OUTPATHS OF ARCS IN REGULAR 3-PARTITE TOURNAMENTS

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Abstract

Guo [Outpaths in semicomplete multipartite digraphs, Discrete Appl. Math. 95 (1999) 273–277] proposed the concept of the outpath in digraphs. An outpath of a vertex \( x \) (an arc \( xy \), respectively) in a digraph is a directed path starting at \( x \) (an arc \( xy \), respectively) such that \( x \) does not dominate the end vertex of this directed path. A \( k \)-outpath is an outpath of length \( k \). The outpath is a generalization of the directed cycle. A \( c \)-partite tournament is an orientation of a complete \( c \)-partite graph.

In this paper, we investigate outpaths of arcs in regular 3-partite tournaments. We prove that every arc of an \( r \)-regular 3-partite tournament has 2- (when \( r \geq 1 \)), 3- (when \( r \geq 2 \)), and 5-, 6-outpaths (when \( r \geq 3 \)). We also give the structure of an \( r \)-regular 3-partite tournament \( D \) with \( r \geq 2 \) that contains arcs which have no 4-outpaths. Based on these results, we conjecture that for all \( k \in \{1, 2, \ldots, r - 1\} \), every arc of \( r \)-regular 3-partite tournaments with \( r \geq 2 \) has \( (3k-1) \)- and \( 3k \)-outpaths, and it has a \( (3k+1) \)-outpath except an \( r \)-regular 3-partite tournament.

Keywords: multipartite tournament, regular 3-partite tournament, outpaths.

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1. Introduction

Throughout the paper, we use the terminology and notation of [1]. The vertex set and the arc set of a digraph \( D \) are denoted by \( V(D) \) and \( A(D) \), respectively. A digraph \( D \) is said to be strongly connected, if for all \( x, y \in V(D) \), there is a directed path from \( x \) to \( y \). A digraph \( D \) is \( r \)-regular, if there is an integer \( r \) such that \( d^{+}(x) = d^{-}(x) = r \) holds for every \( x \in V(D) \). A digraph obtained by
replacing each edge of a complete $c$-partite graph with exactly one arc is called a \textit{$c$-partite tournament} or a \textit{multipartite tournament}. If $D$ is a multipartite tournament and $x \in V(D)$, we denote by $V(x)$ the partite set of $D$ to which $x$ belongs.

An \textit{$l$-outpath} of an arc $x_1x_2$ in a digraph $D$ is a directed path $P = x_1x_2 \cdots x_{l+1}$ with length $l$ starting at $x_1x_2$ such that $x_1$ does not dominate the end vertex $x_{l+1}$ of this directed path $P$. Note that if $D$ is a tournament, an $l$-outpath $P = x_1x_2 \cdots x_{l+1}$ of an arc $x_1x_2$ corresponds in fact to an $(l+1)$-cycle $C = x_1x_2 \cdots x_{l+1}x_1$ through $x_2$, so the concept of the outpath is a generalization of the directed cycle. If $D$ is a multipartite tournament, then $x_1x_2 \cdots x_{l+1}$ is an $l$-outpath of the arc $x_1x_2$ if and only if $x_{l+1} \in V(x_1)$ or $x_{l+1} \rightarrow x_1$ holds.

There are lots of results in multipartite tournaments, see for example \cite{5}. However, the results on 3-partite tournaments are still very few. In 1999, Guo proposed the concept of the outpath in digraphs. At present, outpaths in multipartite tournaments have also been studied by some scholars, see for example \cite{2, 3, 4, 7}. The earliest results are the following two theorems.

\textbf{Theorem 1 (Guo).} Let $D$ be a strongly connected $c$-partite tournament with $c \geq 3$. Then every vertex $v$ of $D$ has a $(k-1)$-outpath for each $k \in \{3, 4, \ldots, c\}$.

\textbf{Theorem 2 (Guo).} Let $D$ be a regular $c$-partite tournament with $c \geq 3$. Then every arc of $D$ has a $(k-1)$-outpath for each $k \in \{3, 4, \ldots, c\}$.

As a generalization of Theorem 2, Cui and the first author proved in \cite{3} that every arc of a regular $c$-partite tournament $D$ with $c \geq 5$ has a $(k-1)$-outpath for each $k \in \{3, 4, \ldots, |V(D)|\}$. In this paper, we investigate outpaths of arcs in regular 3-partite tournaments. However, the following example will show that there exists an infinite family of regular 3-partite tournaments $D$ such that not every arc of $D$ has a $k$-outpath for all $k \in \{3, 4, \ldots, |V(D)|\}$.

\textbf{Example 3.} Let $D$ be an $r$-regular 3-partite tournament with $r \geq 2$ and let $V_1, V_2, V_3$ be three partite sets of $D$ satisfying that $V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow V_1$. (Note that $V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow V_1$ was defined below firstly.) Then it is easy to check that every arc of $D$ has no $(3k+1)$-outpaths for all $k \in \{1, 2, \ldots, r-1\}$.

In this paper, we prove that every arc of an $r$-regular 3-partite tournament has 2- (when $r \geq 1$), 3- (when $r \geq 2$), and 5-, 6-outpaths (when $r \geq 3$). We also give a characterization of regular 3-partite tournaments with at least six vertices whose arcs have no 4-outpaths. We prove that an $r$-regular 3-partite tournament $D$ with $r \geq 2$ contains arcs which have no 4-outpaths if and only if $D$ is the digraph in Example 3. Based on the above results, we conjecture that for all $k \in \{1, 2, \ldots, r-1\}$, every arc of an $r$-regular 3-partite tournament $D$ with $r \geq 2$ has $(3k-1)$- and $3k$-outpaths, and every arc of $D$ has a $(3k+1)$-outpath unless $D$ is the digraph in Example 3.
2. Preliminaries

Let $D$ be a digraph. If $xy$ is an arc of $D$, then we say $x$ dominates $y$ and write $x \rightarrow y$. More generally, if $A$ and $B$ are two disjoint subdigraphs of $D$ such that every vertex of $A$ dominates every vertex of $B$, then we say that $A$ dominates $B$ and denote it by $A \rightarrow B$. Otherwise, we denote it by $A \nrightarrow B$. Let $X$ be a subset of $V(D)$. We use $|X|$ to stand for the number of the vertices of $X$. Let $D'$ be a subdigraph or a vertex set of $D$. The outset $N^+_D(x)$ of a vertex $x$ is the set of vertices of $D'$ dominated by $x$ and the inset $N^-_D(x)$ is the set of vertices of $D'$ dominating $x$. We call the numbers $d^+_D(x) = |N^+_D(x)|$ and $d^-_D(x) = |N^-_D(x)|$ the out-degree and in-degree of $x$ in $D'$, respectively. When $D' = D$, we usually use $N^+(x)$, $N^-(x)$, $d^+(x)$ and $d^-(x)$ instead of $N^+_D(x)$, $N^-_D(x)$, $d^+_D(x)$ and $d^-_D(x)$, respectively.

