SOME RESULTS ON PATH-FACTOR CRITICAL AVOIDABLE GRAPHS

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Abstract

A path factor is a spanning subgraph $F$ of $G$ such that every component of $F$ is a path with at least two vertices. We write $P_{\geq k} = \{ P_i : i \geq k \}$. Then a $P_{\geq k}$-factor of $G$ means a path factor in which every component admits at least $k$ vertices, where $k \geq 2$ is an integer. A graph $G$ is called a $P_{\geq k}$-factor avoidable graph if for any $e \in E(G)$, $G$ admits a $P_{\geq k}$-factor excluding $e$. A graph $G$ is called a $(P_{\geq k}, n)$-factor critical avoidable graph if for any $Q \subseteq V(G)$ with $|Q| = n$, $G - Q$ is a $P_{\geq k}$-factor avoidable graph. Let $G$ be an $(n+2)$-connected graph. In this paper, we demonstrate that (i) $G$ is a $(P_{\geq 2}, n)$-factor critical avoidable graph if $\text{tough}(G) > \frac{n+2}{3}$; (ii) $G$ is a $(P_{\geq 3}, n)$-factor critical avoidable graph if $\text{tough}(G) > \frac{n+1}{2}$; (iii) $G$ is a $(P_{\geq 2}, n)$-factor critical avoidable graph if $\text{I}(G) > \frac{n+2}{3}$; (iv) $G$ is a $(P_{\geq 3}, n)$-factor critical avoidable graph if $\text{I}(G) > \frac{n+3}{2}$. Furthermore, we claim that these conditions are sharp.

Keywords: graph, toughness, isolated toughness, $P_{\geq k}$-factor, $(P_{\geq k}, n)$-factor critical avoidable graph.

2010 Mathematics Subject Classification: 05C70, 05C38, 90B10.

1. Introduction

In this paper, we discuss only finite undirected simple graphs. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ denotes the vertex set of $G$ and $E(G)$ denotes the edge set of $G$. For $x \in V(G)$, the degree of $x$ in $G$ is denoted by $d_G(x)$. For a set $X \subseteq V(G)$, we use $G[X]$ to denote the subgraph of $G$ induced by $X$ and write $G - X$ for $G[V(G) \setminus X]$. We let $i(G)$ and $\omega(G)$ denote the number of isolated vertices and the number of connected components of $G$, respectively. Let $P_n$ and
$K_n$ denote the path and the complete graph of order $n$, respectively. The join $G + H$ denotes the graph with vertex set $V(G) \cup V(H)$ and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{xy : x \in V(G) \text{ and } y \in V(H)\}.$$ 

Chvátal [2] first introduced the toughness of a graph $G$, denoted by $\text{tough}(G)$, namely,

$$\text{tough}(G) = \min \left\{ \frac{|X|}{\omega(G - X)} : X \subseteq V(G), \omega(G - X) \geq 2 \right\},$$

if $G$ is not complete; otherwise, $\text{tough}(G) = +\infty$.

Yang, Ma and Liu [13] first posed isolated toughness of a graph $G$, denoted by $I(G)$, namely,

$$I(G) = \min \left\{ \frac{|X|}{i(G - X)} : X \subseteq V(G), i(G - X) \geq 2 \right\}$$

if $G$ is not a complete graph; otherwise, $I(G) = +\infty$.

A path factor is a spanning subgraph $F$ of $G$ such that every component of $F$ is a path with at least two vertices. We write $P_{\geq k} = \{P_i : i \geq k\}$. Then a $P_{\geq k}$-factor of $G$ means a path factor in which every component admits at least $k$ vertices, where $k \geq 2$ is an integer. A $\{P_k\}$-factor $F$ of $G$ is simply called a $P_k$-factor if every component of $F$ is isomorphic to $P_k$.

A 1-factor of $G$ is a spanning subgraph $F$ of $G$ such that $d_F(x) = 1$ holds for any $x \in V(G)$. A graph $R$ is a factor-critical graph if for any $x \in V(R)$, $R - \{x\}$ admits a 1-factor. Let $R$ be a factor-critical graph with $V(R) = \{x_1, x_2, \ldots, x_n\}$. $n$ new vertices $y_1, y_2, \ldots, y_n$ together with new edges $x_1y_1, x_2y_2, \ldots, x_ny_n$ are added to $R$. Then the resulting graph is said to be a sun. By Kaneko [7], $K_1$ and $K_2$ are also suns. A big sun is a sun of order at least 6. We use $\text{sun}(G)$ to denote the number of sun components of $G$.


