_{0,s,t}(G) \leq 2s + t + 1$. ∎

4. Facial $[r, s, t]$-Colorings with $\max\{r, s, t\} \leq 2$

4.1. Facial $[1, 1, 1]$-coloring

Facial $[1, 1, 1]$-colorings were introduced by Fabrici, Jendrol', and Voigt [11].

Theorem 14 [11]**.** *If G is a plane graph, then*

$$\overline{\chi}_{1,1,1}(G) \leq 6.$$

Conjecture 15 [10]**.** *For every plane graph G it holds $\overline{\chi}_{1,1,1}(G) \leq 5$.*

Conjecture 15 was proved for triangulations, outerplane graphs [11] and bipartite plane graphs [6]; moreover, the upper bound 5 is tight in these classes of plane graphs.

4.2. Facial $[2, 1, 1]$-coloring

An edge list assignment of a graph G is a function L that assigns a set $L(e)$ of colors to each edge $e \in E(G)$. A *facial list edge-coloring* of a plane graph G with list assignment L is a facial edge-coloring c of G such that the color $c(e)$ of each edge e is chosen from the list $L(e)$. If such a coloring exists for every list assignment L with minimum list length at least k, then G is called *facially k-edge-choosable*. Fabrici, Jendroľ, and Voigt [11] proved that every plane graph is facially 4-edge-choosable.

Theorem 16. *If G is a plane graph, then*

$$\overline{\chi}_{2,1,1}(G) \leq 7.$$

Moreover, this bound is tight.

Proof. From Lemma 2 we have $\overline{\chi}_{2,1,1}(G) \geq \overline{\chi}_{2,0,0}(G)$. By Theorem 8, $\overline{\chi}_{2,0,0}(G) = 2 \cdot \overline{\chi}(G) - 1$. Therefore, $\overline{\chi}_{2,1,1}(G) \geq 2 \cdot \overline{\chi}(G) - 1$. Consequently, for every plane graph H with $\overline{\chi}(H) = 4$ it holds $\overline{\chi}_{2,1,1}(H) \geq 7$.

Now we prove that every plane graph admits a facial $[2, 1, 1]$-coloring with colors $1, \ldots, 7$. Let c be a proper vertex-coloring of G with colors $1, 3, 5$, and 7. We assign to every edge $e = uv$ a list of colors $L(e) = \{1, \ldots, 7\} \setminus \{c(u), c(v)\}$. Since $|L(e)| \geq 4$ for every $e \in E(G)$ and plane graphs are facially 4-edge-choosable, the proof is complete. ■

4.3. Facial $[1, 2, 1]$-coloring

Recall that a color c appears at a vertex v in an edge-colored graph G if v is incident with an edge of color c.

Theorem 17. *If G is a plane graph, then*

$$\overline{\chi}_{1,2,1}(G) \leq 7.$$

Moreover, this bound is tight.

Proof. From Lemma 2 we have $\overline{\chi}_{1,2,1}(G) \geq \overline{\chi}_{0,2,0}(G)$. By Theorem 8, $\overline{\chi}_{0,2,0}(G) = 2 \cdot \overline{\chi}'(G) - 1$. Therefore, $\overline{\chi}_{1,2,1}(G) \geq 2 \cdot \overline{\chi}'(G) - 1$. Consequently, for every plane graph H with $\overline{\chi}'(H) = 4$ it holds $\overline{\chi}_{1,2,1}(H) \geq 7$.

Now we prove that every plane graph admits a facial $[1, 2, 1]$-coloring with colors $1, \ldots, 7$. Every plane graph has a proper vertex-coloring with at most 4 colors, therefore the vertex set V of G can be partitioned into four independent sets, say V_1, V_2, V_3, and V_4. Czap and Šugerek [6] proved that G admits a facial edge-coloring with at most four colors such that at most three colors appear at each vertex of G. We use for such an edge-coloring colors $1, 3, 5$, and 7. Every such edge-coloring can be extended to a facial $[1, 2, 1]$-coloring of G. It suffices to color each vertex of V_4 with a color which does not appear on the incident edges, and color all vertices from V_i with color $2i$, for $i = 1, 2, 3$. ■

4.4. Facial $[1, 1, 2]$-coloring

Havet and Yu [15] proved that for every subcubic (not necessarily plane) graph G it holds $\chi_{1,1,2}(G) \geq 5$; moreover, if G is cubic, then $\chi_{1,1,2}(G) \geq 6$. On the other hand, when G is subcubic and bipartite, then $\chi_{1,1,2}(G) \leq 6$. Consequently, if G is subcubic and bipartite, then $5 \leq \chi_{1,1,2}(G) \leq 6$. Havet and Thomassé [14] proved that there is a polynomial-time algorithm that decides whether $\chi_{1,1,2}(G) = 5$ or $\chi_{1,1,2}(G) = 6$ for subcubic bipartite graphs, and they also proved that the same decision problem for not bipartite graphs with maximum degree three is NP-complete.