The following three lemmas are important to prove our main results.

**Lemma 3.** If $D$ is an $r$-regular 3-partite tournament with partite sets $V_1, V_2, V_3$ and $v$ is a vertex of $D$, then $|V_1| = |V_2| = |V_3| = r$ and $d^+(v) = d^-(v) = r$.

**Lemma 4** (Xu, Li, Guo and Li). If $D$ is an $r$-regular 3-partite tournament with partite sets $V_1, V_2, V_3$ and $v$ is a vertex of $V_1$, then $d^+_{v_2}(v) = d^-_{v_3}(v)$ and $d^-_{v_2}(v) = d^+_{v_3}(v)$.

**Lemma 5.** Let $D$ be an $r$-regular 3-partite tournament with $r \geq 2$ and partite sets $V_1, V_2, V_3$. Let $ab$ be an arc of $D$ such that $a \in V_1$ and $b \in V_2$ and $V_3 \rightarrow a \rightarrow V_2$. We divide $V_2$ and $V_3$ into two nonempty parts $V_2^+, V_2^-$ and $V_3^+, V_3^-$, respectively, such that $V_2^+ \rightarrow V_2^-$ and $V_3^+ \rightarrow V_3^-$. Let $V' = V_2^+ \cup V_3^+$ and $V'' = V_2^- \cup V_3^-$. Then the following hold.

1. $N^+(a) = V'$ and $N^-(a) = V''$ and $|V'| = |V''| = r$.
2. $|V_2^+| = |V_3^-|$ and $|V_2^-| = |V_3^+|$.
3. $d^+_{V'}(y) = d^-_{V'}(y)$ and $d^+_{V''}(y) = d^-_{V''}(y)$ for each vertex $y \in V_1$.

**Proof.** Observe $N^+(a) = V'$ and $N^-(a) = V''$. By Lemma 3, we have $d^+(a) = d^-(a) = r$. Therefore, $|V'| = |V''| = r$ holds. This proves (1). By Lemma 3, we get $|V_2| = |V_2^+| + |V_2^-| = r$, $|V_3| = |V_3^+| + |V_3^-| = r$ and $d^+(a) = |V_2^+| + |V_3^+| = r$. Therefore, we have $|V_2^-| = |V_3^+|$ and $|V_2^+| = |V_3^-|$. So (2) is proved. For each vertex $y \in V_1$, by Lemma 3, we have $d^+_{V'}(y) + d^-_{V'}(y) = |V''| = r$, $d^+_{V'}(y) + d^-_{V'}(y) = d^+(y) = r$ and $d^-_{V'}(y) + d^+_{V''}(y) = d^-(y) = r$. So $d^+_{V'}(y) = d^-_{V''}(y)$ and $d^-_{V'}(y) = d^+_{V''}(y)$. This completes the proof of (3).

3. Main Results

By Theorem 2, it is easy to get the following Theorem 6.
Theorem 6. If $D$ is an $r$-regular 3-partite tournament with $r \geq 1$ and $ab$ is an arc of $D$, then $ab$ has a 2-outpath.

Theorem 7. If $D$ is an $r$-regular 3-partite tournament with $r \geq 2$ and $ab$ is an arc of $D$, then $ab$ has a 3-outpath.

Proof. Let $V_1, V_2, V_3$ be three partite sets of $D$. By Lemma 3, we have $|V_1| = |V_2| = |V_3| = r \geq 2$. Without loss of generality, we suppose $a \in V_1$ and $b \in V_2$.

Suppose first that $V_3 \to a \to V_2$. If $V_1 \to b$, then there exists a vertex $y \in V_1 - \{a\}$ such that $b \to y$. By Lemma 4, there is a vertex $x \in V_3$ such that $y \to x$. Then $x \to a$ and $abyx$ is a 3-outpath of $ab$. Assume $V_1 \to b$. Then $b \to V_3$. Let $u \in V_2 - \{b\}$. Then $a \to u$. By Lemma 4, there exists a vertex $x \in V_3$ such that $u \to x$. Obviously, we also have $b \to x$. By $\{b, u\} \to x$ and Lemma 4, there exists a vertex $y \in V_1 - \{a\}$ such that $x \to y$. Then $y \in V(a)$ and $abyx$ is a 3-outpath of $ab$.

Suppose now that $V_3 \to a \to V_2$. We divide the partite set $V_2$ into two nonempty parts $V_2^+, V_2^-$ such that $V_2^- \to a \to V_2^+$. By $a \to b$ and Lemma 4, there exists a vertex $x \in V_3$ such that $b \to x$. If $V_2^- \to x$, then there is an arc $xu$ for some $u \in V_2^-$. Then $u \to a$ and $abxu$ is a 3-outpath of $ab$. Assume $V_2^- \to x$. By $V_2^- \cup \{b\} \to x$ and Lemma 4, there exists a vertex $y \in V_1 - \{a\}$ such that $x \to y$. Then $y \in V(a)$ and $abxy$ is a 3-outpath of $ab$.

Theorem 8. Let $D$ be an $r$-regular 3-partite tournament with $r \geq 2$ and partite sets $V_1, V_2, V_3$. If $ab$ is an arc of $D$, then $ab$ has no 4-outpaths if and only if $V_1 \to V_2 \to V_3 \to V_1$.

Proof. By Lemma 3, we have $|V_1| = |V_2| = |V_3| = r \geq 2$. Suppose, without loss of generality, that $a \in V_1$ and $b \in V_2$. By Example 3, sufficiency is obvious.