**Theorem 1** [11]. A graph $G$ admits a $P_{\geq 2}$-factor if and only if $i(G - X) \leq 2|X|$ for every $X \subseteq V(G)$.


**Theorem 2** [7]. A graph $G$ admits a $P_{\geq 3}$-factor if and only if $\text{sun}(G - X) \leq 2|X|$ for every $X \subseteq V(G)$.

A graph $G$ is called a $P_{\geq k}$-factor avoidable graph if for any $e \in E(G)$, $G$ admits a $P_{\geq k}$-factor excluding $e$. A graph $G$ is called a $(P_{\geq k}, n)$-factor critical avoidable graph if for any $Q \subseteq V(G)$ with $|Q| = n$, $G - Q$ is a $P_{\geq k}$-factor.
Some Results on Path-Factor Critical Avoidable Graphs

avoidable graph. Obviously, a \((P_{\geq k}, 0)\)-factor critical avoidable graph is simply called a \(P_{\geq k}\)-factor avoidable graph.

Kelmas [10] claimed a result on the existence of path factors in subgraphs.

**Theorem 3** [10]. Let \(G\) be a 3-connected claw-free graph and \(|V(G)| \equiv 1 \pmod{3}\). Then for any \(x \in V(G)\) and any \(e \in E(G)\), \(G - \{x, e\}\) has a \(P_3\)-factor, namely, \(G - \{x\}\) is a \(P_3\)-factor avoidable graph.

Motivated by Theorem 3, we consider a more general problem.

**Problem 1.** Find sufficient conditions for a graph to be a \((P_{\geq k}, n)\)-factor critical avoidable graph.

Kano, Lu and Yu [8] verified that a graph \(G\) has a \(P_3\)-factor if \(i(G - S) \leq \frac{2}{3}|S|\) for every \(S \subset V(G)\). Zhou, Yang and Xu [22] proved that an \(n\)-connected graph \(G\) is \((P_{\geq 3}, n)\)-factor critical if its toughness \(tough(G) \geq \frac{n + 1}{2}\). Some other results on path factors can be found in [3, 15, 17, 18]. Lots of authors derived some toughness conditions for the existence of graph factors [4, 5, 9, 20]. Some results on the relationships between isolated toughness and graph factors are obtained by Gao, Liang and Chen [6]. For many other results on graph factors, see [1, 12, 14, 16, 19, 21, 23]. In this paper, we study \((P_{\geq k}, n)\)-factor critical avoidable graphs and get some sufficient conditions for graphs to be \((P_{\geq k}, n)\)-factor critical avoidable graphs depending on toughness and isolated toughness, which are given in Sections 2 and 3.

2. **Toughness and \((P_{\geq k}, n)\)-Factor Critical Avoidable Graphs**

In this section, we explore the relationship between toughness and \((P_{\geq k}, n)\)-factor critical avoidable graphs, and derive two toughness conditions for the existence of \((P_{\geq k}, n)\)-factor critical avoidable graphs for \(k = 2, 3\).

**Theorem 4.** Let \(G\) be an \((n + 2)\)-connected graph, where \(n \geq 0\) is an integer. If its toughness \(tough(G) > \frac{2n^2}{4}\), then \(G\) is a \((P_{\geq 2}, n)\)-factor critical avoidable graph.

**Proof.** Theorem 4 obviously holds for a complete graph. Next, we assume that \(G\) is not complete. Let \(Q \subset V(G)\) with \(|Q| = n\), and \(G' = G - Q\), and let \(e \in E(G')\) and \(H = G' - e\). Since \(G\) is \((n + 2)\)-connected, \(H\) is connected. To prove Theorem 4, it suffices to show that \(H\) admits a \(P_{\geq 2}\)-factor. On the contrary, suppose that \(H\) has no \(P_{\geq 2}\)-factor. Then by Theorem 1, there exists a set \(X \subset V(H)\) such that

\[
i(H - X) \geq 2|X| + 1.
\]
Since $H$ is connected, we have $X \neq \emptyset$. Thus,

$$i(H - X) \geq 2|X| + 1 \geq 3. \tag{2}$$

Note that $\omega(G - (Q \cup X)) \geq \omega(G - (Q \cup X) - e) - 1$. Combining this with (2), we derive

$$\omega(G - (Q \cup X)) \geq (2) \implies \omega(G - (Q \cup X) - e) \geq 2|X| + 1 \geq 3. \tag{3}$$

Claim 1. $|X| \geq 2$.

