THE TURÁN NUMBER OF SPANNING STAR FORESTS

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Abstract

Let \( F \) be a family of graphs. The Turán number of \( F \), denoted by \( ex(n, F) \), is the maximum number of edges in a graph with \( n \) vertices which does not contain any subgraph isomorphic to some graph in \( F \). A star forest is a forest whose connected components are all stars and isolated vertices. Motivated by the results of Wang, Yang and Ning about the spanning Turán number of linear forests [J. Wang and W. Yang, The Turán number for spanning linear forests, Discrete Appl. Math. 254 (2019) 291–294; B. Ning and J. Wang, The formula for Turán number of spanning linear forests, Discrete Math. 343 (2020) 111924]. In this paper, let \( S_{n,k} \) be the set of all star forests with \( n \) vertices and \( k \) edges. We prove that when \( 1 \leq k \leq n-1 \),

\[ ex(n, S_{n,k}) = \left\lfloor \frac{k^2-1}{2} \right\rfloor. \]

Keywords: spanning Turán problem, star forests, Loebl-Komlós-Sós type problems.

2010 Mathematics Subject Classification: 05C05, 05C35, 05C90.

\textsuperscript{1}Supported by the National Natural Science Foundation of China (No. 11871308), the Fundamental Research Funds for the Central Universities (No. 3102019qjhd003) and the Seed Foundation of Innovation and Creation for Graduate Students in Northwestern Polytechnical University (No. CX2020190).

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1. Introduction

All graphs in this paper are finite, undirected and simple. Notations in this paper are standard as [1]. For a graph $G$, let $V(G)$ be the set of vertices, $E(G)$ be the set of edges and $e(G) = |E(G)|$ be the number of edges. For $v \in V(G)$, we define $N(v)$ to be the vertex set whose each vertex is adjacent to the vertex $v$. Furthermore, for a vertex set $S \subseteq V(G)$, we use $N(S)$ to denote the vertex set whose each vertex is in $V(G) \setminus S$ and adjacent to at least one vertex in $S$. Let $N[v] = N(v) \cup \{v\}$. We denote the degree of a vertex $v$ by $d(v)$ and the maximum degree of $G$ by $\Delta(G)$. For a vertex set $U \subseteq V(G)$, the subgraph of $G$ induced by $U$ is denoted by $G[U]$. For a subgraph $H$ of a graph $G$, the graph $G - H$ is the subgraph induced by the vertex set $V(G) \setminus V(H)$, i.e., $G[V(G) \setminus V(H)]$. A star forest is a forest whose connected components are all stars and isolated vertices. We denote the degree of a vertex $v$ by $d(v)$ and the maximum degree of $G$ by $\Delta(G)$. For a vertex set $S \subseteq V(G)$, the subgraph of $G$ induced by $S$ is denoted by $G[S]$. For a subgraph $H$ of a graph $G$, the subgraph induced by the vertex set $V(G) \setminus V(H)$, i.e., $G[V(G) \setminus V(H)]$. A star forest is a forest whose connected components are all stars and isolated vertices. We denote the degree of a vertex $v$ by $\deg_G(v)$ and the maximum degree of $G$ by $\Delta(G)$.