Now, we prove the necessity. Suppose that $ab$ has no 4-outpaths. We consider the following two cases.

Case 1. $V_3 \to a \to V_2$. By $a \to b$ and Lemma 4, there exists a vertex $x \in V_3$ such that $b \to x$. If $V_2 \to x$, then there is a vertex $u \in V_2 - \{b\}$ such that $x \to u$. By $x \to u$ and Lemma 4, there exists a vertex $y \in V_1$ such that $u \to y$. Obviously, $a \to u$ and $y \neq a$. Then $y \in V(a)$ and $abxy$ is a 4-outpath of $ab$, a contradiction.

So $V_2 \to x$ and $x \to V_1$.

If $V_1 \to b$, then there exists a vertex $y \in V_1$ such that $b \to y$. Obviously, $y \neq a$ and $x \to y$. By $b \to y$ and Lemma 4, there is a vertex $w \in V_3$ such that $y \to w$. Obviously, $w \neq x$ and $w \to a$. Then $abwyx$ is a 4-outpath of $ab$, a contradiction. So $V_1 \to b$ and $b \to V_3$.

If $(V_2 - \{b\}) \to (V_3 - \{x\})$, then there exists an arc $x'u'$ for some $x' \in V_3 - \{x\}$ and $u' \in V_2 - \{b\}$. Obviously, $b \to x'$ and $u' \to x \to a$. Thus, $abx'u'x$ is a 4-outpath of $ab$, a contradiction. Therefore, we get $(V_2 - \{b\}) \to (V_3 - \{x\})$. Since $b \to V_3$ and $V_2 \to x$, we have $V_2 \to V_3$. So $V_3 \to V_1$ and $V_1 \to V_2$ hold.
Case 2. \( V_3 \rightarrow a \rightarrow V_2 \). In this case, we prove that \( ab \) always has a 4-outpath, which contradicts our assumption. We divide the partite set \( V_2 \) into two nonempty parts \( V_2^+, V_2^- \) such that \( V_2^- \rightarrow a \rightarrow V_2^+ \). Similarly, the partite set \( V_3 \) is also divided into two nonempty parts \( V_3^+, V_3^- \) such that \( V_3^- \rightarrow a \rightarrow V_3^+ \). Let \( V' = V_2^+ \cup V_3^+ \) and \( V'' = V_2^- \cup V_3^- \). By Lemma 5(1), we have \( N^+(a) = V' \) and \( N^-(a) = V'' \).

If \( V_3^+ \rightarrow b \), then there is an arc \( bx \) for some \( x \in V_3^+ \). By \( b \rightarrow x \) and Lemma 4, there exists a vertex \( y \in V_1 \) such that \( x \rightarrow y \). Obviously, \( a \rightarrow x \) and \( y \neq a \). By \( x \in V' \) and Lemma 5(3), there exists a vertex \( z \in V'' \) such that \( y \rightarrow z \). Then \( z \rightarrow a \) and \( abxyz \) is a 4-outpath of \( ab \), a contradiction. So \( V_3^+ \rightarrow b \). By Lemma 4, there exists a vertex \( y \in V_1 - \{a\} \) such that \( b \rightarrow y \). By \( a \rightarrow b \) and Lemma 4, there is a vertex \( v \in V_3 \) such that \( b \rightarrow v \). It is easy to see that \( v \in V_3^- \).

If \( y \rightarrow v \), then Lemma 4 implies that there is a vertex \( u \in V_2 \) such that \( v \rightarrow u \). Obviously, \( u \neq b \). When \( u \in V_2^- \), we get that \( u \rightarrow a \) and \( abuvu \) is a 4-outpath of \( ab \). When \( u \in V_2^+ \), we have \( a \rightarrow u \). By \( v \rightarrow u \) and Lemma 4, there exists a vertex \( y' \in V_1 \) (\( y' \) may be equal to \( y \)) such that \( u \rightarrow y' \). Since \( a \rightarrow u \), we get \( y' \neq a \). Then \( y' \in V(a) \) and \( abuvy' \) is a 4-outpath of \( ab \), a contradiction. Assume \( v \rightarrow y \). By \( b \rightarrow y \), \( b \in V' \) and Lemma 5(3), there exists a vertex \( z \in V'' \) such that \( y \rightarrow z \). Obviously, \( z \neq v \) and \( z \rightarrow a \). Then \( abvyz \) is a 4-outpath of \( ab \), a contradiction.

Therefore, we have shown that if \( ab \) has no 4-outpaths, then \( V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow V_1 \), and the proof is complete. \( \blacksquare \)

**Theorem 9.** If \( D \) is an \( r \)-regular 3-partite tournament with \( r \geq 3 \) and \( ab \) is an arc of \( D \), then \( ab \) has a 5-outpath and a 6-outpath.

**Proof.** Let \( V_1, V_2, V_3 \) be three partite sets of \( D \). By Lemma 3, we have \( |V_1| = |V_2| = |V_3| = r \) and \( d^+(v) = d^-(v) = r \) for each vertex \( v \) of \( D \). Without loss of generality, suppose \( a \in V_1 \) and \( b \in V_2 \). We distinguish the following two cases.

**Case 1.** \( V_3 \rightarrow a \rightarrow V_2 \). By \( a \rightarrow b \) and Lemma 4, there exists a vertex \( x \in V_3 \) such that \( b \rightarrow x \).

**Case 1.1.** \( (V_2 - \{b\}) \rightarrow x \). By the hypothesis, there is a vertex \( u \in V_2 - \{b\} \) such that \( x \rightarrow u \). By Lemma 4, there is a vertex \( y \in V_1 \) such that \( u \rightarrow y \). Since \( a \rightarrow V_2 \), we have \( a \rightarrow u \) and \( y \neq a \). By \( a \rightarrow u \) and Lemma 4, there exists a vertex \( v \in V_3 \) such that \( u \rightarrow v \). Obviously, \( v \neq x \) and \( v \rightarrow a \). Then \( abxuvy \) (when \( v \rightarrow y \)) or \( abxuyv \) (when \( y \rightarrow v \)) is a 5-outpath of \( ab \). We will prove that \( ab \) has a 6-outpath.