**Proof.** Assume $|X| = 1$. Since $H = G' - e$, we easily know that $i(H - X) = i(G' - e - X) \leq i(G' - X) + 2$. Then by (2), we derive $i(G' - X) \geq i(H - X) - 2 \geq 1$, which implies that there exists an isolated vertex $u$ in $G' - X$, i.e., $d_{G' - X}(u) = 0$. Thus, we have $d_G(u) \leq d_G(u) + |Q| = d_G(u) + n \leq d_{G' - X}(u) + |X| + n = 0 + 1 + n = n + 1$, contradicting that $G$ is $(n + 2)$-connected. Therefore, $|X| \geq 2$. \qed

According to (1), (2), (3), Claim 1 and the definition of $\text{tough}(G)$, we have

$$\text{tough}(G) \leq \frac{|Q \cup X|}{\omega(G - (Q \cup X))} \leq \frac{|Q| + |X|}{i(H - X) - 1} = \frac{n + |X|}{i(H - X) - 1} \leq \frac{n + |X|}{2|X|} = \frac{1}{2} + \frac{n}{2|X|} \leq \frac{1}{2} + \frac{n}{4} = \frac{n + 2}{4},$$

which contradicts $\text{tough}(G) > \frac{n + 2}{4}$. Theorem 4 is verified. \qed

**Remark 5.** Now, we claim that the result in Theorem 4 is sharp. To see this, we construct the graph $G = K_{n+2} + (3K_1 \cup K_2)$. Clearly, $G$ is $(n + 2)$-connected and $\text{tough}(G) = \frac{n + 2}{4}$. Let $Q \subset V(K_{n+2}) \subseteq V(G)$ with $|Q| = n$ and $e$ be the edge of $K_2$. Then $G - Q - e$ is a graph isomorphic to $K_2 + 5K_1$, and it obviously has no $P_{2-2}$-factor. Thus, $G$ is not a $(P_{2-2}, n)$-factor critical avoidable graph.

**Theorem 6.** Let $G$ be an $(n + 2)$-connected graph, where $n \geq 0$ is an integer. If its toughness $\text{tough}(G) > \frac{n + 1}{2}$, then $G$ is a $(P_{2-3}, n)$-factor critical avoidable graph.

**Proof.** Theorem 6 obviously holds for a complete graph. In the following, we assume that $G$ is not complete. Let $Q \subset V(G)$ with $|Q| = n$, and $G' = G - Q$, and let $e \in E(G')$ and $H = G' - e$. Since $G$ is $(n + 2)$-connected, $H$ is connected. To prove Theorem 6, it suffices to show that $H$ admits a $P_{2-3}$-factor. On the contrary, suppose that $H$ has no $P_{2-3}$-factor. Then by Theorem 2, there exists a set $X \subset V(H)$ such that

$$\text{sun}(H - X) \geq 2|X| + 1. \tag{4}$$
Claim 1. \( X \neq \emptyset \).

**Proof.** Assume that \( X = \emptyset \). Then it follows from (4) that
\[
\text{sun}(H) \geq 1.
\]
Since \( H \) is connected, we have \( \text{sun}(H) = 1 \) and \( H \) itself is a sun.

Since \( G \) is \((n+2)\)-connected, \( |V(G)| \geq n + 3 \). Thus, \( |V(H)| = |V(G)| - n \geq 3 \), which implies that \( H \) is a big sun. Hence, \( |V(H)| \geq 6 \). Let \( R \) be the factor-critical graph of \( H \). Then \( |V(R)| \geq 3 \) and there exists \( w \in V(R) \) such that \( \omega(G' - \{w\}) = \omega(H - \{w\}) = 2 \). Thus, we have
\[
\omega(G - Q - \{w\}) = \omega(G' - \{w\}) = 2.
\]

In terms of (6) and the definition of \( \text{tough}(G) \), we get
\[
\text{tough}(G) \leq \frac{|Q \cup \{w\}|}{\omega(G - (Q \cup \{w\}))} = \frac{n + 1}{2},
\]
contradicting to \( \text{tough}(G) > \frac{n+1}{2} \). Hence, \( X \neq \emptyset \). \( \square \)

