Discussiones Mathematicae
Graph Theory xx (xxxx) 1–20
doi:10.7151/dmgt.2218

SUFFICIENT CONDITIONS FOR A DIGRAPH TO ADMIT A $(1, \leq \ell)$-IDENTIFYING CODE

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Abstract

A $(1, \leq \ell)$-identifying code in a digraph D is a subset C of vertices of D such that all distinct subsets of vertices of cardinality at most ℓ have distinct closed in-neighbourhoods within C. In this paper, we give some sufficient conditions for a digraph of minimum in-degree $\delta^- \geq 1$ to admit a $(1, \leq \ell)$-identifying code for $\ell \in \{\delta^-, \delta^- + 1\}$. As a corollary, we obtain the result by Laihonen that states that a graph of minimum degree $\delta \geq 2$ and girth at least 7 admits a $(1, \leq \delta)$-identifying code. Moreover, we prove that every 1-in-regular digraph has a $(1, \leq 2)$-identifying code if and only if the girth of the digraph is at least 5. We also characterize all the 2-in-regular digraphs admitting a $(1, \leq \ell)$-identifying code for $\ell \in \{2, 3\}$.

Keywords: graph, digraph, identifying code.

2010 Mathematics Subject Classification: 05C69, 05C20.

This research has been partially supported by the project 2017SGR1087 of the Agency for the Management of University and Research Grants (AGAUR) of the Generalitat de Catalunya. The last two authors have received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 734922. B. Martínez-Barona has been supported by the National Council of Science and Technology (CONACyT) of Mexico, Scholarship 438356.

1. Introduction

The aim of this paper is to study identifying codes in digraphs. We consider simple digraphs (or directed graphs) without loops or multiple edges. Unless otherwise stated, we follow the book by Bang-Jensen and Gutin [4] for terminology and definitions.

Let $D = (V, A)$ be a digraph with vertex set $V(D) = V$ and arc set $A(D) = A$. A vertex u is *adjacent to* a vertex v if $(u, v) \in A$. If both arcs $(u, v), (v, u) \in A$, then we say that they form a *digon*. A digraph is *symmetric* if $(u, v) \in A$ implies $(v, u) \in A$. A digon is often said a *symmetric arc* of D. A digraph D is said to be *oriented graph* if D has no digon. The *girth* g of a digraph is the length of a shortest directed cycle. Hence, an oriented graph has girth $g \geq 3$. Moreover, observe that every graph G with vertex set V and edge set E can be seen as a symmetric digraph, replacing each edge $uv \in E$ by the digon (u, v) and (v, u). The *out-neighborhood* of a vertex u is $N^+(u) = \{v \in V : (u, v) \in A\}$ and the *in-neighborhood* of u is $N^-(u) = \{v \in V : (v, u) \in A\}$. The *closed in-neighbourhood* of u is $N^-[u] = \{u\} \cup N^-(u)$. The *out-degree* of u is $d^+(u) = |N^+(u)|$ and its *in-degree* $d^-(u) = |N^-(u)|$. We denote by $\delta^+(D)$ the minimum out-degree of the vertices in D, and by $\delta^-(D)$ the minimum in-degree. The *minimum degree* is $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$. A digraph D is said to be *d-in-regular* if $d^-(v) = d$ for all $v \in V$, and *d-regular* if $d^+(v) = d^-(v) = d$ for all $v \in V$.

Given a vertex subset $U \subseteq V$, let $N^-[U] = \bigcup_{u \in U} N^-[u]$. For a given integer $\ell \geq 1$, a vertex subset $C \subseteq V$ is a *$(1, \leq \ell)$-identifying code* in D when for all distinct subsets $X, Y \subseteq V$, with $1 \leq |X|, |Y| \leq \ell$, we have

$$N^-[X] \cap C \neq N^-[Y] \cap C. \tag{1}$$

The definition of a $(1, \leq \ell)$-identifying code for graphs was introduced by Karpovsky, Chakrabarty and Levitin [16], and it is obtained by omitting the superscript sign minus in the neighborhoods in (1). Thus, the definition for digraphs is a natural extension of the concept of $(1, \leq \ell)$-identifying codes in graphs. A $(1, \leq 1)$-identifying code is known as an *identifying code*. Thus, an identifying code of a graph is a set of vertices such that any two vertices of the graph have distinct closed neighborhoods within this set. Identifying codes model fault-diagnosis in multiprocessor systems, and these are used in other applications such as the design of emergency sensor networks. Identifying codes in graphs have received much more attention by researchers. Honkala and Laihonen [15] studied identifying codes in the king grid that are robust against edge deletions. Identifying codes have been considered for vertex-transitive graphs and strongly regular graphs by Gravier *et al.* [14], and for graphs of girth at least five by Balbuena *et al.* [3]. More recently, techniques of spectral graph theory have been applied to the identifying codes [2]. Other results on identifying codes in specific families of graphs, as well as on the smallest cardinality of an identifying code C, can be seen in Bertrand *et al.* [5], Charon *et al.* [6], Exoo *et al.* [9, 10], and the online bibliography of Lobstein [19].

A graph G is said to *admit* a $(1, \leq \ell)$-identifying code if there is a subset of vertices $C \subseteq V(G)$ such that C is a $(1, \leq \ell)$-identifying code in G. Not all graphs admit $(1, \leq \ell)$-identifying codes. For instance, Laihonen [17] pointed out that a graph containing an isolated edge cannot admit a $(1, \leq 1)$-identifying code, because clearly, if $uv \in E(G)$ is isolated, then $N[u] = \{u, v\} = N[v]$. In fact, a graph containing an isolated complete bipartite graph $K_{r,d}$, with $r \leq d$, cannot admit a $(1, \leq d)$-identifying code. It is not difficult to see that if G admits a $(1, \leq \ell)$-identifying code, then $C = V$ is also a $(1, \leq \ell)$-identifying code. Hence, a graph admits a $(1, \leq \ell)$-identifying code if and only if the sets $N[X]$ are mutually different for all $X \subseteq V$, with $|X| \leq \ell$. Laihonen and Ranto [18] proved that if G is a connected graph with at least three vertices admitting a $(1, \leq \ell)$-identifying code, then the minimum degree is $\delta(G) \geq \ell$. Gravier and Moncel [13] showed the existence of a graph with minimum degree exactly ℓ admitting a $(1, \leq \ell)$-identifying code. Laihonen [17] proved the following result.

Theorem 1 [17]**.** *Let $k \geq 2$ be an integer.*

1. *If a k-regular graph has girth $g \geq 7$, then it admits a $(1, \leq k)$-identifying code.*
2. *If a k-regular graph has girth $g \geq 5$, then it admits a $(1, \leq k-1)$-identifying code.*

Araujo-Pardo *et al.* [1] characterized the bipartite k-regular graphs of girth at least 6 having a $(1, \leq k)$-identifying code.

Identifying codes for digraphs were considered by Charon *et al.* [7, 8], and Frieze *et al.* [12] studied identifying codes in random networks. Foucaud *et al.* [11] studied identifying codes in digraphs under the name of separating sets, and they called identifying codes the separating sets that also are dominating sets. These authors characterized the finite digraphs that only admit their whole vertex set as an identifying code in this meaning.

In this paper, we give some sufficient conditions for a digraph of minimum in-degree $\delta^- \geq 1$ to admit a $(1, \leq \ell)$-identifying code for $\ell \in \{\delta^-, \delta^- + 1\}$. As a corollary, we obtain Theorem 1. Moreover, we prove that every 1-in-regular digraph has a $(1, \leq 2)$-identifying code if and only if the girth of the digraph is at least 5. We also characterize all the 2-in-regular digraphs admitting a $(1, \leq \ell)$-identifying code for $\ell \in \{2, 3\}$.

2. Identifying Codes

In this paper we study the concept of a $(1, \leq \ell)$-identifying code for digraphs given in (1). We begin by noting that if C is a $(1, \leq \ell)$-identifying code in a digraph D, then the whole set of vertices V also is. Thus, we have the following straightforward observation.

Lemma 2. *A digraph $D = (V, A)$ admits some $(1, \leq \ell)$-identifying code if and only if for all distinct subsets $X, Y \subseteq V$ with $|X|, |Y| \leq \ell$, we have*

$$N^-[X] \neq N^-[Y]. \tag{2}$$

As already mentioned in the introduction, Laihonen and Ranto [18] proved that if G is a connected graph with at least three vertices admitting a $(1, \leq \ell)$-identifying code, then the minimum degree is $\delta(G) \geq \ell$. We present the following similar result for digraphs.