Let $F$ be a family of graphs. A graph $G$ is called $F$-free if $G$ does not contain any subgraph which is isomorphic with some $F \in F$. The Turán number, denoted by $ex(n, F)$, is the maximum number of edges in an $F$-free graph on $n$ vertices. When $F$ contains only one graph $F$, we denote the Turán number of $F$ by $ex(n, F)$. Let $EX(n, F)$ denote an $F$-free graph on $n$ vertices with $ex(n, F)$ edges. We call this graph an extremal graph for $F$. A traditional starting point of extremal graph theory (a significant branch of graph theory) is the Mantel’s theorem (see e.g. [1]) that the maximum number of edges in a $K_3$-free graph on $n$ vertices is $\left\lfloor \frac{n^2}{4} \right\rfloor$. Turán [15, 16] generalized this result to determine the value $ex(n, K_{r+1})$ and showed that the unique extremal graph for $K_{r+1}$ is the complete $r$-partite graph on $n$ vertices whose all parts are as equal in size as possible (the difference between any two parts is at most 1), denoted by $T_r(n)$. Generally, we call the graph $T_r(n)$ a Turán graph. In 1959, Erdős and Gallai [3] determined the value $ex(n, (k + 1) \cdot K_2)$ and the extremal graph for it. In [14], Simonovits showed that the unique extremal graph for $k \cdot K_{r+1}$ is $K_{k-r+1} \vee T_r(n-p+1)$ when $n$ is sufficiently large. Later, by considering the graph that consists of $p$ disjoint copies of any connected graph $G$ on $n$ vertices, Gorgol [5] gave a lower bound for $ex(m, p \cdot G)$. 
Theorem 1 [3]. When \( n \geq 2k + 1 \),
\[
ex(n, (k + 1) \cdot K_2) = \max \left\{ \binom{2k + 1}{2}, \binom{n}{2} - \binom{n - k}{2} \right\}.
\]

Theorem 2 [5]. Let \( G \) be any connected graph on \( n \) vertices, \( p \) be any positive integer and \( m \) be an integer such that \( m \geq pn \). Then \( ex(m, p \cdot G) \geq \max \left\{ \ex(m - pn + 1, G) + \binom{pn - 1}{2}, \ex(m - p + 1, G) + (p - 1)m - \binom{p}{2} \right\} \).

In [12], Lidický et al. investigated the Turán number of a star forest and determined the value \( \ex(n, F) \) when \( n \) is sufficiently large, where \( F = \bigcup_{i=1}^{k} S_{d_i} \) and \( d_1 \geq d_2 \geq \cdots \geq d_k \). Yin and Rao [18] determined the value \( \ex(n, k \cdot S_l) \) when \( n \geq \frac{1}{2}l^2k(k - 1) + k - 2 + \max\{lk, l^2 + 2l\} \), which improved the results of Lidický et al. Later, Lan et al. [10] determined the Turán number \( \ex(n, 2 \cdot S_l) \) for all positive integers \( n \) and \( l(\geq 4) \) and \( \ex(n, 3 \cdot S_l) \) for all positive integers \( n \) and \( l(\geq 3) \).

Recently, Ning and Wang [13] considered the forbidden family \( \mathcal{L}_{n,k} \) of subgraphs (i.e., a family of all linear forests of order \( n \) with \( k \) edges) and determined the exact value \( \ex(n, \mathcal{L}_{n,k}) \). The Hamiltonian completion number of a graph \( G \) is the minimum number of edges to ensure that the graph \( G \) is Hamiltonian by adding it. Their results determined also the maximum number of edges in a non-Hamiltonian graph with fixed Hamiltonian completion number. Notice that the order of the forbidden linear forest and the order \( n \) are dependent. Motivated by their results, we determine the Turán number \( \ex(n, \mathcal{S}_{n,k}) \) by considering a family of all star forests of order \( n \) with \( k \) edges, denoted by \( \mathcal{S}_{n,k} \).

Theorem 3 [13]. Let \( n, k \) be two positive integers and \( 1 \leq k \leq n - 1 \). Then
\[
ex(n, \mathcal{L}_{n,k}) = \max \left\{ \binom{k}{2}, \binom{n}{2} - \binom{n - \left\lfloor \frac{k-1}{2} \right\rfloor}{2} + c \right\},
\]
where \( c = 0 \) if \( k \) is odd, and \( c = 1 \) otherwise.

For convenience, we would like to give the definition of the almost \( d \)-regular graph. We define the almost \( d \)-regular graph as a graph that contains one vertex of degree \( d - 1 \) and all the other vertices have degree \( d \).