By (4) and Claim 1, we gain \( \omega(G - (Q \cup X)) = \omega(G' - X) \geq \omega(G' - X - e) - 1 = \omega(H - X) - 1 \geq \text{sun}(H - X) - 1 \geq 2|X| \geq 2 \). Combining this with Claim 1 and the definition of \( \text{tough}(G) \), we have
\[
\text{tough}(G) \leq \frac{|Q \cup X|}{\omega(G - (Q \cup X))} \leq \frac{n + |X|}{2|X|} = \frac{1}{2} + \frac{n}{2|X|} \leq \frac{1}{2} + \frac{n}{2} = \frac{n + 1}{2},
\]
this contradicts \( \text{tough}(G) > \frac{n+1}{2} \). This finishes the proof of Theorem 6. \( \blacksquare \)

Remark 7. Now, we show that the conditions in Theorem 6 are best possible, which cannot be replaced by \( G \) being \((n+1)\)-connected and \( \text{tough}(G) \geq \frac{n+1}{2} \).

Let \( G = K_{n+1} + (2K_2) \). We easily see that \( G \) is \((n+1)\)-connected and \( \text{tough}(G) = \frac{n+1}{2} \). Let \( Q \subset V(K_{n+1}) \subseteq V(G) \) with \( |Q| = n \), and \( e \) be an edge of \( 2K_2 \). Then \( G - Q - e \) is a graph isomorphic to \( K_1 + (2K_1 \cup K_2) \), and it obviously has no \( P_{\geq 3} \)-factor, and so \( G \) is not a \((P_{\geq 3}, n)\)-factor critical avoidable graph.

3. **Isolated Toughness and \((P_{\geq k}, n)\)-Factor Critical Avoidable Graphs**

In this section we give two sufficient conditions using isolated toughness for a graph to be a \((P_{\geq k}, n)\)-factor critical avoidable graph for \( k = 2, 3 \).

**Theorem 8.** Let \( G \) be an \((n+2)\)-connected graph, where \( n \geq 0 \) is an integer. If its isolated toughness \( I(G) > \frac{n+2}{3} \), then \( G \) is a \((P_{\geq 2}, n)\)-factor critical avoidable graph.
**Proof.** Theorem 8 obviously holds for a complete graph. In what follows, we assume that \( G \) is not complete. Let \( Q \subset V(G) \) with \(|Q| = n\), and \( G' = G - Q \), and let \( e \in E(G') \) and \( H = G' - e \). Since \( G \) is \((n+2)\)-connected, \( H \) is connected.

To prove Theorem 8, it suffices to show that \( H \) admits a \( P_{\geq 2} \)-factor. On the contrary, suppose that \( H \) has no \( P_{\geq 2} \)-factor. Then by Theorem 1, there exists a set \( X \subset V(H) \) such that

\[
i(H - X) \geq 2|X| + 1.
\]

**Claim 1.** \(|X| \geq 2\).

**Proof.** If \( X = \emptyset \), then by (7) and \( H \) being connected, we obtain

\[
1 \leq i(H) = 0,
\]

which is a contradiction.

Next, we consider \(|X| = 1\). Note that \( i(H - X) = i(G' - e - X) \leq i(G' - X) + 2 \). Combining this with (7), we derive \( i(G' - X) \geq i(H - X) - 2 \geq 2|X| + 1 - 2 = 2|X| - 1 = 1 \), which hints that there exists \( w \in V(G') \setminus X \) with \( d_{G' - X}(w) = 0 \). Therefore, we admit \( d_G(w) = d_{G' + Q}(w) \leq d_{G'}(w) + |Q| = d_{G'}(w) + n \leq d_{G' - X}(w) + |X| + n = 0 + 1 + n = n + 1 \), which contradicts that \( G \) is \((n+2)\)-connected. Thus, we derive \(|X| \geq 2\).