Proposition 3. *Let D be a digraph admitting a $(1, \leq \ell)$-identifying code. Let u be a vertex such that $d^+(u) \geq 1$. Then, $\ell \leq d^-(u) + 1$. Furthermore, if u belongs to a digon, then $\ell \leq d^-(u)$.*

Proof. Let $u \in V(D)$ be such that $d^+(u) \geq 1$ and $v \in N^+(u)$. Then, both sets $X = N^-(u) \cup \{u, v\}$ and $Y = N^-(u) \cup \{v\}$ have the same closed in-neighbourhood. Consequently, $\ell \leq d^-(u) + 1$. Furthermore, if $v \in N^-(u)$, then $X' = N^-(u) \cup \{u\}$ and $Y' = N^-(u)$ have the same closed in-neighbourhood implying that $\ell \leq d^-(u)$. ■

Corollary 4. *Let D be a digraph of minimum in-degree δ^- admitting a $(1, \leq \delta^- + 1)$-identifying code. Then, any vertex u with $d^-(u) = \delta^-$ does not lay on a digon.*

Corollary 5. *Let D be a digraph admitting a $(1, \leq \ell)$-identifying code. Then, $\ell \leq \min\{d^-(u) + 1 \mid u \in V(D)$ and $d^+(u) \geq 1\}$.*

We recall that a *transitive tournament* of three vertices is denoted by TT_3, see F_1 of Figure 1.

Remark 6. Let D be a TT_3-free digraph. Then, for every arc (x, y) of D, we have $N^-(x) \cap N^-(y) = \emptyset$ and $N^+(x) \cap N^+(y) = \emptyset$.

Remark 7. Two distinct vertices u and v of D are called *twins* if $N^-[u] = N^-[v]$. Hence, a digraph D admits a $(1, \leq 1)$-identifying code if and only if D is twin-free.

Theorem 8. *Let D be a twin-free digraph with minimum in-degree $\delta^- \geq 1$.*

(i) *Suppose that $\delta^- \geq 2$ and D does not contain any subdigraph as F_1 nor F_2 of Figure 1, then D admits a $(1, \leq \delta^- - 1)$-identifying code.*

(ii) *Suppose that D is an oriented graph and does not contain any subdigraph as F_1 nor F_2 of Figure 1, then D admits a $(1, \leq \delta^-)$-identifying code.*

(iii) *If D does not contain any subdigraph from F_1 to F_9 of Figure* 1, *then D admits a $(1, \leq \delta^-)$-identifying code.*

(iv) *Suppose that $\delta^- \geq 2$ and the vertices of in-degree δ^- do not lay on a digon. If D does not contain any subdigraph as those of Figure* 1, *then D admits a $(1, \leq \delta^- + 1)$-identifying code.*

(v) *Suppose that $\delta^- = 1$ and the vertices of in-degree* 1 *do not lay on directed cycles of length less than five. If D does not contain any subdigraph as F_1, F_3, F_4, F_5, F_6 nor F_{11} of Figure* 1, *then D admits a $(1, \leq 2)$-identifying code.*

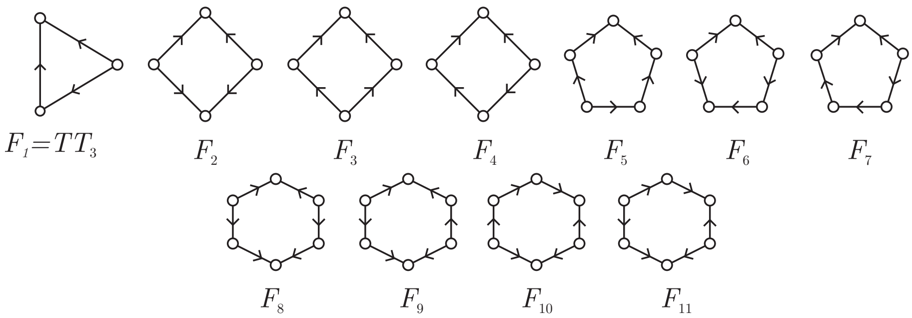

Figure 1. All the forbidden subdigraphs of Theorem 8.

Proof. By Remark 7, D admits a $(1, \leq 1)$-identifying code because D is twin-free. In what follows, for brevity, we made reference to the different cases F_1–F_{11} of Figure 1 without mentioning the figure.

We reason assuming that D does not admit a $(1, \leq \ell)$-identifying code for $\ell \in \{\delta^- - 1, \delta^-, \delta^- + 1\}$. Then there are two different subsets X and Y with $|Y|, |X| \leq \ell$ such that $N^-[X] = N^-[Y]$. Since $X \neq Y$ we can choose $x \in X \setminus Y$ with $N^-(x) = \{v_1, \ldots, v_\tau\}$ for $\tau \geq \delta^-$. As $N^-(x) \subseteq N^-[X] = N^-[Y]$, for all v_i, $i = 1, \ldots, \tau$, there exists a vertex $y_i \in Y$ such that $y_i \in N^+(v_i)$ or $y_i = v_i \in Y$. Moreover, all vertices y_i are mutually different, since otherwise some subdigraph F_1 or F_2 would be contained in D. Hence, $|Y| \geq \delta^-$, which contradicts the hypothesis of (i), and the proof of (i) is completed.

Observe that both (ii) and (iii) are proved if $\delta^- = 1$, so we may assume that $\delta^- \geq 2$ in these two cases.

We continue the proof assuming that $\ell \in \{\delta^-, \delta^- + 1\}$. Since $x \in N^-[X] = N^-[Y]$, there is $y \in Y$ such that $y \in N^+(x)$. Observe that $y \notin N^-(x)$ because by hypothesis of (ii) the digraph is an oriented graph. Moreover, y is different from each y_i because D is free of F_1, implying that $|Y| \geq \delta^- + 1$, which contradicts the hypothesis of (ii), and the proof of (ii) is completed.

Next, to see (iii) let us show that $|X| \geq \delta^- + 1$. To do that, let us see that for each $v_i \in N^-(x)$ one can associate to it a vertex $z_i \in X \setminus \{x\}$ in such a way that $z_i \neq z_j$ for all $i \neq j$. Let us consider the following partition of $N^-(x) : N^-(x) \cap (Y \setminus X)$, $N^-(x) \cap X$ and $N^-(x) \cap (V \setminus (X \cup Y))$. We have the following cases (see Figure 8).

Case 1. $v_i \in N^-(x) \cap (Y \setminus X)$. Since $\delta^- \geq 2$, there is $w_i \in N^-(v_i) \setminus \{x\} \subseteq N^-[Y] \setminus \{x\} = N^-[X] \setminus \{x\}$. Hence, if $w_i \in X$, then $z_i = w_i$ and $z_i \neq x$; and if $w_i \notin X$, since $w_i \in N^-[Y] = N^-[X]$, there exists $z_i \in X$ such that $z_i \in N^+(w_i)$. In this case we may assume that $z_i \neq x$, because D is free of F_1.

Case 2. $v_i \in N^-(x) \cap X$. Then $z_i = v_i$ and $z_i \neq x$.

Case 3. $v_i \in N^-(x) \cap (V \setminus (X \cup Y)) \subseteq N^-[X] \setminus (X \cup Y) = N^-[Y] \setminus (X \cup Y)$. Then we consider the vertices $y_i \in Y$ such that $y_i \in N^+(v_i)$ and $y_i \neq y_j$ for $i \neq j$. If $y_i \in X$, then $z_i = y_i$, and $y_i \neq x$ because $x \in X \setminus Y$. If $y_i \in Y \setminus X$, then there exists $z_i \in X$ such that $z_i \in N^+(y_i)$. Observe that z_i is different from x, because D is free of F_1.

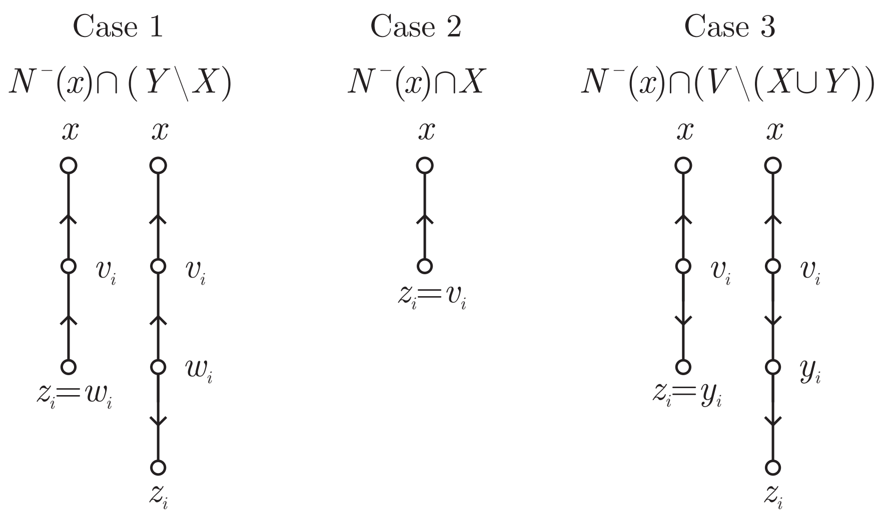

Figure 2. All the cases in the proof of (iii) of Theorem 8.