Havet and Yu [15] posed the conjecture that every subcubic graph different from K_4 admits a $[1, 1, 2]$-coloring with at most six colors.

Conjecture 18 [15]**.** *Let G be a subcubic graph. If $G \neq K_4$, then $\chi_{1,1,2}(G) \leq 6$.*

Note that $\chi_{1,1,2}(K_4) = 7$, see [17]. Conjecture 18 was proved for outerplanar graphs [4,13] and for graphs such that vertices are covered by a set of independent triangles [23].

Lemma 6 implies that $\overline{\chi}_{1,1,2}(G) \leq 9$ for every plane graph G. Clearly, $\overline{\chi}_{1,1,2}(K_4) = \chi_{1,1,2}(K_4) = 7$.

Problem 19. Is there a plane graph G with $\overline{\chi}_{1,1,2}(G) \geq 8$?

4.5. Facial $[2,2,1]$-coloring

The fifth case of Theorem 7 implies that $\overline{\chi}_{2,2,1}(G) \leq 8$ for every plane graph G. Moreover, there are infinitely many plane graphs with $\overline{\chi}_{2,2,1}(G) = 7$, because $\overline{\chi}_{2,2,1}(G) \geq \overline{\chi}_{2,1,1}(G)$.

Problem 20. Is there a plane graph G with $\overline{\chi}_{2,2,1}(G) = 8$?

4.6. Facial $[2,1,2]$-coloring

Theorem 21. *If G is a plane graph, then*

$$\overline{\chi}_{2,1,2}(G) \leq 10.$$

Proof. Let c be a proper vertex-coloring of G with colors $1, 3, 5$, and 7. We assign to every edge $e = uv$ a list $L(e) = \{1, \dots, 10\} \setminus \{x-1, x, x+1, y-1, y, y+1\}$, where $x = c(u)$ and $y = c(v)$. Since $|L(e)| \geq 4$ for every $e \in E(G)$ and plane graphs are facially 4-edge-choosable, the proof is complete. ■

There are infinitely many plane graphs with $\overline{\chi}_{2,1,2}(G) = 7$, since $\overline{\chi}_{2,1,2}(G) \geq \overline{\chi}_{2,1,1}(G)$.

Problem 22. Is there a plane graph G with $\overline{\chi}_{2,1,2}(G) \geq 8$?

4.7. Facial $[1,2,2]$-coloring

Corollary 5 implies that $\overline{\chi}_{1,2,2}(G) \leq 11$ for every plane graph G.

Lemma 23. *For the complete graph on four vertices it holds $\overline{\chi}_{1,2,2}(K_4) = 8$.*

Proof. The fact $\overline{\chi}_{1,1,2}(K_4) = 7$ (see [17]) implies that $\overline{\chi}_{1,2,2}(K_4) \geq 7$. Suppose to the contrary, that K_4 admits a facial $[1,2,2]$-coloring c with colors $1, \dots, 7$. Observe that, if a facial $[1,2,2]$-coloring uses only odd colors on the vertices, then it is also a facial $[2,2,2]$-coloring. Kemnitz and Marangio [18] proved that $\chi_{2,2,2}(K_4) = 9$, therefore the coloring c uses at least one of the colors $2, 4, 6$ on a vertex v of K_4. In this case, the edges incident with v cannot be colored in a required way, a contradiction.

A facial $[1,2,2]$-coloring of K_4 with colors $1, \dots, 8$ is depicted in Figure 1.

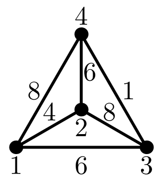

Figure 1. A facial $[1,2,2]$-coloring of K_4 with colors $1,\ldots,8$.