**Subcase 1.1.1.** \( v \rightarrow y \). If \( (V_3 - \{x, v\}) \rightarrow y \), then there exists a vertex \( w \in V_3 - \{x, v\} \) such that \( y \rightarrow w \). Thus, \( w \rightarrow a \) and \( abxuwv \) is a 6-outpath of \( ab \). Assume \( (V_3 - \{x, v\}) \rightarrow y \). Note that \( \{u, v\} \rightarrow y \). We have \( N^-(y) = (V_3 - \{x\}) \cup \{u\} \) and \( N^+(y) = (V_2 - \{u\}) \cup \{x\} \). Let \( u' \in V_2 - \{b, u\} \). Then
y → u'. If (V₁ − {a, y}) → u', then there is an arc u'y' for some y' ∈ V₁ − {a, y}. Thus, y' ∈ V(a) and ab has a 6-outpath abxuyu'y'. Assume (V₁ − {a, y}) → u'. Note {a, y} → u'. We get V₁ → u' and u' → V₃. Then u' → v → a and abxuyu'v is a 6-outpath of ab.

Subcase 1.1.2. y → v. If (V₁ − {a, y}) → v, then there exists a vertex y' ∈ V₁ − {a, y} such that v → y'. Thus, y' ∈ V(a) and abxuyu'y' is a 6-outpath of ab. Assume (V₁ − {a, y}) → v. Note that {u, y} → v. We have N⁻(v) = (V₁ − {a}) ∪ {u} and N⁺(v) = (a) ∪ (V₂ − {u}). Let u' ∈ V₂ − {b, u}. Then v → u'. If (V₃ − {x, v}) → u', then there is an arc u'v' for some v' ∈ V₃ − {x, v}. Thus, v' → a and ab has a 6-outpath abxuyu'y'. Assume (V₃ − {x, v}) → u'. Since {a, v} → u', we get N⁻(u') = {a} ∪ (V₃ − {x}) and N⁺(u') = (V₁ − {a}) ∪ {x}. Then u' → y and abxuyu'y is a 6-outpath of ab.

Case 1.2. (V₂ − {b}) → x. In this case, we have V₂ → x → V₁ since b → x.

Subcase 1.2.1. (V₁ − {a}) → b. By the hypothesis, there exists a vertex y ∈ V₁ − {a} such that b → y. By Lemma 4, there is a vertex w ∈ V₃ − {x} such that y → w. Obviously, we have x → y. Then Lemma 4 implies that there exists a vertex u ∈ V₂ such that y → u. It is easy to see u ≠ b and u → x. Then y ∈ V(a).

Suppose first that u → w. If w → y', then abxuyu'y is a 6-outpath of ab. Assume y' → w. By {y, y'} → w and Lemma 4, there exists a vertex v ∈ V₂ − {b} such that w → v. Since u → w, we have v ≠ u. Obviously, v → x → a and abxuyu'y is a 6-outpath of ab.

Suppose now that w → u. If u → y', then abxuyu'y is a 6-outpath of ab. Assume y' → u. By {a, y, y'} → u and Lemma 4, there exists a vertex w' ∈ V₃ − {x, w} such that u → w'. Then w' → a and abxuyu'y is a 6-outpath of ab.

Subcase 1.2.2. (V₁ − {a}) → b. Since a → b, we have V₁ → b and b → V₃. Let w ∈ V₃ − {x}. Then b → w.

Suppose first that (V₂ − {b}) → w. Then there is a vertex u ∈ V₂ − {b} such that w → u. By Lemma 4, there exists a vertex y ∈ V₁ such that u → y. Obviously, a → u, y ≠ a and y ∈ V(a). Recalling that V₂ → x → V₁, we get u → x → y. Then abxuyu'y is a 5-outpath of ab. Let w' ∈ V₃ − {x, v}. Then w' → a. If y → w', then abxuyu'y is a 6-outpath of ab. Assume w' → y. By {x, w'} → y and Lemma 4, there is a vertex u' ∈ V₂ − {b} such that y → u'. Then u' → x → a. Since u → y, we have that u' ≠ u and abxuyu'y is a 6-outpath of ab.

Suppose now that (V₂ − {b}) → w. Since b → w, we have V₂ → w and w → V₁. Then w → a. Let y ∈ V₁ − {a}. Then {x, w} → y. By Lemma 4,
there is a vertex $z \in V_2 - \{b\}$ such that $y \to z$. Obviously, $z \to w$ and $ab$ has a 5-outpath $abxyzw$. Let $y' \in V_1 - \{a, y\}$. Then $y' \in V(a)$ and $w \to y'$. Now, $ab$ has a 6-outpath $abxyzwy'$.

Subcase 2. $V_3 \to a \to V_2$. We divide the partite set $V_2$ into two nonempty parts $V_2^+, V_2^-$ such that $V_2^- \to a \to V_2^+$. Similarly, the partite set $V_3$ is divided into two nonempty parts $V_3^+, V_3^-$ such that $V_3^- \to a \to V_3^+$. Let $V' = V_2^+ \cup V_3^+$ and $V'' = V_2^- \cup V_3^-$. By Lemma 5(1), we have $N^+(a) = V'$, $N^-(a) = V''$ and $|V'| = |V''| = r$.

Subcase 2.1. $V_3^+ \to b$. By the hypothesis, there is a vertex $x \in V_3^+$ such that $b \to x$. By Lemma 4, there is a vertex $y \in V_1 - \{a\}$ such that $x \to y$. Obviously, $y \in V(a)$. Similarly, by $a \to x$ and Lemma 4, there is a vertex $u \in V_2 - \{b\}$ such that $x \to u$.

We first show that $ab$ has a 5-outpath. If $u \to y$, then by $x \to V'$, $x \to y$ and Lemma 5(3), there is a vertex $z \in V''$ such that $y \to z$. Obviously, $z \neq u$ and $z \to a$. Then $ab$ has a 5-outpath $abxyz$. Assume $y \to u$. Then $(V_1 - \{a, y\}) \cup V_3^- \to u$ (as otherwise, we have $(V_1 - \{a, y\}) \cup V_3^- \cup \{x, y\} \subseteq N^-(u)$ and $d^-(u) \geq r + 1$, a contradiction). Therefore, there exists a vertex $z' \in V_1 - \{a, y\} \cup V_3^-$ such that $u \to z'$. Then $z' \in V(a)$ or $z' \to a$. Now, $ab$ has a 5-outpath $abxyz'$.