Now let us see that all z_i are different. For this, let $i, j \in \{1, \ldots, \tau\}$ such that $i \neq j$. If $v_i, v_j \in N^-(x) \cap (Y \setminus X)$ and $z_i = z_j$, then (see Figure 2, Case 1) it could be $w_j = z_j = z_i = w_i \in X$, and D would contain the subdigraph F_3, contradicting the hypothesis of (iii). It could be $z_j = z_i = w_i \in X$ and $w_j \notin X$, then D would contain the subdigraph F_5, a contradiction. Finally, it could be $w_i, w_j \notin X$, $z_i = z_j$ and $z_i \in N^+(w_i) \cap N^+(w_j)$, then D would contain the subdigraph F_8, a contradiction. Therefore, all the z_i are different in Case 1. If $v_i, v_j \in N^-(x) \cap X$, then it is clear that $z_i \neq z_j$ in Case 2. If $v_i, v_j \in N^-(x) \cap (V \setminus (X \cup Y))$ and $z_i = z_j$, then (see Figure 2, Case 3) it could be $z_j = y_i \in X$, and D contains the subdigraph F_6. Hence, $y_i, y_j \in Y \setminus X$ and D contains the subdigraph F_8. Therefore, all the z_i are different in Case 3. It remains to prove that for all $i, j \in \{1, \ldots, \tau\}$, with $i \neq j$, $z_i \neq z_j$ when v_i and v_j are in a different partite subsets of the considered partition of $N^-(x)$. Thus, if $z_i = z_j$ for some $i \neq j$, with $v_i \in N^-(x) \cap (Y \setminus X)$ and $v_j \in N^-(x) \cap X$, then D contains one of the subdigraphs F_1 or F_3 (see Figure 2, Cases 1 and 2); if $v_i \in N^-(x) \cap (Y \setminus X)$ and $v_j \in N^-(x) \cap (V \setminus (X \cup Y))$, then D contains one of the subdigraphs F_4, F_6, F_7 or F_9 (see Figure 2, Cases 1 and 3); and finally, if $v_i \in N^-(x) \cap X$ and $v_j \in N^-(x) \cap (V \setminus (X \cup Y))$, then D contains one of the subdigraphs F_1 or F_4 (see Figure 2, Cases 2 and 3). In any case, we can conclude that X has at least $\delta^- + 1$ vertices, which is a contradiction because $|Y|, |X| \leq \delta^-$ in case (iii), and the proof of this case is completed.

To prove (iv), we assume that $|X| = \delta^- + 1$ and $|Y| \leq \delta^- + 1$. Since by hypothesis $\delta^- \geq 2$, reasoning as in (iii) it follows that $X = \{z_1, z_2, \ldots, z_{\delta^-}, x\}$ and $d^-(x) = \delta^-$. Hence, by hypothesis x does not lay on a digon. Let $y \in N^+(x)$ with $y \in Y \setminus N^-(x)$. First, let us show that $y \in Y \cap X$. Suppose that $y \in Y \setminus X$. Observe that for all $u \in Y \setminus X$, it can be proved analogously that $d^-(u) = \delta^-$. Since $\delta^- \geq 2$, there is $z \in N^-(y) \setminus \{x\}$. Let us show that $z \notin X$. Otherwise, suppose $z \in X$, then $z = z_j$ for some $j = 1, \ldots, \delta^-$. If $v_j \in N^-(x) \cap (Y \setminus X)$, then D contains F_4 or F_5 (see Figure 2, Case 1), if $v_j \in N^-(x) \cap X$, then D contains F_1 (see Figure 2, Case 2); and if $v_j \in N^-(x) \setminus (X \cup Y)$, then D contains F_3 or F_5 (see Figure 2, Case 3). Therefore, $z \notin X$. Hence, $z \in N^-(z_i)$ for some $i \in \{1, \ldots, \delta^-\}$. If $v_i \in N^-(x) \cap (Y \setminus X)$, then D contains F_7 or F_{10}; if $v_i \in N^-(x) \cap X$, then D contains F_4; and if $v_i \in N^-(x) \setminus (X \cup Y)$, then D contains F_6 or F_{11}, a contradiction. This implies that $y \in X \cap Y$ as we claimed. So $y = z_i$ for some $i = 1, \ldots, \delta^-$. Notice that if $v_i \notin N^-(x) \cap (Y \setminus X)$, then x would be contained in a digon, or D contains F_1 or F_3, a contradiction. If $v_i \in Y \setminus X$, then reasoning for v_i as for x, we obtain that every $t \in N^+(v_i) \cap X$ satisfies that $t \in X \cap Y$. However, $x \in N^+(v_i) \cap X$, but $x \notin Y$, which is a contradiction and the proof of (iv) is done.

To prove (v) we assume that $\delta^- = 1$ and $|X| = 2$. Clearly, the following claims holds if $\delta^- \geq 2$; moreover, since there are no vertices of in-degree 1 laying on a digon and by Remark 6, the claim follows.

Claim 9. *Let $(u, v) \in A(D)$. Then, there is $w \in N^-(u) \setminus N^-[v]$.*

First observe that if $|Y| = 1$, say $Y = \{y\}$, then $x \in N^-(y)$ and by Claim 9, there is $w \in N^-(x) \setminus N^-[y]$, implying that $N^-[X] \neq N^-[Y]$, a contradiction. Then $|Y| = |X| = 2$.

Let $X = \{x, x'\}$, $x \in X \setminus Y$, and $Y = \{y, y'\}$ with $y \in N^+(x)$. Let us prove that the arc (x, y) is not on a digon. Otherwise, suppose that $(x, y), (y, x) \in A(D)$. By Claim 9, there exist $w, z \in V(D)$ such that $z \in N^-(x) \setminus N^-[y]$, and $w \in N^-(y) \setminus N^-[x]$. Hence, $z \in N^-[y']$ and $w \in N^-[x']$. If $z \notin Y$, then $z \neq y'$ and $z \in N^-(y')$. Moreover, since D is free of F_1, $y' \in N^-[x'] \subset N^-[X]$. If $x' = y'$, then $w \neq x'$ because D is free of F_3; $w \in N^-(x')$, implying that D contains F_6, therefore $x' \neq y'$ (and so $y' \in Y \setminus X$). Moreover, we can assume that $w \notin \{y', x'\}$, otherwise D contains F_4 or F_5. Thus, $w \in N^-(x')$ implying that D contains F_{11}, concluding that $z \in Y$. Hence, let us assume that $Y = \{y, z\}$, and analogously $X = \{x, w\}$. By Claim 9, there is $u \in (N^-(z) \setminus N^-[x]) \cap N^-[w]$, because $N^-[Y] = N^-[X]$, then D contains F_3 if $u = w$ or F_5 if $u \in N^-(w)$. Therefore, the arc (x, y) is not on a digon.

Suppose that $X \cap Y \neq \emptyset$. First assume that $X = \{x, x'\}$ and $Y = \{y, x'\}$. Taking into account that $N^-[Y] = N^-[X]$ we have $x \in N^-(y) \cup N^-(x')$ and $y \in N^-(x')$ because (x, y) is not on a digon. By Claim 9 there is $w \in N^-(x) \setminus N^-[y]$ and $w \in N^-[x']$ (because $N^-[X] = N^-[Y]$). If $w = x'$, then $(xyx'x)$ is a 3-cycle in D, and by hypothesis there is $u \in N^-(x) \setminus \{x'\}$. By Remark 6, $u \notin N^-(y) \cup N^-(x')$, a contradiction. Then, $w \neq x'$, implying that D contains F_4. Secondly, assume that $X = \{x, y\}$ and $Y = \{y, y'\}$. By Claim 9 there is $w \in N^-(x) \setminus N^-[y]$ and $w \in N^-[y']$. If $w = y'$, there is $w' \in N^-(y') \setminus N^-[x]$ by Claim 9, and D would contain a F_4. Thus $w \neq y'$ and $w \in N^-(y')$, and since $y' \in N^-(x) \cup N^-(y)$, D would contain a F_1 or F_3, a contradiction.

Suppose that $X \cap Y = \emptyset$. Let $X = \{x, x'\}$ and $Y = \{y, y'\}$. Then, $y \in N^-(x')$, and since $y \in Y \setminus X$, reasoning for y as for x, the arc (y, x') is like the arc (x, y) and so it is not lying on a digon. Then $x' \in N^-(y')$ and similarly, $y' \in N^-(x)$. By hypothesis there are no vertices of in-degree 1 lying on a 4-cycle, it follows that there is $z \in N^-(x) \setminus \{y'\}$, but by Remark 6, $N^-(x) \cap (N^-(y) \cup N^-(y')) = \emptyset$ implying that $N^-[X] \neq N^-[Y]$, a contradiction. ■

If for each graph G, we consider its corresponding symmetric digraph obtained by replacing each edge uv of G by the arcs (u, v) and (v, u), then we obtain the following corollary from Theorem 8.