■

Problem 24. Is there a plane graph G with $\overline{\chi}_{1,2,2}(G) \geq 9$?

5. FACIAL $[r,s,t]$-COLORINGS OF TREES WITH $\max\{r,s,t\} \leq 2$

Let v be a vertex of a tree. If the degree of v equals one, then it is a *leaf*, otherwise we call it an *internal vertex*. An internal vertex is *even* if its degree is even.

Theorem 25 [6]. *If T is a nontrivial tree (i.e., T has at least one edge), then*

$$\overline{\chi}_{1,1,1}(T) = \begin{cases} 3 & \text{if each internal vertex is even,} \\ 4 & \text{otherwise.} \end{cases}$$

Theorem 26. *If T is a tree on at least two edges, then*

$$\overline{\chi}_{2,2,2}(T) = \overline{\chi}_{1,2,2}(T) = \begin{cases} 5 & \text{if each internal vertex is even,} \\ 7 & \text{otherwise.} \end{cases}$$

Proof. Corollary 4 implies that $\overline{\chi}_{2,2,2}(T) = 2 \cdot \overline{\chi}_{1,1,1}(T) - 1$, hence the exact value for $\overline{\chi}_{2,2,2}(T)$ can be obtained from Theorem 25.

From Theorem 12 and Theorem 13 we have $\overline{\chi}_{0,2,2}(T) = 5$ if each internal vertex of T is even and $\overline{\chi}_{0,2,2}(T) = 7$ otherwise. Since $\overline{\chi}_{0,2,2}(T) \leq \overline{\chi}_{1,2,2}(T) \leq \overline{\chi}_{2,2,2}(T)$ the proof is complete. ■

Theorem 27. *If T is a nontrivial tree, then*

$$\overline{\chi}_{2,1,2}(T) = \begin{cases} 5 & \text{if each internal vertex is even,} \\ 6 & \text{otherwise.} \end{cases}$$

Proof. Theorem 9 implies that $\overline{\chi}_{2,0,2}(T) = 5$, hence $\overline{\chi}_{2,1,2}(T) \geq 5$. If each internal vertex of T is even, then $\overline{\chi}_{2,1,2}(T) \leq 5$, since $\overline{\chi}_{2,2,2}(T) \leq 5$.

If T has an internal odd-vertex, then $\overline{\chi}_{2,1,2}(T) \geq 6$. Suppose to the contrary, that T admits a facial $[2,1,2]$-coloring with colors $1,2,3,4$, and 5. Then every internal odd-vertex v has color either 1 or 5, since we need at least three colors for the edges incident with v. If v has color 1 (respectively, 5), then the incident edges are colored with colors $3,4,5$ (respectively, $1,2,3$). In this case there is no admissible color for the second endvertex of the edge colored with 4 (respectively, 2), a contradiction.

It remains to show that $\overline{\chi}_{2,1,2}(T) \leq 6$. We prove that every tree admits a facial $[2,1,2]$-coloring with colors $1,\ldots,6$ such that the colors 2 and 5 are not used on the vertices. We pick any vertex of T to be the root. We color the edges and vertices of T starting from the root to the leaves.

In the first step we color the root u of T with 1, then we color the incident edges with colors $3,4,5$ such that facially adjacent edges receive distinct colors. Thereafter we color the endvertex v of every edge uv with a color given by Table 1.

the color of u	the color of uv	the color of v
1	3 or 4	6
1	5 or 6	3
3	1	6
3	5 or 6	1
4	1 or 2	6
4	6	1
6	1 or 2	4
6	3 or 4	1

Table 1. Possible colors for the edges and their endvertices.

In each next step it is sufficient to find a suitable facial $[2,1,2]$-coloring of a star S with one precolored edge (whose endvertices are also precolored). The central vertex u_S of S now plays the role of u. Similarly as in the first step, first we color the edges incident with u_S and then the uncolored endvertices (see Table 1). The set of admissible colors for the edges incident with u_S depends on the color of u_S, see Table 2.

the color of u_S	admissible colors for the edge u_Sv
1	$3,4,5,6$
3	$1,5,6$
4	$1,2,6$
6	$1,2,3,4$

Table 2. Possible colors for the vertex u_S and the incident edges.