According to (7) and Claim 1, we get

\[
i(G - (Q \cup X)) \geq i(G - (Q \cup X) - e) - 2 = i(H - X) - 2 \geq 2|X| - 1 \geq 3.
\]

It follows from (8), Claim 1 and the definition of \( I(G) \) that

\[
I(G) \leq \frac{|Q \cup X|}{i(G - (Q \cup X))} \leq \frac{|Q| + |X|}{2|X| - 1} = \frac{n + \frac{1}{2}}{2|X| - 1} + \frac{|X| - \frac{1}{2}}{2(|X| - \frac{1}{2})} = \frac{n + \frac{1}{2}}{2|X| - 1} + \frac{1}{2} \leq \frac{n + \frac{1}{2}}{3} + \frac{1}{2} = \frac{n + 2}{3},
\]

which contradicts \( I(G) > \frac{n+2}{3} \). Theorem 8 is proved.

**Remark 9.** Now, we explain that the result in Theorem 8 is sharp. To see this, we construct the graph \( G = K_{n+2} + (3K_1 \cup K_2) \). Obviously, \( G \) is \((n+2)\)-connected and \( I(G) = \frac{n+2}{3} \). Let \( Q \subset V(K_{n+2}) \subset V(G) \) with \(|Q| = n\), and \( e \) be the edge of \( K_2 \). Then \( G - Q - e \) is a graph isomorphic to \( K_2 + (5K_1) \), and it obviously has no \( P_{\geq 2} \)-factor. Thus, \( G \) is not a \((P_{\geq 2}, n)\)-factor critical avoidable graph.
**Theorem 10.** Let $G$ be an $(n+2)$-connected graph, where $n$ is a positive integer. If its isolated toughness $I(G) > \frac{n+3}{2}$, then $G$ is a $(P_{\geq 3}, n)$-factor critical avoidable graph.

**Proof.** Theorem 10 obviously holds for a complete graph. Next, we assume that $G$ is not complete. Let $Q \subset V(G)$ with $|Q| = n$, and $G' = G - Q$, and let $e = xy \in E(G')$ and $H = G' - e$. Since $G$ is $(n+2)$-connected, $H$ is connected. To prove Theorem 10, it suffices to show that $H$ admits a $P_{\geq 3}$-factor. On the contrary, suppose that $H$ has no $P_{\geq 3}$-factor. Then by Theorem 2, there exists a set $X \subset V(H)$ such that

$$\text{sun}(H - X) \geq 2|X| + 1.$$  

**Claim 1.** $X \neq \emptyset$.

**Proof.** Assume $X = \emptyset$. Then $\text{sun}(H) \geq 1$. This implies $\text{sun}(H) = 1$ since $H$ is connected.

Note that $G$ is $(n+2)$-connected. Hence, $|V(G)| \geq n + 3$. Thus, $|V(H)| = |V(G)| - n \geq (n + 3) - n = 3$, which implies that $H$ is a big sun. Therefore, $|V(H)| \geq 6$. Let $R$ be the factor-critical subgraph of $H$. Then $i(H - V(R)) = |V(R)| \geq 3$. Next, we consider two cases.

**Case 1.** $x, y \in V(H) \setminus V(R)$. Clearly, there exists $z \in V(R)$ with $yz \in E(G)$. Thus, we easily see

$$i(G - (Q \cup (V(R) \setminus \{z\}) \cup \{y\})) = i(G' - ((V(R) \setminus \{z\}) \cup \{y\}))$$

$$= i(G' - ((V(R) \setminus \{z\}) \cup \{y\}) - e)$$

$$= i(H - ((V(R) \setminus \{z\}) \cup \{y\}))$$

$$= |V(R)| \geq 3.$$

Combining this with the definition of $I(G)$ and $I(G) > \frac{n+3}{2}$, we admit. Clearly, there exists $z \in V(R)$ with $yz \in E(G)$. Thus, we easily get

$$\frac{n + 3}{2} < I(G) \leq \frac{|Q \cup (V(R) \setminus \{z\}) \cup \{y\}|}{i(G - (Q \cup (V(R) \setminus \{z\}) \cup \{y\}))}$$

$$= \frac{|Q| + |V(R)|}{|V(R)|} = \frac{n}{|V(R)|} + 1 \leq \frac{n}{3} + 1 = \frac{n+3}{3},$$

which is a contradiction.