Corollary 10. *Let G be a graph of girth g and minimum degree $\delta \geq 2$. Then*

1. *If $g \geq 7$, then G admits a $(1, \leq \delta)$-identifying code.*
2. *If $g \geq 5$, then G admits a $(1, \leq \delta - 1)$-identifying code.*

Observe that Theorem 1 by Laihonen is a consequence of Corollary 10.

3. 1-IN-REGULAR AND 2-IN-REGULAR DIGRAPHS

In this section, we characterize the d-in-regular digraphs admitting a $(1, \leq d)$-identifying code and a $(1, \leq d+1)$-identifying code for $d \in \{1, 2\}$. Recall that by Proposition 3, if D is a d-in-regular digraph admiting a $(1, \leq \ell)$-identifying code, then $\ell \leq d+1$. We start by giving a characterization of 1-in-regular digraphs admitting a $(1, \leq 2)$-identifying code. Observe that every 1-in-regular digraph D admits a $(1, \leq 1)$-identifying code if and only if D does not contain digons.

Theorem 11. *Every* 1*-in-regular digraph D admits a* $(1, \leq 2)$*-identifying code if and only if the girth of D is at least* 5*.*

Proof. Let $(u_1u_2u_3u_1)$ be a directed triangle or $(v_1v_2v_3v_4v_1)$ a 4-cycle in D. Then the sets $X_1 = \{u_1, u_3\}$, $Y_1 = \{u_2, u_3\}$, $X_2 = \{v_1, v_3\}$ and $Y_2 = \{v_2, v_4\}$ are such that $N^-[X_i] = N^-[Y_i]$, for $i = 1, 2$. Therefore, if D contains a k-cycle, for some $k = 2, 3$ or 4, then D does not admit a $(1, \leq 2)$-identifying code. Conversely, suppose that the girth of D is at least 5. Since D is 1-in-regular, it follows that D does not contain any subdigraph isomorphic to F_1, F_3, F_4, F_5, F_6 nor F_{11} of Figure 1, then by Theorem 8, D admits a $(1, \leq 2)$-identifying code. This completes the proof. ∎

The following result gives a complete characterization of all 2-in-regular digraphs admitting a $(1, \leq 1)$-identifying code and a characterization of all 2-in-regular digraphs admitting a $(1, \leq 2)$-identifying code.

Theorem 12. *Let D be a* 2*-in-regular digraph.*

(i) *D admits a* $(1, \leq 1)$*-identifying code if and only if it does not contain any subdigraph isomorphic to H_1 of Figure* 3*.*

(ii) *D admits a* $(1, \leq 2)$*-identifying code if and only if it does not contain any subdigraph isomorphic to one of the digraphs of Figure* 3*.*

Proof. In what follows, for brevity, we made reference to the different cases H_1–H_{13} of Figure 3 without mentioning the figure. First note that any digraph with twins and minimum in-degree at least 2, necessarily contains H_1. Hence, the proof of (i) follows by Remark 7, because the vertices x, y of H_1 are twins. To prove (ii), first let $X = \{x, x'\}$ (or $X = \{x\}$) and $Y = \{y, y'\}$. It is straightforward to check that $N^-[X] = N^-[Y]$ in each one of the digraphs shown in Figure 3. For the converse, we assume that D does not contain any subdigraph isomorphic to the digraphs depicted in Figure 3, and $N^-[X] = N^-[Y]$ for $X \neq Y$ such that $1 \leq |Y| \leq |X| \leq 2$. According to (i), we can assume that $|X| = 2$, consequently $3 \leq |N^-[X]| \leq 6$. Notice that if $|Y| = 1$, then $|N^-[Y]| = 3$, and so $|N^-[X]| = 3$ yielding that D contains H_1. Therefore, we assume that $|Y| = |X| = 2$. Let $X = \{x, x'\}$ and $Y = \{y, y'\}$ with $x \in X \setminus Y$. Let $N^-(x) = \{v_1, v_2\}$ and $y \in Y$

such that $y \in N^+(x)$. As we did in the proof of Theorem 8, we consider the different cases according to the partition of $N^-(x)$: $N^-(x) \cap (Y \setminus X)$, $N^-(x) \cap X$ and $N^-(x) \cap (V \setminus (X \cup Y))$.

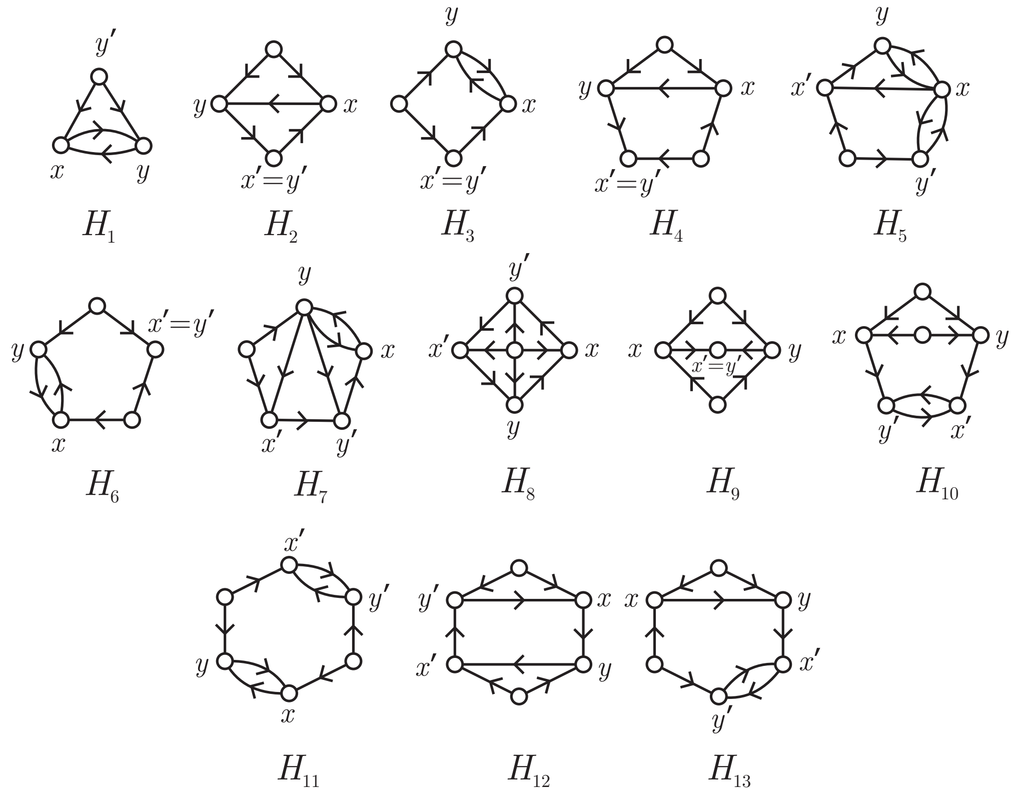

Figure 3. The forbidden subdigraphs in a 2-in-regular digraph admitting a $(1, \leq 2)$-identifying code.

Case 1. $v_1, v_2 \in Y \setminus X$. Let $y = v_1$ and $y' = v_2$ and observe that in this case $x' \notin Y$. As D is H_1-free and H_3-free, $(N^-(y) \setminus \{x\}) \cap N^-[y'] = \emptyset$ and there is no arc between y' and $N^-(y) \setminus \{x\}$. Let $w \in N^-(y) \setminus \{x\}$ and $w' \in N^-(y') \setminus \{x\}$. Then $w, w' \in N^-[x']$.

Subcase 1.1. Suppose that $\{w, w'\} \cap \{x'\} = \emptyset$. Hence, $N^-(x') = \{w, w'\}$. Since $x' \in N^-[Y]$, it follows that $x' \in N^-(y')$ implying that D contains H_{13}, a contradiction.

Subcase 1.2. Suppose that $x' = w$. Hence, $w' \in N^-(x')$. If there is $z \in N^-(x') \setminus (X \cup Y \cup \{w'\})$, then $z \in N^-(y')$, implying that D contains H_{10}, a contradiction. Therefore, $N^-[X] = X \cup Y \cup \{w'\}$, implying that $N^-(x') = \{w', x\}$

or $N^-(x') = \{w', y\}$. First suppose that $N^-(x') = \{w', x\}$. If $x \in N^-(y')$, then D contains H_5 and if $y \in N^-(y')$, then D contains H_4, a contradiction. Therefore, $N^-(x') = \{w', y\}$. If $x \in N^-(y')$, then D contains H_6 and if $y \in N^-(y')$, then D contains H_5, a contradiction.