Since for each possible color of u_S there are at least three admissible colors for the incident edges the precoloring extension is always possible. ■

Theorem 28. *If T is a tree on at least two edges, then*

$$\overline{\chi}_{2,2,1}(T) = \begin{cases} 4 & \text{if each internal vertex is even,} \\ 5 & \text{otherwise.} \end{cases}$$

Proof. Since every facial path on two edges has no facial $[2,2,1]$-coloring with colors $1,2$, and 3 we have $\overline{\chi}_{2,2,1}(T) \geq 4$.

If each internal vertex of T is even, then a proper vertex-coloring with colors $2,4$ and a facial edge-coloring with colors $1,3$ induce a facial $[2,2,1]$-coloring of T.

If T has an internal odd-vertex, then, by Theorem 11, it holds $\overline{\chi}_{0,2,1}(T) = 5$, consequently, $\overline{\chi}_{2,2,1}(T) \geq 5$. In this case, a proper vertex-coloring with colors $2,4$ and a facial edge-coloring with colors $1,3,5$ induce a required coloring of T. ■

Theorem 29. *If T is a tree on at least two edges, then*

$$\overline{\chi}_{1,2,1}(T) = \begin{cases} 3 & \text{if } T \text{ is a star on an even number of edges,} \\ 4 & \text{if each internal vertex of } T \text{ is even and } T \text{ is not a star,} \\ 5 & \text{otherwise.} \end{cases}$$

Proof. It is easy to see that every star on an even number of edges admits a facial $[1,2,1]$-coloring with colors $1,2$, and 3. In the other cases we can use the same arguments as in the proof of Theorem 28 and a simple observation that facial paths on (at least) three edges have no facial $[1,2,1]$-coloring with colors $1,2$, and 3. ■

Theorem 30. *If T is a tree on at least two edges, then*

$$5 \leq \overline{\chi}_{1,1,2}(T) \leq 6.$$

Moreover, these bounds are sharp.

Proof. First we prove that $\overline{\chi}_{1,1,2}(T) \geq 5$ for every tree with at least two edges. Clearly, it suffices to show that the path P on two edges admits no facial $[1,1,2]$-coloring with colors $1,2,3$, and 4. If an edge of P is colored with 2 or 3, then its endvertices cannot be colored in a required way. On the other hand, if the two edges of P are colored with 1 and 4, then there is no admissible color for the common endvertex.

Lemma 6 implies that $\overline{\chi}_{1,1,2}(T) \leq 6$; moreover, if each internal vertex of T is even, then $\overline{\chi}_{1,1,2}(T) \leq 5$. Consequently, if each internal vertex of T is even, then $\overline{\chi}_{1,1,2}(T) = 5$.

Wang and Chen [26] determined the exact value of $\overline{\chi}_{1,1,2}(T)$ for trees with maximum degree three. From their result it follows, that there are infinitely many trees T with internal odd-vertices such that $\overline{\chi}_{1,1,2}(T) = 5$ and also infinitely many trees T' with $\overline{\chi}_{1,1,2}(T') = 6$. ■

Theorem 31. *If T is a tree on at least two edges, then*

$$4 \leq \overline{\chi}_{2,1,1}(T) \leq 5.$$

Moreover, these bounds are sharp.

Proof. It is easy to see that any facial path on two edges has no facial $[2,1,1]$-coloring with colors $1,2$, and 3. Therefore, $\overline{\chi}_{2,1,1}(T) \geq 4$.

Since $\overline{\chi}_{2,1,1}(T) \leq \overline{\chi}_{2,2,1}(T)$, Theorem 28 implies that $\overline{\chi}_{2,1,1}(T) \leq 5$; moreover, if each internal vertex of T is even, then $\overline{\chi}_{2,1,1}(T) \leq 4$. Consequently, if each internal vertex of T is even, then $\overline{\chi}_{2,1,1}(T) = 4$.

From Lemma 32 it follows that there are infinitely many trees T with internal odd-vertices such that $\overline{\chi}_{2,1,1}(T) = 4$ and Lemma 33 implies that there are infinitely many trees T' with $\overline{\chi}_{2,1,1}(T') = 5$. ∎

Lemma 32. *If T is a tree such that each internal odd-vertex is adjacent to a leaf or to at least two even-vertices, then $\overline{\chi}_{2,1,1}(T) = 4$.*

Proof. From Theorem 31 it follows that $\overline{\chi}_{2,1,1}(T) \geq 4$, so it suffices to show that T admits a facial $[2,1,1]$-coloring with colors 1, 2, 3 and 4.