**Case 2.** $x \in V(R)$ or $y \in V(R)$. In this case, $i(G - (Q \cup (V(R)))) = i(G' - V(R)) = i(G' - V(R) - e) = i(H - V(R)) = |V(R)| \geq 3$. Thus, we get

$$I(G) \leq \frac{|Q \cup V(R)|}{i(G - (Q \cup V(R)))} = \frac{|Q| + |V(R)|}{|V(R)|} = \frac{n}{|V(R)|} + 1 \leq \frac{n}{3} + 1 = \frac{n+3}{3},$$

which contradicts $I(G) > \frac{n+3}{2}$. Hence, $X \neq \emptyset$. \hfill $\square$
Let $Sun(H - X)$ denote the union of sun components of $H - X$, which consists of $a$ isolated vertices, $b$ $K_2$-components and $c$ big sun components $S_1, S_2, \ldots, S_c$. Let $R_i$ be the factor-critical subgraph of $S_i$ for $1 \leq i \leq c$, and write $Z = \bigcup_{1 \leq i \leq c} V(R_i)$. We select one vertex from every $K_2$ component of $H - X$, and the set of such vertices is denoted by $Y$. Clearly, $|Y| = b$. Then $i(H - (X \cup Y \cup Z)) = a + b + |Z|$ and it follows from (9) and Claim 1 that

$$\text{(10)} \quad \text{sun}(H - X) = a + b + c \geq 2|X| + 1 \geq 3.$$

**Claim 2.** $0 \leq a \leq 1$.

**Proof.** Assume that $a \geq 2$. By (10), $c \geq 0$ and $|V(R_i)| \geq 3$, we derive

$$i(G - (Q \cup X \cup Y \cup Z \cup \{x\})) = i(G' - (X \cup Y \cup Z \cup \{x\}) - e)$$

$$= i(H - (X \cup Y \cup Z \cup \{x\}))$$

$$\geq i(H - (X \cup Y \cup Z)) - 1$$

$$= a + b + |Z| - 1 \geq a + b + 3c - 1$$

$$\geq a + b + c - 1 \geq 2.$$

Combining this with the definition of $I(G)$ and $I(G) > \frac{n + 3}{2}$, we derive

$$\frac{n + 3}{2} < I(G) \leq \frac{|Q \cup X \cup Y \cup Z \cup \{x\}|}{i(G - (Q \cup X \cup Y \cup Z \cup \{x\}))} \leq \frac{n + |X| + b + |Z| + 1}{a + b + |Z| - 1},$$

namely,

$$\text{(11)} \quad 0 > \frac{n + 1}{2} (a + b + |Z|) + a - |X| - \frac{3n + 5}{2}.$$

It follows from (10), (11), $a \geq 2$, $c \geq 0$, $|Z| = \sum_{i=1}^c |V(R_i)| \geq 3c$ and Claim 1 that

$$0 > \frac{n + 1}{2} (a + b + |Z|) + a - |X| - \frac{3n + 5}{2}$$

$$\geq \frac{n + 1}{2} (a + b + 3c) + 2 - |X| - \frac{3n + 5}{2}$$

$$\geq \frac{n + 1}{2} (a + b + c) - |X| - \frac{3n + 1}{2}$$

$$\geq \frac{n + 1}{2} (2|X| + 1) - |X| - \frac{3n + 1}{2}$$

$$= n(|X| - 1) \geq 0,$$

which is a contradiction. Therefore, $0 \leq a \leq 1$. □

We easily see that $x \notin V(aK_1)$ or $y \notin V(aK_1)$ since $0 \leq a \leq 1$ (by Claim 2).
**Claim 3.** \(x \in V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)\) or \(y \in V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)\).

**Proof.** Assume that \(x, y \notin V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)\). Note that \(x \notin V(aK_1)\) or \(y \notin V(aK_1)\). Hence, there is at least one vertex in \(\{x, y\}\) such that the vertex does not belong \(V(aK_1) \cup V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)\). Without loss of generality, we let \(x \notin V(aK_1) \cup V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)\). Then \(x \in V(G) \setminus (Q \cup V(aK_1) \cup V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c))\). Thus, we easily deduce
\[
i(G - (Q \cup X \cup Y \cup Z \cup \{x\})) \geq a + b + |Z| \geq a + b + 3c \geq 3
\]
by (10), \(c \geq 0\) and \(|Z| = \sum_{i=1}^c |V(R_i)| \geq 3c\). In terms of the definition of \(I(G)\), we derive
\[
I(G) \leq \frac{|Q \cup X \cup Y \cup Z \cup \{x\}|}{i(G - (Q \cup X \cup Y \cup Z \cup \{x\}))} \leq \frac{n + |X| + b + |Z| + 1}{a + b + |Z|}.
\]
It follows from (10), (12), \(a \geq 0, c \geq 0, |Z| = \sum_{i=1}^c |V(R_i)| \geq 3c\) and \(I(G) > \frac{n + 3}{2}\) that
\[
0 \geq (I(G) - 1)(a + b + |Z|) + a - n - |X| - 1
\]
\[
\geq (I(G) - 1)(a + b + 3c) - n - |X| - 1
\]
\[
\geq (I(G) - 1)(a + b + c) - n - |X| - 1
\]
\[
\geq (I(G) - 1)(2|X| + 1) - n - |X| - 1
\]
\[
= I(G)(2|X| + 1) - n - 3|X| - 2,
\]
which implies
\[
I(G) \leq \frac{3|X| + n + 2}{2|X| + 1}.
\]