Subcase 1.3. Suppose that $x' = w'$. Hence, $w \in N^-(x')$. If there is $z \in N^-(x') \setminus (X \cup Y \cup \{w'\})$, then $z \in N^-(y')$, implying that D contains H_{13}, a contradiction. Therefore, $N^-[X] = X \cup Y \cup \{w'\}$. Hence, $N^-(x') = \{w, x\}$ or $N^-(x') = \{w, y\}$. First suppose that $N^-(x') = \{w, x\}$. If $x \in N^-(y')$, then D contains H_4 and if $y \in N^-(y')$, then D contains H_9, a contradiction. Therefore, $N^-(x') = \{w, y\}$. Hence, if $x \in N^-(y')$, then D contains H_4 and if $y \in N^-(y')$, then D contains H_7, a contradiction.

Case 2. $v_1, v_2 \in X$. Since $|X| = 2$ this case is not possible.

Case 3. $v_1, v_2 \notin (X \cup Y)$. Since $x \in N^-(y)$, we have $|N^-(y) \cap \{v_1, v_2\}| \leq 1$ implying that $\{v_1, v_2\} \cap N^-(y') \neq \emptyset$. Without loss of generality, suppose that $v_1 \in N^-(y')$.

Subcase 3.1. If $y \in Y \setminus X$, then $y \in N^-(x')$. If $y' \in X \cap Y$, i.e., $y' = x'$, then $v_2 \in N^-(y)$, implying that D contains H_4. If $y' \in Y \setminus X$, then $N^-(x') = \{y, y'\}$ and $x' \in N^-(y) \cup N^-(y')$. If $x' \in N^-(y)$, then $v_2 \in N^-(y')$, implying that D contains H_{10}. And, if $x' \in N^-(y')$, then $v_2 \in N^-(y)$, implying that D contains H_{13}.

Subcase 3.2. If $y \in X \cap Y$ i.e. $x' = y$, then $y' \in N^-(y)$ and $v_1, v_2 \in N^-(y')$, hence D contains H_9, a contradiction.

Case 4. $v_1 \in Y \setminus X$ and $v_2 \in X$, that is, $v_2 = x'$. Observe that if $v_1 \in N^+(x)$, since D is H_1-free, there is $w \in V(D) \setminus X$ such that $w \in N^-(v_1) \subset N^-[Y]$. Thus, $w \in N^-(x')$, implying that D contains H_3, a contradiction. Then $v_1 \notin N^+(x)$ and so $v_1 = y'$, and moreover $y \in N^-(x')$. If $x' \in N^+(x)$, then $N^-[X] = \{x, x', y, y'\}$, yielding that $y \in N^-(y')$, contradicting that D is H_3-free. Therefore, $N^+(x) \cap \{y', x'\} = \emptyset$ and recall that $y \in N^-(x')$. Moreover, reasoning for y as for x in Case 1, we get that $x' \notin N^-(y)$. Moreover, if $y' \in N^-(y)$, then D contains H_2, a contradiction. Therefore, there is $w \in N^-(y) \setminus (X \cup Y)$. Hence, $w \in N^-(x')$, implying that D contains H_2, a contradiction.

Case 5. $v_1 \in Y \setminus X$ and $v_2 \notin (X \cup Y)$.

Subcase 5.1. If $v_1 \in N^+(x)$, then we can assume that $v_1 = y$. Since D is H_1-free, $v_2 \in N^-(y')$ and there is $w \in V(D) \setminus \{x, v_2\}$ such that $N^-(y) = \{x, w\}$. Observe that since D is H_3-free, $v_2 \notin N^-(w)$, we have $w \neq y'$. Moreover, since D is H_6-free, $w \notin N^-(y')$. Hence, $w \in N^-[x']$, implying that $x' \neq y'$. Observe that reasoning for y as for x in Case 1, we get that $w \neq x'$. Then, $w \in N^-(x')$

and, since $x', y' \in N^-[X] = N^-[Y]$, it follows that $x' \in N^-(y')$ and $y' \in N^-(x')$, therefore D contains H_{11}, a contradiction.

Subcase 5.2. If $v_1 \notin N^+(x)$, then $v_1 = y'$ and $y \in N^-[x']$. First suppose that $y = x'$. If $N^-(y') \subseteq X \cup \{v_2\}$, then $N^-(y') = \{x', v_2\}$ implying that D contains H_2. Hence, there is $w \in N^-(y') \setminus (X \cup \{v_2\})$. Then $w \in N^-(x')$ and $v_2 \in N^-(y')$, implying that D contains H_4, a contradiction. Therefore, $y \neq x'$, implying that $y \in N^-(x')$. Reasoning for y as for x in Case 1 and Case 4, it follows that $N^-(y) \cap \{x', y'\} = \emptyset$. Then $x' \in N^-(y')$. Moreover, since $v_2 \in N^-(x)$, $v_2 \in N^-(y) \cup N^-(y')$. Also, reasoning for x' as for x in Case 1 and Case 4 it follows that $N^-(x') \cap \{x, y'\} = \emptyset$. Hence, if $v_2 \in N^-(y) \cap N^-(y')$, then $N^-[Y] = X \cup Y \cup \{v_2\}$, implying that $v_2 \in N^-(x')$. Then D contains H_8, a contradiction. If $v_2 \in N^-(y') \setminus N^-(y)$, then there is $z \in N^-(y) \setminus (X \cup Y \cup \{v_2\})$, implying that $N^-(x') = \{y, z\}$ and D contains H_{12}. Analogously if $v_2 \in N^-(y) \setminus N^-(y')$. And the proof of this case is completed.

Case 6. $v_1 \in X$ and $v_2 \notin (X \cup Y)$. That is, $v_1 = x'$. If $x' \in X \setminus Y$, then $y \in N^-(x')$. Since $y \in Y \setminus X$, reasoning for x' as for x in Cases 1, 4 and 5, we reach a contradiction. Hence, $x' \in X \cap Y$. If $x' = y$, then $y' \in N^-(x')$ and $v_2 \in N^-(y')$, implying that D contains H_3. Therefore, $x' \neq y$ and hence, $y \in Y \setminus X$. Since $x \in N^-(y)$, reasoning for y as for x in Cases 1, 4 and 5, we reach a contradiction. ∎

Corollary 13. *Every TT_3-free 2-in-regular oriented graph admits a $(1, \leq 2)$-identifying code if and only if it does not contain any subdigraph isomorphic to H_9 of Figure* 3.

Observe that Corollary 13 is an improvement of Theorem 8(ii) for 2-in-regular oriented digraphs. Now, a TT_3-free and 2-in-regular oriented graph can have two distinct vertices u, v with $|N^-(u) \cap N^-(v)| = 2$, that is, a subdigraph F_2 of Figure 1, but in this case there is no vertex $w \in V$ such that $u, v \in N^-(w)$.

In the following theorem we characterize the 2-in-regular digraphs admitting a $(1, \leq 3)$-identifying code.

Theorem 14. *Let D be a 2-in-regular digraph. Then D has a $(1, \leq 3)$-identifying code if and only if it is a TT_3-free oriented graph, and does not contain any subdigraph isomorphic to one of the digraphs of Figure* 4.

Proof. By Proposition 3, if D contains a digon, then D does not admit a $(1, \leq 3)$-identifying code. Suppose that D contains a TT_3, let say $w \in N^-(u) \cap N^-(v)$ and $(u, v) \in A(D)$, and let $z \in V(D)$ be such that $N^-(u) = \{w, z\}$. Hence, the sets $X = \{z, u, v\}$ and $Y = \{z, v\}$ have the same closed in-neiborhood. Furthermore, for every digraph shown in Figure 4 let $X = \{x_1, x_2, x_3\}$ (or $X = \{x_1, x_2\}$)