Theorem 4. Let \( n, k \) be two positive integers and \( 1 \leq k \leq n - 1 \). Then
\[
ex(n, \mathcal{S}_{n,k}) = \left\lfloor \frac{k^2 - 1}{2} \right\rfloor.
\]
Moreover, the unique extremal graph contains a connected component of size \( k + 1 \) and \( n - k - 1 \) isolated vertices. And the connected component is the almost \( (k - 1) \)-regular graph or the \((k - 1)\)-regular graph on \( k + 1 \) vertices.
Considering the median degree of a graph $G$, Loebl, Komlós and Sós [2] conjectured that every graph $G$ of order $n$ with at least $n/2$ vertices of degree at least $k$ contains each tree $T$ of order $k + 1$ as a subgraph. This is a median degree version of the famous Erdős-Sós conjecture. For more results about Loebl-Komlós-Sós conjecture, we refer the reader to [6, 7, 8, 9].

**Conjecture 5** (The Erdős-Sós Conjecture). Every graph $G$ order $n$ with average degree greater than $k - 1$ contains each tree $T$ of order $k + 1$ as a subgraph.

**Conjecture 6** (The Loebl-Komlós-Sós Conjecture, [2]). Every graph $G$ order $n$ with median degree greater than $k - 1$ contains each tree $T$ of order $k + 1$ as a subgraph.

Füredi and Simonovits called it Loebl-Komlós-Sós type problem or the Median problem [4] as follows: for a given graph $G$ of order $n$, which $m$ and $d$ ensure that if $G$ has at least $m$ vertices of degree $≥ d$, then $G$ contains some subgraph $H$. One could analogously define such problems for families $H$ of graphs. We solved a problem of this type for $H = S_{n,k}$.

**Theorem 7.** If a simple graph $G$ on $n$ ($n ≥ k + 2$) vertices has at least $\left\lceil \frac{3k}{2} + 2 \right\rceil$ vertices of degree at least $\left\lceil \frac{k}{2} \right\rceil$, then $G$ contains some graph in $S_{n,k}$ as its subgraph.

2. The Proof of Theorem 4

**Proof.** We prove this result mainly by induction on $k$. First, if $k = 1$, then we know that $S_{n,1}$ contains only one edge, i.e., $S_{n,1} = K_2 ∪ E_{n-2}$. Notice that if a graph $G$ contains at least one edge, it must contain $S_{n,1}$ as its subgraph. Then we have that $ex(n, S_{n,1}) = 0 = \left\lceil \frac{1^2 - 1}{2} \right\rceil$ and the unique extremal graph is $E_n$. If $k = 2$, we can see that $S_{n,2} = \{S_2 ∪ E_{n-3}, 2 \cdot K_2 ∪ E_{n-4}\}$. Notice that the graph which does not contain any graph in $S_{n,2}$ as its subgraph must have maximum degree at most 1. If there are two edges in distinct connected components of $G$, then we can find a copy of $2 \cdot K_2$ in $G$, a contradiction. Thus, we have that $ex(n, S_{n,2}) ≤ 1$. And, the graph $K_2 ∪ E_{n-2}$ does not contain any graph in $S_{n,2}$ as its subgraph and $e(K_2 ∪ E_{n-2}) = 1$. Then $ex(n, S_{n,2}) = 1 = \left\lceil \frac{2^2 - 1}{2} \right\rceil$ and the unique extremal graph is $K_2 ∪ E_{n-2}$. We can see that $S_{n,3} = \{S_3 ∪ E_{n-4}, S_2 ∪ K_2 ∪ E_{n-5}, 3 \cdot K_2 ∪ E_{n-6}\}$ when $k = 3$. It is easy to know that the graph which does not contain any graph in $S_{n,3}$ as its subgraph must have maximum degree at most 2. Also, we can see that there are at most two connected components in a extremal graph for $S_{n,3}$ which is denoted by $G$. If there are exactly two connected components in $G$, then both of two connected components contain only an edge, thus $e(G) = 2$. If there is only one connected component in $G$, then we have that $Δ(G) ≤ 2$. If there are exactly two connected components in $G$, then both of two connected components contain only an edge, thus $e(G) = 2$. If there is only one connected component in $G$, then we have that $Δ(G) ≤ 2$. If there are exactly two connected components in $G$, then both of two connected components contain only an edge, thus $e(G) = 2$. If there is only one connected component in $G$, then we have that $Δ(G) ≤ 2$. If there are exactly two connected components in $G$, then both of two connected components contain only an edge, thus $e(G) = 2$. If there is only one connected component in $G$, then we have that $Δ(G) ≤ 2$. If there are exactly two connected components in $G$, then both of two connected components contain only an edge, thus $e(G) = 2$. If there is only one connected component in $G$, then we have that $Δ(G) ≤ 2$. If there are exactly two connected components in $G$, then both of two connected components contain only an edge, thus $e(G) = 2$. If there is only one connected component in $G$, then we have that $Δ(G) ≤ 2$. If there are exactly two connected components in $G$, then both of two connected components contain only an edge, thus $e(G) = 2$. If there is only one connected component in $G$, then we have that $Δ(G) ≤ 2$. If there are exactly two connected components in $G$, then both of two connected components contain only an edge, thus $e(G) = 2$. If there is only one connected component in $G$, then we have that $Δ(G) ≤ 2$. If there are exactly two connected components in $G$, then both of two connected components contain only an edge, thus $e(G) = 2$. If there is only one connected component in $G
If $\Delta(G) = 1$, then $e(G) = 1$. If $\Delta(G) = 2$ and $v$ is a vertex in $G$ with degree 2, then let $N(v) = \{v_1, v_2\}$. If $v_1v_2$ is an edge of $G$, then $e(G) = 3$. If $v_1v_2$ is not an edge of $G$, then $v_1$ and $v_2$ can only be adjacent to the same vertex in $V(G) \setminus \{v, v_1, v_2\}$ and this fourth vertex is the last in the connected component. Otherwise, we can find a copy of $S_2 \cup K_2$ in $G$, a contradiction. Thus, we have that $ex(n, S_{n,3}) = 4 = \left\lceil \frac{3n-1}{2} \right\rceil$ and $K_{2,2} \cup E_{n-4}$ is the unique extremal graph.