From (13), Claim 1 and \(n \geq 1\), we have
\[
I(G) \leq \frac{3|X| + n + 2}{2|X| + 1} = \frac{3}{2} + \frac{n + 1}{2} \cdot \frac{1}{3} \leq \frac{3}{2} + \frac{n + 1}{3} = \frac{n + 3}{2} + \frac{1 - n}{6} \leq \frac{n + 3}{2},
\]
which contradicts \(I(G) > \frac{n + 3}{2}\). Claim 3 is verified. \(\square\)

Without loss of generality, we let \(x \in V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)\) by Claim 3. Then there exists \(z \in V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)\) such that \(xz \in E(G)\) and there is at least one vertex of \(\{x, z\}\) with degree 1 in the subgraph \((bK_2) \cup S_1 \cup \cdots \cup S_c\). Thus, we obtain
\[
i(G - (Q \cup X \cup (Y \cup Z) \setminus \{z\}) \cup \{x\})) = a + b + |Z| \geq a + b + 3c \geq 3
\]
by (10), \( c \geq 0 \) and \( |Z| = \sum_{i=1}^{c} |V(R_i)| \geq 3c \). Combining this with the definition of \( I(G) \) and \( I(G) > \frac{n+3}{2} \), we obtain
\[
\frac{n + 3}{2} < I(G) \leq \frac{|Q \cup X \cup ((Y \cup Z) \setminus \{z\}) \cup \{x\}|}{i(G - (Q \cup X \cup ((Y \cup Z) \setminus \{z\}) \cup \{x\}))} = \frac{n + |X| + b + |Z|}{a + b + |Z|},
\]
that is,
\[
0 > \frac{n + 1}{2} (a + b + |Z|) - n - |X| + a.
\]
Combining this with (10), \( a \geq 0, c \geq 0, n \geq 1, |Z| = \sum_{i=1}^{c} |V(R_i)| \geq 3c \) and Claim 1, we derive
\[
0 > \frac{n + 1}{2} (a + b + |Z|) - n - |X| + a \geq \frac{n + 1}{2} (a + b + c) - n - |X|
\geq \frac{n + 1}{2} (2|X| + 1) - n - |X| = n|X| + \frac{1}{2} - \frac{n}{2} \geq n + \frac{1}{2} - \frac{n}{2} = \frac{n + 1}{2} \geq 1,
\]
which is a contradiction. This finishes the proof of Theorem 10.

**Remark 11.** Next, we elaborate that the conditions in Theorem 10 are best possible, which cannot be replaced by \( G \) being \((n+1)\)-connected and \( I(G) \geq \frac{n+3}{2} \).

Let \( G = K_{n+1} + (2K_2) \). It is clear that \( G \) is \((n+1)\)-connected and \( I(G) = \frac{n+3}{2} \).

Let \( Q \subset V(K_{n+1}) \subseteq V(G) \) with \( |Q| = n \), and \( e \) be an edge of \( 2K_2 \). Then \( G - Q - e \) is a graph isomorphic to \( K_1 + (2K_1 \cup K_2) \), and it obviously has no \( P_{\geq 3} \)-factor. Therefore, \( G \) is not a \((P_{\geq 3}, n)\)-factor critical avoidable graph.

**Acknowledgements**

The authors would like to thank the anonymous referees for their comments on this paper. This work is supported by Six Big Talent Peak of Jiangsu Province (Grant No. JY–022).

**References**


Received 9 June 2020
Revised 3 September 2020
Accepted 3 September 2020