If each internal vertex of T is even, then $\overline{\chi}_{2,1,1}(T) \leq \overline{\chi}_{2,2,1}(T) = 4$, see Theorem 28.

So we can assume that T contains an internal odd-vertex. We pick any internal odd-vertex v to be the root and color the edges and vertices of T starting from the root to the leaves. In the first step we color v with 1, then color an adjacent even-vertex or leaf u with 3 and color the edge uv with 4. Next we color the other vertices adjacent to v with 4 and color the uncolored incident edges with colors 2 and 3 so that facially adjacent edges receive distinct colors. In each next step we find a suitable facial $[2,1,1]$-coloring of a star with one precolored edge.

Let S be a star with central vertex v and let e be the incident edge with a prescribed color.

First assume that v is an even-vertex. In this case we color S in the following:

– if v has color 1, then we color the adjacent uncolored vertices with 4 and color the incident uncolored edges with 2 and 3;

– if v has color 2, then we color the adjacent uncolored vertices with 4 and color the incident uncolored edges with 1 and 3;

– if v has color 3, then we color the adjacent uncolored vertices with 1 and color the incident uncolored edges with 2 and 4;

– if v has color 4, then we color the adjacent uncolored vertices with 1 and color the incident uncolored edges with 2 and 3.

Observe that until now each odd-vertex received color 1 and 4.

Now assume that v is an odd-vertex. We color S as follows:

– if v has color 1 and e has color 4, then we color the adjacent uncolored vertices with 4 and color the incident uncolored edges with 2 and 3;

– if v has color 1 and e has color 2 or 3, then we color one adjacent uncolored even-vertex u with 3 and color the edge uv with 4; thereafter we color the other adjacent uncolored vertices with 4 and color the incident uncolored edges with 2 and 3;

– if v has color 4 and e has color 1, then we color the adjacent uncolored vertices with 1 and color the incident uncolored edges with 2 and 3;

– if v has color 4 and e has color 2 or 3, then we color one adjacent uncolored even-vertex u with 2 and color the edge uv with 1; thereafter we color the other adjacent uncolored vertices with 1 and color the incident uncolored edges with 2 and 3. ■

Lemma 33. *If a tree T has an internal odd-vertex v which is adjacent to only internal odd-vertices, then* $\overline{\chi}_{2,1,1}(T) = 5$.

Proof. From Theorem 31 it follows that $4 \leq \overline{\chi}_{2,1,1}(T) \leq 5$. Suppose to the contrary that T admits a facial $[2, 1, 1]$-coloring with colors 1, 2, 3, and 4. Then each internal odd-vertex is colored with 1 or 4, since three colors appear at each internal odd-vertex.

W.l.o.g. we can assume that the vertex v has color 1. Then the incident vertices have color 4. Consequently, the edges incident with v are colored with 2 and 3, a contradiction. ■

Note that the facial [2,1,1]-chromatic number depends on the embedding of the tree. For example, the tree depicted in Figure 2 has different facial [2,1,1]-chromatic number depending on its embedding. With the embedding on the left, its facial [2,1,1]-chromatic number is four and with the embedding on the right its facial [2,1,1]-chromatic number is five.

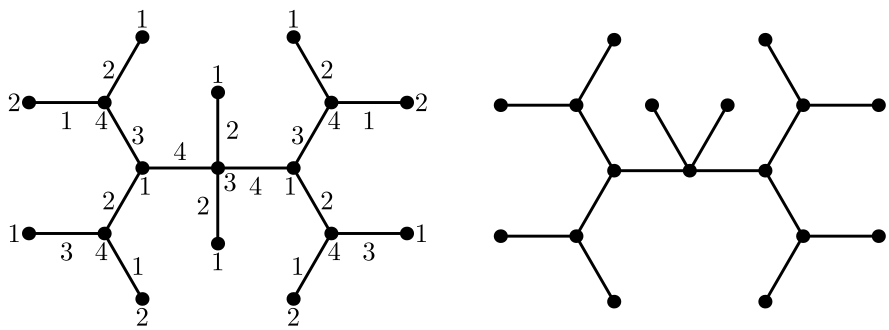

Figure 2. Two embeddings of the same tree with different facial [2,1,1]-chromatic numbers.

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