Now, we assume that the result holds for all $k' < k$. In the following, we will show that the result holds for $k$. From the above analyses, we can see that the extremal graph has maximum degree $k - 1$ when $k \in \{1, 2, 3\}$. We claim that this conclusion holds for $k$. In order to prove that the claim is true, we first would like to construct an extremal graph for $S_{n,k}$, denoted by $G$, and then to prove any graph $G'$ with $\Delta(G') = k - 1$ which contains no $S_{n,k}$ as its subgraph can contain at most $e(G)$ edges. Then we would like to show that all other graphs with maximum degree less than $k - 1$ have less edges than the constructed graph $G$.

The constructed extremal graph is as follows. $G$ contains only one connected component containing an edge which is the almost $(k - 1)$-regular graph or the $(k - 1)$-regular graph on $k + 1$ vertices. We denote the connected component of $G$ by $G_1$. Notice that if $G$ contains any graph in $S_{n,k}$, then all edges of it must be contained in $G_1$. There is only one graph in $S_{n,k}$ which contains only one connected component, that is $S_k \cup E_{n-k-1}$ and the only connected component is $S_k$. Since any graph in $S_{n,k}$ other than $S_k \cup E_{n-k-1}$ has at least two connected components which contain at least $k + 2$ vertices, $G_1$ cannot contain any of them as a subgraph. If $G$ is not $S_{n,k}$-free, then $G_1$ must contain $S_k$ as its subgraph. It is easy to deduce that $G_1$ does not contain $S_k$ as its subgraph since $G_1$ is the almost $(k - 1)$-regular graph or the $(k - 1)$-regular graph on $k + 1$ vertices. Thus $G$ is $S_{n,k}$-free.