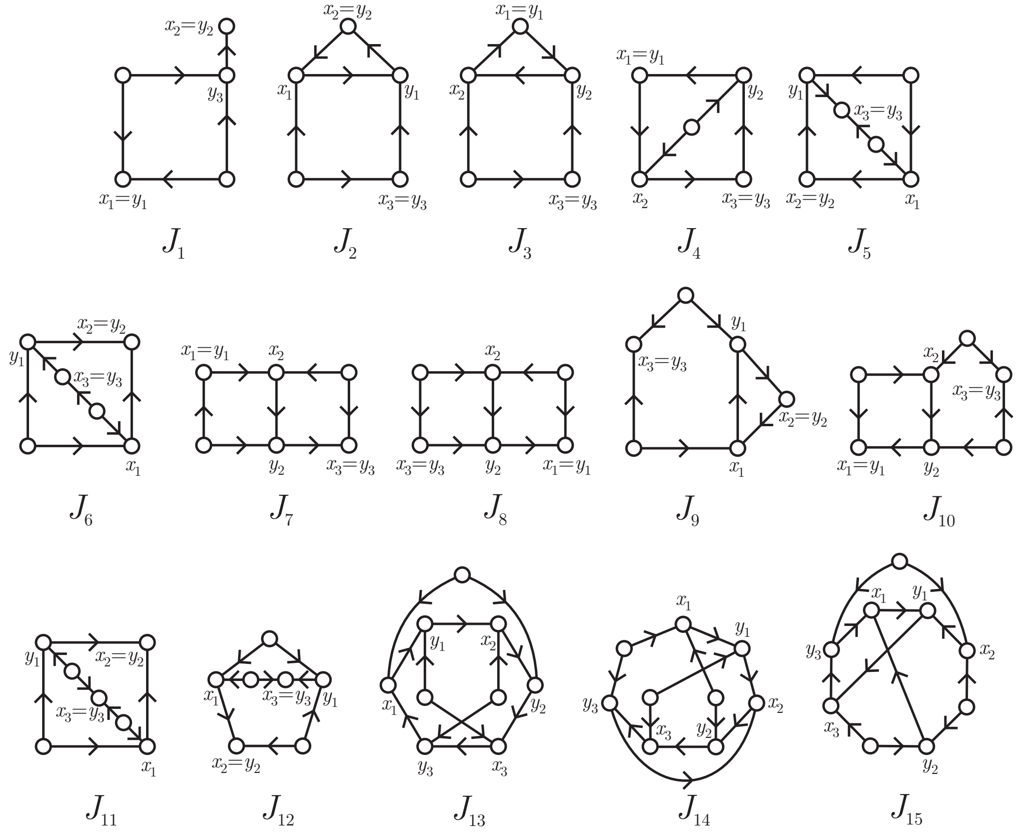

Figure 4. All the forbidden subdigraphs of Theorem 14.

and $Y = \{y_1, y_2, y_3\}$. It is straightforward to check that $N^-[X] = N^-[Y]$ in each case. To the converse, we reason by a contradiction. Let D be a TT_3-free oriented graph without the subdigraphs of Figure 4. Let $X, Y \subseteq V(D)$, $X \neq Y$, with $N^-[X] = N^-[Y]$ and such that $1 \leq |X| \leq |Y| \leq 3$. Since D does not contain a subdigraph isomorphic to J_1 of Figure 4, then it does not contain a subdigraph H_9 of Figure 3. By Corollary 13, D admits a $(1, \leq 2)$-identifying code. Hence, $|Y| = 3$, $|N^-[Y]| \geq 6$ and $|X| \geq 2$. In what follows, for brevity, we always make reference to the different cases J_1–J_{15} of Figure 4 without mentioning the figure. Let us prove the following claim.

Claim 15. *Let $a, b \in V(D)$, $a \neq b$, be such that $N^-(a) \subseteq N^-[b]$. Then, $N^-(a) = N^-(b)$ and $N^+(a) = N^+(b) = \emptyset$.*

Proof. If $b \in N^-(a)$, then D contains a TT_3, which is a contradiction. Hence, $N^-(a) = N^-(b)$ and $N^+(a) = N^+(b) = \emptyset$, because otherwise D contains J_1. $\square$

Suppose $X = \{x_1, x_2\}$. Then $|N^-[X]| = 6$ (because $N^-[X] = N^-[Y]$) and $N^-[x_1] \cap N^-[x_2] = \emptyset$. Let $N^-(x_1) = \{u, v\}$ and $N^-(x_2) = \{z, t\}$, so that $N^-[X] = \{x_1, x_2, u, v, z, t\} = N^-[Y]$. Without loss of generality, we may assume that $u \in Y$. Since D has neither digon nor TT_3, $N^-(u) \subseteq N^-[x_2]$, which implies by Claim 15 that $N^-(u) = N^-(x_2)$ and $N^+(u) = \emptyset$, a contradiction. Therefore, $|X| = |Y| = 3$. Let us denote $X = \{x_1, x_2, x_3\}$. We prove the following claims.

Claim 16. *Let $a, b, c \in V(D)$. If $N^-[a] \subseteq N^-[b] \cup N^-[c]$, then $a \in \{b, c\}$.*

Proof. If $a \notin \{b, c\}$, then without loss of generality, let us assume that $a \in N^-(b)$. Hence, by Remark 6, $N^-(a) \subseteq N^-[c]$, which contradicts Claim 15 because $N^+(a) \neq \emptyset$. □

Claim 17. *$N^-(x_i) \neq N^-(x_j)$ for all $1 \leq i < j \leq 3$.*

Proof. Suppose that $N^-(x_1) = N^-(x_2)$. Then $N^+(x_1) = N^+(x_2) = \emptyset$, because D is J_1-free, which implies $x_1, x_2 \in Y$. Since $|N^-[X]| \geq 6$, there is $z \in N^-(x_3) \setminus (N^-[x_1] \cup N^-[x_2])$. Because $\{x_3, z\} \subseteq N^-[Y]$, D must contain a digon if $z = y_3 \in Y$, or a TT_3 if $\{x_3, z\} = N^-(y_3)$, which is a contradiction. Therefore, $N^-(x_1) \neq N^-(x_2)$. □

Claim 18. *If $7 \leq |N^-[X]| \leq 8$, $N^-(x_i) \cap N^-(x_j) = \{v\}$, $i \neq j$, and there are exactly two or no arc between the elements of X, then $|Y \cap \{x_i, x_j\}| \leq 1$.*

Proof. We proceed by contradiction. Assume $Y = \{x_1, x_2, y\}$. First suppose that there is no arc between the elements of X. If $v \in N^-(x_1) \cap N^-(x_2) \cap N^-(x_3)$, then according to Claim 17, $|N^-[X]| = 7$ and $N^-[x_3] \subseteq N^-[x_1] \cup N^-[y]$, which contradicts Claim 16. Hence, $N^-(x_1) \cap N^-(x_2) \cap N^-(x_3) = \emptyset$. If $|N^-[X]| = 7$, let $N^-(x_1) = \{u, v\}$, $N^-(x_2) = \{v, z\}$ and $N^-(x_3) = \{z, w\}$. Since $N^-(v) \cap N^-[X] \subseteq \{x_3, w\}$, by Remark 6, $v \notin Y$, and analogously $z \notin Y$. Consequently, $N^-[x_3] \subseteq N^-[x_2] \cup N^-[y]$, which contradicts Claim 16. If $|N^-[X]| = 8$, then $N^-(x_3) \subseteq N^-[y]$, a contradiction to Claim 15 because $y \notin X$ and so $N^+(y) \neq \emptyset$. Finally assume that there are two arcs between the elements of X. Notice that by Remark 6, both arcs between the elements of X are incident in x_3. Furthermore, since $7 \leq |N^-[X]| \leq 8$ and $N^-(x_1) \cap N^-(x_2) = \{v\}$, $v = x_3$ and $|N^-[X]| = 7$, we have $N^-(x_3) \subseteq N^-[y]$, a contradiction to Claim 15. □

Let $N^-(x_1) = \{u, v\}$. We distinguish the following cases according to the number of arcs between the vertices of X.

Case 1. There are no arcs between the elements of X.

Subcase 1.1. Suppose $|N^-[X]| = 6$. Then, $N^-[X] = \{x_1, x_2, x_3, u, v, z\}$, so Claim 17 implies that $|N^-(x_i) \cap N^-(x_j)| = 1$ for all $i \neq j$. Let $N^-(x_2) = \{v, z\}$. Observe that $v \notin N^-(x_3)$, otherwise $N^-(x_3) = N^-(x_i)$ for some $i \in \{1, 2\}$, contradicting Claim 17. Therefore $N^-(x_3) = \{u, z\}$. Let $y \in Y \setminus X$. Then $y \in \{u, v, z\}$. We can check that $|N^-(y) \cap N^-[X]| \leq 1$ for all $y \in \{u, v, z\}$, because D is a TT_3-free oriented graph, which is a contradiction.

Subcase 1.2. Suppose $|N^-[X]| = 7$. Then $N^-[X] = \{x_1, x_2, x_3, u, v, z, w\}$. By Claim 17, there are two cases to be considered, namely, $|N^-(x_1) \cap N^-(x_2) \cap N^-(x_3)| = 1$ and $|N^-(x_1) \cap N^-(x_2) \cap N^-(x_3)| = 0$.

Subsubcase 1.2.1. If $|N^-(x_1) \cap N^-(x_2) \cap N^-(x_3)| = 1$, without loss of generality, $N^-(x_2) = \{v, z\}$ and $N^-(x_3) = \{v, w\}$. Since D is an oriented graph and does not contain TT_3, $N^-(v) \cap N^-[X] = \emptyset$, which means that $v \notin Y$ and $v \in N^-(Y)$. Since $N^+(v) \cap \{u, z, w\} = \emptyset$, it follows that $Y \cap X \neq \emptyset$. By Claim 18, $|X \cap Y| = 1$. Without loss of generality, suppose that $X \cap Y = \{x_1\}$. If $Y = \{x_1, z, w\}$, then $x_2 \in N^-(w)$ and $x_3 \in N^-(z)$, implying that D contains J_4. If $Y = \{x_1, u, z\}$, then $x_2 \in N^-(u)$ and $N^-(u) \subseteq \{x_2, x_3, w\}$. If $N^-(u) = \{x_2, x_3\}$, then $w \in N^-(z)$ and hence D contains J_6. If $N^-(u) = \{x_2, w\}$, then $x_3 \in N^-(z)$ and D contains J_5.