Next, we will prove that $ab$ has a 6-outpath. We discuss the following two subcases.

Subcase 2.1.1. $|V_3^+| = 1$. By Lemma 5(2), we have $|V_2^+| = |V_3^-| = 1$, $|V_2^-| = |V_3^+| = r - 1 \geq 2$. Obviously, $V_2^+ = \{b\}$ and $V_2^- = V_2 - \{b\}$. Let $V_3^- = \{v\}$.

Then $v \to a$ and $V_3^+ = V_3 - \{v\}$.

Suppose first that $y \to u$. If $V_3^+ \to u$, then $N^+(u) = V_3^+ \cup \{y\} = (V_3 - \{v\}) \cup \{y\} \cup \{v\}$ and $N^-(u) = (V_1 - \{y\}) \cup \{v\}$. Let $y' \in V_1 - \{a, y\}$. Then $y' \in V(a)$ and $u \to y'$. Thus, $abxyuy'v$ (when $y' \to v$) or $abxyuyv'$ (when $v \to y'$) is a 6-outpath of $ab$. Assume $V_3^+ \to u$. Then there exists a vertex $w \in V_3^+$ such that $u \to w$. Obviously, $w \neq x$ and $a \to w$. If $(V_1 - \{a, y\}) \to w$, then there is a vertex $y' \in V_1 - \{a, y\}$ such that $w \to y'$. Then $y' \in V(a)$ and $ab$ has a 6-outpath $abxyuyw'y$. Assume $(V_1 - \{a, y\}) \to w$. Since $\{a, u\} \to w$, we have $N^-(w) = (V_1 - \{y\}) \cup \{u\}$ and $N^+(w) = \{y\} \cup (V_2 - \{u\})$. Let $u' \in V_2^- - \{u\}$. Then $w \to u'\to a$ and $ab$ has a 6-outpath $abxyuyw'y$.

Suppose now that $u \to y$. If $V_3^+ \to y$, then $N^-(y) = V_3^+ \cup \{x\} = (V_2 - \{b\}) \cup \{x\}$ and $N^+(y) = \{b\} \cup (V_3 - \{x\})$. Let $w \in V_3^+ - \{x\}$. Then $\{a, y\} \to w$. When $(V_1 - \{a, y\}) \to w$, there is a vertex $y' \in V_1 - \{a, y\}$ such that $w \to y'$. Then $y' \in V(a)$ and $abxyuyw'y$ is a 6-outpath of $ab$. When $(V_1 - \{a, y\}) \to w$, we have $V_1 \to w \to V_2$ since $\{a, y\} \to w$. Let $u' \in V_2^- - \{u\}$. Then $w \to u' \to a$ and $abxyuyw'y$ is a 6-outpath of $ab$.

Assume $V_3^- \to y$. Then there is a vertex $z \in V_3^- \text{ such that } y \to z$. Clearly, $z \neq u$. If $(V_1 - \{a, y\}) \to z$, then there exists a vertex $y_0 \in V_1 - \{a, y\}$ such
that \( z \to y_0 \). Thus, \( y_0 \in V(a) \) and \( abxyzy_0 \) is a 6-outpath of \( ab \). Assume 
\((V_1 - \{a, y\}) \to z\). Since \( y \to z \), we get \((V_1 - \{y\}) \to z\) and 
\(d_{V_1}(z) \geq r - 1 \geq 2\). By Lemma 4, there is a vertex \( w \in V_3 - \{x\} \) such that \( z \to w \). When \( w = v \), we 
know that \( w \to a \) and \( abxyzw \) is a 6-outpath of \( ab \). When \( w \neq v \), we have 
\( w \in V_3^+ - \{x\} \) and \( a \to w \). If \( w \to u \), then \( abxyzw \) is a 6-outpath of \( ab \). If 
\( u \to w \), then by \( \{u, z\} \to w \) and Lemma 4, there exists a vertex \( y_1 \in V_1 - \{a, y\} \) 
such that \( w \to y_1 \). Thus, \( y_1 \in V(a) \) and \( ab \) has a 6-outpath \( abxyzy_1 \).

Subcase 2.1.2. \( 2 \leq |V_2^+| \leq r - 1 \). By Lemma 5(2), we have \( 2 \leq |V_2^+| = 
|V_2^-| \leq r - 1, 1 \leq |V_3^+| \leq |V_3^-| \leq r - 2\).

Suppose first that \((V' - \{b\}) \to y\). Then there exists a vertex \( v \in V' - \{b\} \) 
such that \( y \to v \). Obviously, \( v \in V_3^+ \) or \( v \in V_2^+ - \{b\} \).

When \( v \in V_3^+, \) by \( (a, y) \to v \) and Lemma 4, there exists a vertex \( w \in V_2 - \{b\} \) 
such that \( v \to w \). If \( V_3^- \to w \), then there is an arc \( wz \) for some vertex \( z \in V_3^- \) 
and \( abxyzw \) is a 6-outpath of \( ab \). Assume \( V_3^- \to w \). By \( V_3^- \cup \{v\} \to w \) and 
Lemma 4, there is a vertex \( y' \in V_1 - \{a, y\} \) such that \( w \to y' \). Now, \( y' \in V(a) \) and 
\( abxywy' \) is a 6-outpath of \( ab \).

When \( v \in V_2^+ - \{b\}, \) by \( (a, y) \to v \) and Lemma 4, there exists a vertex \( v' \in V_3 - \{x\} \) such that \( v \to v' \). If \( V_2^- \to v' \), then there is an arc \( v'w' \) for some vertex \( u' \in V_2^- \) 
and \( abxyv'w' \) is a 6-outpath of \( ab \). Assume \( V_2^- \to v' \). If 
\((V_1 - \{a, y\}) \to v' \), then there is a vertex \( y' \in V_1 - \{a, y\} \) such that \( v' \to y' \). Now, 
\( y' \in V(a) \) and \( abxyv'y' \) is a 6-outpath of \( ab \).