Next, we will prove that any graph $G'$ with $\Delta(G') = k - 1$ which does not contain any graph in $S_{n,k}$ as its subgraph can contain at most $e(G)$ edges. Without loss of generality, we assume that $d(v) = k - 1$ for $v \in V(G')$ and let $N(v) = \{v_1, v_2, \ldots, v_{k-1}\}$. Notice that there are no edges in $G'[V(G') \setminus N[v]]$, otherwise we can find a $S_{k-1} \cup K_2 \cup E_{n-k-2} \in S_{n,k}$ in $G'$, a contradiction. Thus, all edges of $G'$ must be contained in the connected component which contains all vertices in $N(v)$. Any $v_i$ can be adjacent to at most one vertex in $V(G') \setminus N[v]$, thus there are at most $k - 1$ edges between $N(v)$ and $V(G') \setminus N[v]$. For any $v_i \in N(v)$, if there is an edge between the vertex $v_i$ and the vertex set $V(G') \setminus N[v]$, then by the definition of $\Delta$, there exist at most $k - 3$ edges between the vertex $v_i$ and the vertex set $N(v)$. Therefore, if there are $y \leq k - 1$ edges between the two vertex sets $N(v)$ and $V(G') \setminus N[v]$, then at least $\left\lceil \frac{y}{2} \right\rceil$ edges are missing inside $N(v)$. Therefore, the total number of edges is
We can explain the first inequality of Equation (1) as follows. The first two terms are an upper bound on the number of edges inside $N[v]$, and the last term is the number of other edges by definition.

Notice that the extremal graph can contain only one vertex in $V(G') \setminus N[v]$ which is adjacent to every vertex in $N(v)$. Assume that there are two vertices which are adjacent to the vertex set $N(v)$, denoted by $v_x, v_y$. Then we have that $d_{N(v)}(v_x) + d_{N(v)}(v_y) = k - 1$. Since $G'[N(v)]$ is an almost $(k - 3)$-regular graph, we can find a copy of $S_2 \cup S_{k-2}$, a contradiction. Thus, we know that the constructed graph $G$ is the unique extremal graph $G'$ for $\Delta(G') = k - 1$.

Our third step is to prove the following conclusion. All other graphs with maximum degree less than $k - 1$ contain less edges than $G$.

First, we claim that the extremal graph for $S_{n,k}$ contains only one connected component. Assume that there are two connected components in the extremal graph for $S_{n,k}$ and there is one connected component such that the largest star forest in it has $x$ edges. By the induction hypothesis, we have that the extremal graph contains at most $\left\lfloor \frac{(x+1)^2 - 1}{2} \right\rfloor + \left\lfloor \frac{(k-x)^2 - 1}{2} \right\rfloor \leq \left\lfloor \frac{(k)^2 - 1}{2} \right\rfloor$ edges. Thus we have that the extremal graph for $S_{n,k}$ contains only one connected component. Let $H$ be an extremal graph for $S_{n,k}$ that contains only one connected component $H'$ containing an edge which has maximum degree $3 \leq \Delta(H') = t < k - 1$. Let $d(u) = t$ and $N(u) = \{u_1, u_2, \ldots, u_t\}$. In the following, we divide the edge set of the graph $H$ into two parts: the edges of the graph $H[V(H') \setminus N[u]]$ and other edges. We denote the graph which is induced by the second set of edges by $H_1$.

Case 1. $H_1$ contains no star forest with more than $t$ edges. Any $u_i$ can be adjacent to at most one vertex in $V(H') \setminus N[u]$, otherwise we can find a copy of $S_2 \cup S_{t-1}$, a contradiction. Each vertex $u_i$ can be adjacent to at most $t - 1$ vertices in $N(u) - u_i$ for $i \in [1,t]$. For any $u_i \in N(u)$, if there is an edge between the vertex $u_i$ and the vertex set $V(H_1) \setminus N[u]$, then by the definition of $\Delta$, there exist at most $t - 2$ edges between the vertex $u_i$ and the vertex set $N(u)$. Therefore, if there are $z \leq t$ edges between the two vertex sets $N(u)$ and $V(H_1) \setminus N[u]$, then at least $\left\lceil \frac{z}{2} \right\rceil$ edges are missing inside $N(u)$. Therefore, the total number of edges is

\begin{equation}
(2)
e(H_1) \leq \left( \frac{t + 1}{2} \right) - \left\lceil \frac{z}{2} \right\rceil + z \leq \left\lfloor \frac{t(t + 1)}{2} \right\rfloor.
\end{equation}

