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FACIAL $[r,s,t]$-COLORINGS OF PLANE GRAPHS

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Abstract

Let G be a plane graph. Two edges are facially adjacent in G if they are consecutive edges on the boundary walk of a face of G. Given nonnegative integers r, s, and t, a facial $[r, s, t]$-coloring of a plane graph $G = (V, E)$ is a mapping $f : V \cup E \to \{1, \dots, k\}$ such that $|f(v_1) - f(v_2)| \geq r$ for every two adjacent vertices v_1 and v_2, $|f(e_1) - f(e_2)| \geq s$ for every two facially adjacent edges e_1 and e_2, and $|f(v) - f(e)| \geq t$ for all pairs of incident vertices v and edges e. The facial $[r, s, t]$-chromatic number $\overline{\chi}_{r,s,t}(G)$ of G is defined to be the minimum k such that G admits a facial $[r, s, t]$-coloring with colors $1, \dots, k$. In this paper we show that $\overline{\chi}_{r,s,t}(G) \leq 3r + 3s + t + 1$ for every plane graph G. For some triplets $[r, s, t]$ and for some families of plane graphs this bound is improved. Special attention is devoted to the cases when the parameters r, s, and t are small.

Keywords: plane graph, boundary walk, edge-coloring, vertex-coloring, total-coloring.

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