Subsubcase 1.2.2. If $|N^-(x_1) \cap N^-(x_2) \cap N^-(x_3)| = 0$, without loss of generality, $N^-(x_2) = \{v, z\}$ and $N^-(x_3) = \{z, w\}$. By Claim 18, $|Y \cap \{x_1, x_2\}| \leq 1$ and $|Y \cap \{x_2, x_3\}| \leq 1$. Moreover, if $\{x_1, x_3\} \subseteq Y$, then since $x_2 \in N^-[Y]$, we have $\{u, w\} \cap Y \neq \emptyset$; without loss of generality, let us assume that $Y = \{x_1, x_3, u\}$. Then $x_2 \in N^-(u)$ and $N^-(u) \subseteq \{x_2, x_3, w\}$. If $N^-(u) = \{x_2, x_3\}$, then D contains J_8, and if $N^-(u) = \{x_2, w\}$, then D contains J_{10}. Therefore, $|Y \cap X| \leq 1$. First, assume that $|Y \cap X| = 1$. Suppose that $X \cap Y = \{x_1\}$ and let $Y = \{x_1, y, y'\}$. Then $N^-[x_3] \subseteq N^-[y] \cup N^-[y']$, which contradicts Claim 16. Hence, $X \cap Y \neq \{x_1\}$, and similarly $X \cap Y \neq \{x_3\}$. Then $X \cap Y = \{x_2\}$. If $v \in Y$, then there is $y \in Y \setminus \{x_2, v\}$ such that $N^-(y) = \{x_1, u\}$ contradicting Remark 6. Hence, $v \notin Y$, and analogously $z \notin Y$. Therefore $Y = \{x_2, u, w\}$, and then $x_1 \in N^-(w)$, implying that $N^-(u) = \{x_3, x_2\}$. Consequently, D contains J_8. Finally if $|Y \cap X| = 0$, by symmetry, we only have to consider the following two cases. If $Y = \{u, v, z\}$, then $x_2 \in N^-(u)$ and $x_1 \in N^-(z)$, implying that D contains J_4. If $Y = \{u, z, w\}$, then $x_3 \in N^-(u)$ implying that $N^-(u) = \{x_2, x_3\}$, and D contains J_8.

Subcase 1.3. Suppose $|N^-[X]| = 8$. Without loss of generality, $N^-(x_2) = \{v, z\}$, and $N^-(x_3) = \{t, w\}$. Observe that $v \notin Y$, otherwise $N^-(v) \subseteq N^-[x_3]$ in a contradiction to Claim 15. If $Y \cap X = \emptyset$, then we can assume that $t \in Y$ and $v \in N^-(t)$. Consequently, $\{u, z\} \cap N^-(t) = \emptyset$, otherwise D contains J_1, therefore $\{x_3, w\} \cap N^-(t) \neq \emptyset$, a contradiction. Therefore $Y \cap X \neq \emptyset$. If $|Y \cap X| = 2$, then by Claim 18, $\{x_1, x_3\} \subseteq Y$ or $\{x_2, x_3\} \subseteq Y$. If $Y = \{y, x_2, x_3\}$, then $N^-[x_1] \subseteq N^-[x_2] \cup N^-[y]$, contradicting Claim 16. Then $Y \neq \{y, x_2, x_3\}$, and similarly $Y \neq \{y, x_1, x_3\}$. Thus $|Y \cap X| = 1$. If $Y = \{x_1, y, y'\}$ or $Y = \{x_3, y, y'\}$, then $N^-[x_3] \subseteq N^-[y] \cup N^-[y']$ or $N^-[x_1] \subseteq N^-[y] \cup N^-[y']$, respectively, which contradicts Claim 16.

Subcase 1.4. Suppose $|N^-[X]| = 9$. Hence, the in-neighborhoods of the elements of X must be disjoint, the same is true for Y. Let $N^-(x_i) = \{u_i, v_i\}$, for $i = 1, 2, 3$. Observe that if $1 \leq |X \cap Y| \leq 2$, then $N^-[x_i] \subseteq N^-[y] \cup N^-[y']$ for some $i \in \{1, 2, 3\}$ and $y, y' \in Y \setminus \{x_i\}$, in a contradiction to Claim 16. Therefore, $X \cap Y = \emptyset$. Without loss of generality, there are two cases to be considered.

Subsubcase 1.4.1. If $Y = \{u_1, v_1, u_2\}$, then $x_1 \in N^-(u_2)$. If $x_3 \in N^-(u_1)$, then without loss of generality, $u_3 \in N^-(v_1)$ and $v_3 \in N^-(u_2)$; moreover, $x_2 \in N^-(v_1)$ and $v_2 \in N^-(u_1)$ or $x_2 \in N^-(u_1)$ and $v_2 \in N^-(v_1)$, implying that D contains J_{14} or J_{15}, respectively. If $x_3 \in N^-(u_2)$, then we may assume that $u_3 \in N^-(u_1)$ and $v_3 \in N^-(v_1)$, and so $x_2 \in N^-(u_1)$ and $v_2 \in N^-(v_1)$, implying that D contains J_{15}.

Subsubcase 1.4.2. Let $Y = \{u_1, u_2, u_3\}$. Without loss of generality, suppose $x_2 \in N^-(u_1)$. Then by Remark 6, $N^-(u_1) \setminus \{x_2\} \subseteq N^-[x_3]$. Since there is no arc between the elements of Y, there are two possibilities to be considered.

- If $N^-(u_1) = \{x_2, x_3\}$, then $v_3 \in N^-(u_2)$ and $v_2 \in N^-(u_3)$. Hence, $x_1 \in N^-(u_2)$ and $v_1 \in N^-(u_3)$, or $v_1 \in N^-(u_2)$ and $x_1 \in N^-(u_3)$; in any case D contains J_{14}.

- If $N^-(u_1) = \{x_2, v_3\}$, then $x_3 \in N^-(u_2)$, and $v_2 \in N^-(u_3)$. If $x_1 \in N^-(u_2)$, then $v_1 \in N^-(u_3)$, implying that D contains J_{14}. Finally, if $x_1 \in N^-(u_3)$, then $v_1 \in N^-(u_2)$, implying that D contains J_{13}.

Case 2. There is just one arc between the elements of X, say $(x_1, x_2) \in A(D)$. Then $|N^-(X)| \in \{6, 7, 8\}$, and $N^-(x_1) \cap N^-(x_2) = \emptyset$ by Remark 6. Let $N^-(x_2) = \{x_1, z\}$ and let us distinguish the following cases.

Subcase 2.1. $|N^-[X]| = 6$. Hence, $N^-[X] = \{x_1, x_2, x_3, u, v, z\}$, and by Claim 17 let us assume without loss of generality, that $N^-(x_3) = \{v, z\}$. Moreover, since D is an oriented graph and does not contain J_1, $N^-(z) \cap N^-[X] \subseteq \{u\}$ and $N^-(v) \cap N^-[X] \subseteq \{x_2\}$, therefore $z, v \notin Y$; hence $u \in Y$. Since D is a TT_3-free oriented graph, $N^-(u) \subseteq \{x_2, x_3, z\}$. Moreover, by Remark 6, $z \notin N^-(u)$. Hence, $N^-(u) = \{x_2, x_3\}$, implying that D contains J_2.

Subcase 2.2. $|N^-[X]| = 7$. Then there is $w \in N^-(x_3) \setminus (X \cup \{u, v, z\})$. By symmetry, $N^-(x_3) = \{z, w\}$ or $N^-(x_3) = \{v, w\}$. First suppose that $N^-(x_3) = \{z, w\}$. Since D is a TT_3-free oriented graph, if $z \in Y$, then $N^-(z) = N^-(x_1)$, which is a contradiction by Claim 15. Hence $z \notin Y$. Analogously, if $w \in Y$ and $x_2 \in N^-(w)$, then $N^-(w) \subseteq \{x_2, u, v\}$, implying that D contains J_7; and if $x_2 \notin N^-(w)$, then $N^-(w) \subseteq N^-[x_1]$, contradicting Claim 15. Thus, $w \notin Y$. If $v \in Y$, then $N^-(v) \subseteq (N^-[x_3] \cup \{x_2\})$. Hence, by Claim 15, $x_2 \in N^-(v)$, implying that $N^-(v) \subseteq \{x_2, x_3, w\}$, but if $N^-(v) = \{x_2, x_3\}$ or $N^-(v) = \{x_2, w\}$, then D contains J_2 or J_9, respectively. Therefore, $v \notin Y$, and by symmetry we can conclude also that $u \notin Y$, a contradiction.