We can explain the first inequality of Equation (2) as follows. The first two terms are an upper bound on the number of edges inside $N[u]$, and the last term is the number of other edges by definition.
Then by the induction hypothesis and Equation (2),

\[ e(H) = e(H') \leq \left\lfloor \frac{(k - t)^2 - 1}{2} \right\rfloor + \left\lfloor \frac{t(t + 2)}{2} \right\rfloor \]

\[ = \left\lfloor k^2 - 1 + 2t^2 + 2t - 2kt \right\rfloor \leq \left\lfloor k^2 - 1 \right\rfloor = e(G). \tag{3} \]

**Case 2.** \(H_1\) contains a star forest with \(y + t - 1\) edges. But \(H_1\) contains no star forest with more than \(y + t - 1\) edges. Similarly, any \(u_i\) can be adjacent to at most \(y\) vertices in \(V(H') \setminus N[u]\), otherwise we can find a copy of \(S_{y+1} \cup S_{t-1}\), a contradiction. Therefore, there are at most \(ty\) edges between the set \(N(u)\) and the set \(V(H') \setminus N[u]\). We assume that \(u_i\) is adjacent to \(y_i\) vertices in \(V(H') \setminus N[u]\) for \(i \in [1, t]\). Then

\[ e(H_1) \leq \sum_{i=1}^{t} \frac{(t - 1 - y_i)}{2} + \sum_{i=1}^{t} y_i + t \]

\[ \leq \left\lfloor \frac{t(t - 1 - y)}{2} \right\rfloor + t(y + 1). \tag{4} \]

Then by the induction hypothesis and Equation (4),

\[ e(H) \leq \left\lfloor \frac{t(t - 1 - y)}{2} \right\rfloor + t(y + 1) + \left\lfloor \frac{(k - y - t + 1)^2 - 1}{2} \right\rfloor \]

\[ = \left\lfloor \frac{y^2 + (4t - 2k - 2)y}{2} \right\rfloor + \left\lfloor \frac{k^2 + 2t^2 - t + 1 - 2kt + 2k}{2} \right\rfloor. \tag{5} \]

Further, we analyze two subcases as follows.

**Subcase 2.1.** \(t \leq \left\lfloor \frac{k-1}{2} \right\rfloor + 1\). Considering the term \(y\) and \(2 \leq y \leq t - 1\), we can see that the right part of the above inequality about \(y\) is a parabola with an upward opening, which obtains a value of 0 at points \(y = 0\) and \(y = 2k + 2 - 4t\). If \(t - 1 \geq 2k + 2 - 4t\) (i.e., \(t \geq (2k + 3)/5\)), then \(e(H)\) can attain its maximum value at \(y = t - 1\). Otherwise, \(e(H)\) can attain its maximum value at \(y = 2\). When \(t \geq (2k + 3)/5\), we can calculate the maximum number of edges as \(\left\lfloor \frac{k^2 + 3t^2 - 9t + 4 - 4kt + 4k}{2} \right\rfloor\). And the maximum number of edges is \(\left\lfloor \frac{k^2 + 2t^2 + 7t + 1 - 2kt - 2k}{2} \right\rfloor\) when \(t < (2k + 3)/5\). The two values are both less than \(e(G) = \left\lfloor \frac{k^2 - 1}{2} \right\rfloor\). Thus, we prove the above conclusion when \(t \leq \left\lfloor \frac{k-1}{2} \right\rfloor + 1\).