When \( y_0 \to v' \to a \). Then \( abxyv'y' \) is a 6-outpath of \( ab \). Assume \((V_1 - \{a, y\}) \to v \). 
Since \( \{a, y\} \to v \), we get \( V_1 \to v \to V_3 \). Then \( v \to x \). By \( (b, v) \to x \) and 
Lemma 4, there exists a vertex \( y_1 \in V_1 - \{y\} \) such that \( x \to y_1 \). Since \( a \to x \), we 
get \( y_1 \neq a \) and \( y_1 \in V_1 - \{a, y\} \). Note \( y_1 \to v \) and \( y \in V(a) \). Then \( abxy_1vy' \) is 
a 6-outpath of \( ab \).

Suppose now that \((V' - \{b\}) \to y\). If \( V_2^- \to y \), then \( N^{-}(y) = (V_2 - \{b\}) \cup V_3^- \). 
So we have \( |V_3^-| = |V_2^-| = 1 \) and \( V_2^+ = \{x\} \). Thus, \( N^{+}(y) = \{b\} \cup V_3^- \). 
Let \( V_2^- = \{w\} \). By \( w \to \{a, y\} \) and Lemma 4, there exists a vertex \( z \in V_3^- \) such that 
\( z \to w \). Clearly, we have \( y \to z \). Let \( u' \in V_2^+ - \{b\} \). Then \( u' \to y \). If \( x \to u' \), then 
\( abxyzuv \) is a 6-outpath of \( ab \). Assume \( u' \to x \). By \( (b, u') \to x \) and Lemma 4, 
there exists a vertex \( y' \in V_1 - \{y\} \) such that \( x \to y' \). Obviously, \( y' \neq a \), \( y' \in V(a) \) 
and \( z \to a \). Then \( abxywy'z \) (when \( y' \to w \)) or \( abxyzw' \) (when \( w \to y' \)) is a 
6-outpath of \( ab \).

Assume \( V_2^- \to y \). Then there is an arc \( yw \) for some vertex \( w \in V_2^- \). By 
\((V_2^+ - \{b\}) \to y \) and Lemma 4, there exists a vertex \( v_1 \in V_3 \) such that \( y \to v_1 \).

Since \( V_3^+ \to y \), we get \( v_1 \in V_3^- \) and \( v_1 \to a \).
If \( v_1 \to w \), then \((V_1 - \{a, y\}) \cup (V_3^- - \{v_1\}) \not\to w\) (as otherwise, we get 
\((V_1 - \{a, y\}) \cup (V_3^- - \{v_1\}) \cup \{y, v_1\} \subseteq N^-(w)\) and \(d^-(w) \geq r+1\), a contradiction).

So there is a vertex \( z_1 \) in \((V_1 - \{a, y\}) \cup (V_3^- - \{v_1\})\) such that \( w \to z_1 \). Note \( z_1 \in V(a) \) or \( z_1 \to a \). Then \( abxyv_1wz_1 \) is a 6-outpath of \( ab \).

Assume \( w \to v_1 \). If \((V_1 - \{a, y\}) \to v_1 \), then there exists a vertex \( y' \in V_1 - \{a, y\} \) such that \( v_1 \to y' \). Then \( y' \in V(a) \) and \( abxyuv_1y' \) is a 6-outpath of \( ab \).

Assume \((V_1 - \{a, y\}) \to v_1 \). Since \( \{y, w\} \to v_1 \), we have \( N^-(v_1) = (V_1 - \{a, y\}) \cup \{y, w\} = (V_1 - \{a\}) \cup \{w\} \) and \( N^+(v_1) = \{a\} \cup (V_2 - \{w\}) \).

Let \( u_1 \in V_2^+ - \{b\} \). Then \( v_1 \to u_1 \to y \). If \( x \to w \), then \( abxuwv_1uy \) is a 6-outpath of \( ab \). Assume \( w \to x \). By \( \{b, w\} \to x \) and Lemma 4, there exists a vertex \( y_1 \in V_1 - \{y\} \) such that \( x \to y_1 \). Obviously, \( y_1 \neq a \) and \( y_1 \to v_1 \). Then \( abxyv_1uy_1y \) is a 6-outpath of \( ab \).

Subcase 2.2. \( V_3^+ \to b \).

Subcase 2.2.1. \( |V_3^+| = 1 \). By Lemma 5(2), we have \( |V_2^+| = |V_3^-| = 1 \) and 
\[ |V_2^-| = |V_3^+| = r - 1 \geq 2 \]. Obviously, we have \( V_2^+ = \{b\} \). Let \( V_3^- = \{v\} \).

Then \( v \to a \), \( V_2^- = V_2 - \{b\} \) and \( V_3^- = V_3 - \{v\} \). Since \( V_3^+ \to b \), we get 
\[ N^-(b) = V_3^+ \cup \{a\} \] and \( N^+(b) = (V_1 - \{a\}) \cup \{v\} \).

If \( V_3^+ \to (V_1 - \{a\}) \), then we have \( V_3^+ \cup \{b\} \to (V_1 - \{a\}) \cup (V_2 - \{b\}) \cup \{v\} \). 
\( V_3^+ \to (V_1 - \{a\}) \cup \{b\} \) and \( \{a\} \cup (V_2 - \{b\}) \to V_3^+ \). Let \( y, y' \) be two distinct vertices in \( V_1 - \{a\} \) and let \( u \) and \( x \) be two arbitrary vertices in \( V_2 - \{b\} \) and \( V_3^+ \), respectively. Then \( y' \in V(a) \) and \( ab \) has a 5-outpath \( abxyxy' \) and a 6-outpath \( abxyxy' \).

Assume \( V_3^+ \to (V_1 - \{a\}) \). Then there is an arc \( xy \) for some \( y \in V_1 - \{a\} \) and \( x \in V_3^- \). Clearly, we get \( b \to y \).

If \((V_1 - \{a, y\}) \to x \), then there exists a vertex \( y' \in V_1 - \{a, y\} \) such that \( x \to y' \). By Lemma 4, there is a vertex \( u \in V_2 \) such that \( y' \to u \). Note \( b \to y' \).

We have \( u \neq b \) and \( u \in V_2^- \). Then \( u \to a \) and \( abxyuy \) is a 5-outpath of \( ab \).