Assume now that $N^-(x_3) = \{v, w\}$. Observe that $N^-(v) \cap N^-[X] \subseteq \{x_2, z\}$. Then $v \notin Y$. If $u \in Y$, then $N^-(u) \subseteq \{x_2, x_3, z, w\}$, but it could not be neither $\{x_2, z\}$ nor $\{x_3, w\}$ (by Remark 6). If $N^-(u) = \{x_2, x_3\}$, then D contains J_3; if $N^-(u) = \{x_2, w\}$, then D contains J_9; if $N^-(u) = \{x_3, z\}$, then D contains J_7; and if $N^-(u) = \{z, w\}$, then D contains J_{10}. Therefore, $u \notin Y$. If $w \in Y$, then $N^-(w) \subseteq \{x_1, x_2, u, z\}$. Hence, by Remark 6 and Claim 15, $N^-(w) = \{u, z\}$ or $N^-(w) = \{u, x_2\}$, implying that D contains J_{12} or J_5, respectively. Therefore, $w \notin Y$. If $z \in Y$, then $N^-(z) \subseteq (N^-[x_3] \cup N^-(x_1))$. Hence, by Claim 15 and Remark 6, $N^-(z) = \{u, w\}$ or $N^-(z) = \{u, x_3\}$, yielding that D contains J_{11} or J_6, respectively. Hence, $z \notin Y$, a contradiction.

Subcase 2.3. $|N^-[X]| = 8$. In this case, $N^-(x_3) = \{t, w\}$ for $t, w \notin N^-[x_1] \cup N^-[x_2]$. First, observe that if $Y \cap \{x_1, x_2\} = \emptyset$, then without loss of generality, $t \in Y$, $x_1 \in N^-(t)$, yielding that $N^-(t) = N^-(x_2)$, a contradiction to Claim 15. Therefore $Y \cap \{x_1, x_2\} \neq \emptyset$. Hence, since $N^-[x_3] \cap (N^-[x_1] \cup N^-[x_2]) = \emptyset$, it follows that $N^-[x_3] \subseteq N^-[y] \cup N^-[y']$, with $y, y' \in Y$, yielding by Claim 16 that $x_3 \in Y$. If $x_2 \notin Y$, then $Y = \{x_1, x_3, y\}$ and $\{x_2, z\} = N^-(y)$, which is a contradiction to Remark 6. Therefore, $Y = \{x_2, x_3, y\}$, yielding that $N^-(x_1) \subseteq N^-[y]$, contradicting Claim 16.

Case 3. There are exactly two arcs between the elements of X. Then $|N^-[X]| \in \{6, 7\}$. Let us distinguish the following cases.

Subcase 3.1. Assume that $(x_1x_2x_3)$ is a path of D. Then $N^-(x_2) \cap N^-(x_3) = N^-(x_2) \cap N^-(x_1) = \emptyset$ by Remark 6. Hence, $N^-(x_2) = \{z, x_1\}$.

Subsubcase 3.1.1. $|N^-[X]| = 6$. Without loss of generality, we may assume that $N^-(x_3) = \{x_2, u\}$. Observe that if $u \in Y$, then $N^-(u) = \{x_2, z\}$, a contradiction to Remark 6, and then $u \notin Y$. If $v \in Y$, then $x_2 \notin N^-(v)$ again by Remark 6. Hence, if $v \in Y$, then $N^-(v) = \{x_3, z\}$, yielding that D contains J_4. Therefore, $z \in Y$ and $|Y \cap X| = 2$. By Remark 6 and Claim 15, $N^-(z) = \{x_3, v\}$, implying that D contains J_3.

Subsubcase 3.1.2. $|N^-[X]| = 7$. Then $N^-(x_3) = \{x_2, w\}$ for some $w \notin N^-[x_1] \cup N^-[x_2]$. If $w \in Y$, then $N^-(w) \subseteq (N^-[x_1] \cup \{z\})$ and, by Claim 15 and Remark 6, $z \in N^-(w)$ and $N^-(w) \subseteq \{u, v, z\}$. This implies that D contains J_6. Therefore, $w \notin Y$. If $z \in Y$, then $N^-(z) \subseteq N^-(x_1) \cup \{x_3, w\}$. Hence, by Claim 15 and Remark 6, without loss of generality, $N^-(z) = \{v, w\}$ or $N^-(z) = \{v, x_3\}$, implying that D contains J_8 or J_2, respectively. Therefore, $z \notin Y$. If $u \in Y$, then $N^-(u) \subseteq N^-[x_3] \cup \{z\}$. By Claim 15 and Remark 6, $N^-(u) = \{z, x_3\}$ or $N^-(u) = \{z, w\}$, yielding that D contains J_4 or J_5, respectively. Therefore, $u \notin Y$ and, by symmetry $v \notin Y$, hence, $Y \setminus X = \emptyset$, a contradiction.

Subcase 3.2. Assume that $N^-(x_2) = \{x_1, x_3\}$. If $|N^-[X]| = 6$, then, without loss of generality, suppose $N^-(x_3) = \{v, z\}$. Observe that $v \notin Y$, otherwise $N^-(v) = \{x_2\}$. If $z \in Y$, then $N^-(z) \subseteq \{x_1, x_2, u\}$ and by Remark 6, $N^-(z) = \{x_2, u\}$, yielding that D contains J_3. Hence, $z \notin Y$, and reasoning similarly, $u \notin Y$, a contradiction. If $|N^-[X]| = 7$, then $N^-(x_3) = \{z, w\}$ for some $w \notin \{x_1, x_2, x_3, u, v, z\}$. If $u \in Y$, then $N^-(u) \subseteq N^-[x_3] \cup \{x_2\}$, and by Claim 15 and Remark 6, $x_2 \in N^-(u)$ and $N^-(u) \subseteq \{x_2, z, w\}$, implying that D contains J_3. Therefore, $u \notin Y$. Analogously, $v, z, w \notin Y$, yielding that $Y \setminus X = \emptyset$, a contradiction.

Subcase 3.3. Without loss of generality, let us assume that $(x_1, x_2), (x_1, x_3) \in A$. If $|N^-[X]| = 6$, then $N^-(x_2) = \{x_1, z\} = N^-(x_3)$, which contradicts Claim 17. Hence, $|N^-[X]| = 7$. Let $N^-(x_2) = \{x_1, z\}$ and $N^-(x_3) = \{x_1, w\}$. Observe that we also may assume that there are exactly two arcs between the elements of Y and there is some $y \in Y$ satisfying the same as x_1, that is, $N^+[y] \cap Y = Y - y$. Therefore, if $x_1 \in Y$, we can assume that $Y = \{x_1, u, w\}$ and $N^+(u) \cap Y = \{x_1, w\}$. Then $N^-(u) \subseteq \{x_3, x_2, z\}$ and by Remark 6, $x_3 \in N^-(u)$, implying that $N^-(u) = \{x_3, x_2\}$ or $N^-(u) = \{x_3, z\}$ yielding that D contains J_2 or J_9, respectively. Moreover, since $N^+(x_1) \cap N^-[X] = \{x_2, x_3\}$, it follows that $Y \cap \{x_2, x_3\} \neq \emptyset$. Furthermore, by Claim 18, $|Y \cap \{x_2, x_3\}| = 1$. Without loss of generality, suppose $Y \cap X = \{x_2\}$. If $Y = \{x_2, z, u\}$, then $u \in N^+(z)$, yielding that $N^-(z) \subseteq \{v, x_3, w\}$. By Remark 6, $N^-(z) = \{v, w\}$ or $N^-(z) = \{v, x_3\}$, implying that D contains J_5 or J_4, respectively. If $Y = \{x_2, z, w\}$, then $z \in N^-(w)$ and $x_3 \in N^-(z)$. Then, without loss of generality, $u \in N^-(z)$ yielding that D contains J_3. Therefore, $z \notin Y$. If $Y = \{x_2, u, v\}$, then, without loss of generality, $x_3 \in N^-(u)$ and $w \in N^-(v)$, implying that D contains J_3. Finally, if $Y = \{x_2, u, w\}$, then $x_3 \in N^-(u)$ and $v \in N^-(w)$, yielding that D contains J_2.

Case 4. There are three arcs between the elements of X. Hence, $|N^-[X]| = 6$ and since D is TT_3-free, we may assume that $(x_1 x_2 x_3 x_1)$ is a directed triangle. Then $N^-(x_i) \cap N^-(x_j) = \emptyset$, for all $i \neq j$. Let $N^-(x_1) = \{x_2, u\}$, $N^-(x_2) = \{x_3, v\}$ and $N^-(x_3) = \{x_1, z\}$. Notice that if $z \in N^-(u)$ or $v \in N^-(u)$, then D contains J_2 or J_3, respectively. Therefore, since D is a TT_3-free oriented graph, $N^-(u) \cap N^-[X] \subseteq \{x_2\}$ and $u \notin Y$. Observe that, by symmetry, we can conclude the same for v and z, obtaining again a contradiction. ■

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Received 9 November 2018
Revised 5 March 2019
Accepted 5 March 2019