**Subcase 2.2.** \(\left\lfloor \frac{k-1}{2} \right\rfloor + 1 < t \leq k - 1\). It is easy to know that \(2 \leq y \leq k - t\). By a simple calculation for Equation (5), we know that the maximum value of the right part of Equation (5) can be obtained when \(y = k - t\). Substituting
\[ y = k - t \] into Equation (5), we obtain that the maximum value is \( \left\lfloor \frac{kt + t}{2} \right\rfloor \). We can obtain that it is smaller than \( e(G) \) by a simple calculation. Thus we prove that the conclusion holds for all \( k \).

Thus, our claim is true that the extremal graph has maximum degree \( k - 1 \). This completes our proof.

3. The Proof of Theorem 7

**Proof.** To prove this conclusion, we just need to prove that there are at most \( \frac{3k}{2} + 1 \) vertices of degree at least \( \left\lceil \frac{k}{2} \right\rceil \) if \( G \) contains no graph in \( S_{n,k} \) as its subgraph. We define \( d \) as the smallest number that is at least \( \left\lceil \frac{k}{2} \right\rceil \) and there is a vertex of degree exactly \( d \). Without loss of generality, let \( v \in V(G) \), \( d(v) = d \) and \( N(v) = \{v_1, v_2, \ldots, v_d\} \). Notice that every vertex not in \( N[v] \) nor in \( N(N(v)) \) has degree at most \( k - d - 1 \), otherwise we can find a copy of \( S_d \cup S_{k-d} \cup E_{n-k-2} \in S_{n,k} \), a contradiction. Thus we know that every vertex which has degree at least \( \left\lceil \frac{k}{2} \right\rceil \) can be contained only in \( N[v] \) or \( N(N(v)) \). Every vertex in \( N(v) \) has at most \( k - d \) neighbors in \( V(G) \setminus N[v] \). Thus we know that there are at most \( d(k-d) \) edges between the vertex set \( N(v) \) and \( N(N(v)) \).

**Claim 8.** There are at most \( \frac{3k}{2} - d \) vertices in \( N(N(v)) \) which have degree at least \( d \).

In the following proof, we call a vertex which has degree at least \( \left\lceil \frac{k}{2} \right\rceil \) as the large degree vertex.

**Proof.** Suppose that there are at least \( \frac{3k}{2} - d + 1 \) large degree vertices in \( N(N(v)) \). By Theorem 4, there are at most \( \left\lceil \frac{(k-d)^2-1}{2} \right\rceil \) edges in \( G[V(G) \setminus N[v]] \). It follows that the large degree vertices in \( N(N(v)) \) are incident to at least \( \left( \frac{3k}{2} - d + 1 \right) \) edges, but there are at most \( (k-d)^2 - 1 \) incidences inside \( G[V(G) \setminus N[v]] \), thus at least \( \left( \frac{3k}{2} - d + 1 \right) d - (k-d)^2 - 1 = \frac{7kd}{2} - 2d^2 - k^2 + d + 1 \) incidences are by edges between \( N(v) \) and \( N(N(v)) \). Thus, the number of edges between \( N(v) \) and \( N(N(v)) \) must be no less than \( \frac{7kd}{2} - 2d^2 - k^2 + d + 1 \), which is greater than \( d(k-d) \). This contradicts with the fact that there are at most \( d(k-d) \) edges between the vertex set \( N(v) \) and \( N(N(v)) \). Thus, we have that there are at most \( \frac{3k}{2} - d \) vertices in \( G \) which have degree at least \( d \).

Combining Claim 8 and the fact that there are at most \( d + 1 \) vertices in \( N[v] \) which have degree at least \( d \), we conclude that there are at most \( \frac{3k}{2} + 1 \) vertices of degree at least \( \left\lceil \frac{k}{2} \right\rceil \) if \( G \) contains no graph in \( S_{n,k} \) as its subgraph. This completes our proof.
Acknowledgement

The authors would like to thank an anonymous referee for providing valuable comments and suggestions which improved the presentation of this paper.

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doi:10.37236/3142


doi:10.4064/cm-3-1-19-30

doi:10.1016/j.dam.2018.07.014


Received 1 June 2020
Revised 26 September 2020
Accepted 26 September 2020