We will seek for a 6-outpath of \( ab \). If \( y' \to v \), then \( abxyuvw \) (when \( u \to v \)) or \( abxyuyw \) (when \( v \to w \)) is a 6-outpath of \( ab \). Assume \( v \to y' \). By \( b \to y' \) and Lemma 4, there exists a vertex \( w \in V_3 \) such that \( y' \to w \). Since \( \{x, v\} \to y', \) we have \( w \neq x \) and \( w \neq v \). Then \( w \in V_3^+ - \{x\} \) and \( a \to w \). By \( \{a, y'\} \to w \) and Lemma 4, there exists a vertex \( u' \in V_2 - \{b\} \) (\( u' \) may be equal to \( u \)) such that \( w \to u' \). Then \( u' \to a \) and \( abxyuwy' \) is a 6-outpath of \( ab \).

If \((V_1 - \{a, y\}) \to x \), we have \( V_1 \to x \to V_2 \) since \( \{a, y\} \to x \).

In the case when \( (V_1 - \{a\}) \to (V_2 - \{b\}) \), we have \((V_1 - \{a\}) \to (V_2 - \{b\}) \cup \{x\} \) and \( \{b\} \cup (V_2 - \{x\}) \to (V_1 - \{a\}) \). In addition, we also have \( (V_1 - \{a\}) \cup \{x\} \to (V_2 - \{b\}) \to \{a\} \cup (V_3 - \{x\}) \). Let \( y' \) and \( u \) be two arbitrary vertices in \( V_1 - \{a, y\} \) and \( V_2 - \{b\} \), respectively. Then \( x \to u \to v \to y' \). Note \( y' \in V(a) \) and \( v \to a \).

We have that \( ab \) has a 5-outpath \( abxyuv \) and a 6-outpath \( abxyuy' \).

In the other case when \((V_1 - \{a\}) \to (V_2 - \{b\}) \), there exists an arc \( uy' \) from \( V_2 - \{b\} \) to \( V_1 - \{a\} \) (\( y' \) may be equal to \( y \)). Let \( y_1 \in V_1 - \{a, y\} \) (when \( y' \neq y, \)
$y_1$ may be equal to $y$). Then $b \to y_1 \to x \to u$. By $b \to y'$ and Lemma 5(3), there exists a vertex $z \in V''$ such that $y' \to z$. Since $u \to y'$, we have $z \neq u$. Note $y' \in V(a)$ and $z \to a$. Then $ab$ has a 5-outpath $abyxuy'$ and a 6-outpath $aby_1xuy'z$.

**Subcase 2.2.2.** $2 \leq |V_2^+| \leq r - 1$. By Lemma 5(2), we have $2 \leq |V_2^+| = |V_3^-| \leq r - 1$, $1 \leq |V_2^-| = |V_3^+| \leq r - 2$. By $V_3^+ \to b$ and Lemma 4, there is an arc $by$ for some $y \in V_1 - \{a\}$. Obviously, $y \in V(a)$.

**Subcase 2.2.2.1.** $V_3^+ \to y$. By the hypothesis, there is a vertex $x \in V_3^+$ such that $y \to x$.

Suppose first that $(V_2^+ - \{b\}) \to x$. Then there exists a vertex $u \in V_2^+ - \{b\}$ such that $x \to u$. If $(V_1 - \{a, y\}) \to u$, then there is a vertex $y' \in V_1 - \{a, y\}$ such that $u \to y'$. By $u \in V'$ and Lemma 5(3), there exists a vertex $v \in V''$ such that $y' \to v$. Note $y' \in V(a)$ and $v \to a$. Then $ab$ has a 5-outpath $abyxuy'$ and a 6-outpath $aby_1xuy'z$.

Assume $V_1 - \{a, y\} \to u$. Since $\{a, x\} \to u$, we get $N^-(u) = (V_1 - \{y\}) \cup \{x\}$ and $N^+(u) = \{y\} \cup (V_3 - \{x\})$. Let $z \in V_3^-$. Then $u \to z \to a$ and $abyuza$ is a 5-outpath of $ab$. We will seek for a 6-outpath of $ab$. Let $w \in V_2^-$ be arbitrary. If $(V_1 - \{a, y\}) \cup \{w\} \to z$, then there is an arc $zy'$ or $zw$ for some $y' \in V_1 - \{a, y\}$. Note $y' \in V(a)$ and $w \to a$. Then $abyuza$ or $abyuzw$ is a 6-outpath of $ab$. Assume $(V_1 - \{a, y\}) \cup \{w\} \to z$. Then it is easy to see that $N^-(z) = (V_1 - \{a, y\}) \cup \{u, w\}$ and $N^+(z) = \{a, y\} \cup (V_2 - \{u, w\})$. So $z \to \{b, y\}$. By $(x, z) \to b$ and Lemma 4, there exists a vertex $y_0 \in V_1 - \{y\}$ such that $b \to y_0$. Obviously, $y_0 \neq a$ and $y_0 \to u$. Then $abyxuzy$ (when $y_0 \to x$) or $abyx_0uzy$ (when $x \to y_0$) is a 6-outpath of $ab$.

Suppose now that $(V_2^+ - \{b\}) \to x$. By Lemma 4, there exists a vertex $y' \in V_1$ such that $x \to y'$. Since $\{a, y\} \to x$, we have $y' \neq a$ and $y' \neq y$. By $x \in V'$, $x \to y'$ and Lemma 5(3), there is a vertex $z \in V''$ such that $y' \to z$. Then $z \to a$ and $aby_1y'z$ is a 5-outpath of $ab$. We will prove that $ab$ has a 6-outpath.

By $(a, y) \to x$ and Lemma 4, there exists a vertex $v \in V_2 - \{b\}$ such that $x \to v$. Since $(V_2^- - \{b\}) \to x$, we get $v \in V_2^-$. When $v \to y'$, we have that $v \neq z$ and $abyxvz$ is a 6-outpath of $ab$. When $y' \to v$, it is easy to see that $(V_1 - \{a, y, y'\}) \cup V_3^- \to v$ (as otherwise, $(V_1 - \{a, y, y'\}) \cup V_3^- \cup \{y', x\} \subseteq N^-(v)$ and $d^-(v) \geq r + 1$, a contradiction). So there is a vertex $v' \in (V_1 - \{a, y, y'\}) \cup V_3^-$ such that $v \to v'$. Note $v' \in V(a)$ or $v' \to a$. Then $abyv'v'v'$ is a 6-outpath of $ab$.

**Subcase 2.2.2.2.** $(V_2^+ - \{b\}) \to y$. By the hypothesis, there is an arc $yu$ for some vertex $u \in V_2^+ - \{b\}$.

Suppose first that $V_3^- \to u$. Then there exists a vertex $x \in V_3^+$ such that $u \to x$. If $(V_1 - \{a, y\}) \to x$, then there is a vertex $y' \in V_1 - \{a, y\}$ such that $x \to y'$. By $x \in V'$ and Lemma 5(3), there exists a vertex $w \in V''$ such that $y' \to w$. Note $y' \in V(a)$ and $w \to a$. Then $ab$ has a 5-outpath $abyxuy'$ and a 6-
outpath \( abyuxy'w \). Assume \((V_1 - \{a, y\}) \to x\). Since \(\{a, u\} \to x\), we get \(N^-(x) = (V_1 - \{y\}) \cup \{u\}\) and \(N^+(x) = \{y\} \cup (V_2 - \{u\})\). Let \(z \in V_2^-\). Then \(x \to z \to a\) and \(abyuxz\) is a 5-outpath of \(ab\). In addition, we also have \((V_1 - \{a, y\}) \cup V_3^- \to z\) (as otherwise, \((V_1 - \{a, y\}) \cup V_3^- \cup \{x\} \subseteq N^-(z)\) and \(d^-(z) \geq r + 1\), a contradiction). So there is an arc \(zy_1\) or \(zy\) for some \(y_1 \in V_1 - \{a, y\}\) and \(v \in V_3^-\). Note \(y_1 \in V(a)\) and \(v \to a\). Then \(abyuxzy_1\) or \(abyuxzv\) is a 6-outpath of \(ab\).

Suppose now that \(V_3^+ \to u\). By Lemma 4, there exists a vertex \(y' \in V_1\) such that \(u \to y'\). Since \(\{a, u\} \to x\), we have \(y' \neq a\) and \(y' \neq y\). By \(V_1 \to y'\), \(u \to y'\), and Lemma 5(3), there is a vertex \(z' \in V''\) such that \(y' \to z'\). Then \(z' \to a\) and \(abyuy'z'\) is a 5-outpath of \(ab\). We will prove that \(ab\) has a 6-outpath.

By \(\{a, y\} \to u\) and Lemma 4, there exist two distinct vertices \(v, v' \in V_3\) such that \(u \to \{v, v'\}\). Since \(V_3^+ \to u\), we get \(v, v' \in V_3^-\). If \(v \to y'\), then \(v \neq z'\) and \(abyuy'z'\) is a 6-outpath of \(ab\). Assume \(y' \to v\). If \((V_1 - \{a, y, y'\}) \cup V_3^- \to v\), then there is a vertex \(w \in (V_1 - \{a, y, y'\}) \cup V_3^-\) such that \(v \to w\). Note \(w \in V(a)\) or \(w \to a\). Then \(abyuy'w\) is a 6-outpath of \(ab\). Assume \((V_1 - \{a, y, y'\}) \cup V_3^- \to v\). Note \((V_1 - \{a, y, y'\}) \cup V_3^- \cup \{y', u\} \to v\) and \(d^-(v) = r\). We have \(|V_3^-| = 1\) and \(N^+(v) = \{a, y\} \cup (V_3^- - \{u\})\). Then \(v \to b\). By \(V_3^+ \cup \{v\} \to b\) and Lemma 4, there is a vertex \(y_0 \in V_1 - \{y\}\) (\(y_0\) may be equal to \(y'\)) such that \(b \to y_0\). Obviously, \(y_0 \neq a\) and \(y_0 \to v\). Thus, \(abyuy'vy'v\) is a 6-outpath of \(ab\).

**Subcase 2.2.2.3.** \((V_2^+ - \{b\}) \cup V_3^+ \to y\). In this case, we have \(V' \to y \to V''\) since \(b \to y\). By \(a \to b\) and Lemma 4, there exists a vertex \(c \in V_3\) such that \(b \to c\). Note \(V_3^+ \to b\). We get \(c \in V_3^-\). Let \(x \in V_3^+\) be arbitrary. Note that \(\{b, y\} \to c \to a\) and \(a \to x \to \{b, y\}\). Then there exists a vertex \(u \in (V_1 - \{a, y\}) \cup (V_2 - \{b\})\) such that \(c \to u \to x\) (as otherwise, we have \(d^-(c) < d^+(x)\), this is impossible). Then \(ab\) has a 5-outpath \(abcuw\). Let \(v \in V_3^- - \{c\}\). Then \(y \to v\) and \(ab\) has a 6-outpath \(abcyuv\).

The proof of Theorem 9 is complete.

Theorems 6–9 give support to the following conjecture.

**Conjecture 10.** Let \(D\) be an \(r\)-regular 3-partite tournament with \(r \geq 2\) and partite sets \(V_1, V_2, V_3\). If \(ab\) is an arc of \(D\), then the following hold for all \(k \in \{1, 2, \ldots, r - 1\}\).

1. \(ab\) has a \((3k - 1)\)-outpath.
2. \(ab\) has a 3k-outpath.
3. \(ab\) has a \((3k + 1)\)-outpath unless \(V_1 \to V_2 \to V_3 \to V_1\).

Note that the length of the longest path in an \(r\)-regular 3-partite tournament is at most \(3r - 1\). So the value of \(k\) cannot exceed \(r - 1\) in (2) and (3) of Conjecture 10. However, the following example show that (1) of Conjecture 10 is not always true when \(k = r\).
Example 11. Let $V_1 = \{a, y\}, V_2 = \{b, u\}$ and $V_3 = \{x, v\}$ be the partite sets of a 3-partite tournament $D$ such that \{u, v\} → $a$ → \{b, x\}, \{a, x\} → $b$ → \{y, v\}, \{y, b\} → $v$ → \{a, u\} $V_2$ → $y$ → $V_3$, $V_3$ → $u$ → $V_1$ and $V_1$ → $x$ → $V_2$. Then $D$ is 2-regular, but the arc $ab$ has no 5-outpath since there is only one path $abvuyx$ of length 5 starting from $ab$, which is not an outpath of $ab$.

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