THE EXISTENCE OF PATH-FACTOR COVERED GRAPHS

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Abstract

A spanning subgraph $H$ of a graph $G$ is called a $P_{\geq k}$-factor of $G$ if every component of $H$ is isomorphic to a path of order at least $k$, where $k \geq 2$. A graph $G$ is called a $P_{\geq k}$-factor covered graph if there is a $P_{\geq k}$-factor of $G$ covering $e$ for any $e \in E(G)$. In this paper, we obtain two special classes of $P_{\geq 2}$-factor covered graphs. We also obtain two special classes of $P_{\geq 3}$-factor covered graphs. Furthermore, it is shown that these results are all sharp.

Keywords: path-factor, $P_{\geq 2}$-factor covered graph, $P_{\geq 3}$-factor covered graph, claw-free graph, isolated toughness.

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1. Introduction

We consider only finite simple graph, unless explicitly stated. We refer to [6] for the notation and terminologies not defined here. Let $G = (V(G), E(G))$ be a simple graph, where $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. A subgraph $H$ of $G$ is called a spanning subgraph of $G$ if $V(H) = V(G)$ and $E(H) \subseteq E(G)$. A subgraph $H$ of $G$ is called an induced subgraph of $G$ if every pair of vertices in $H$ which are adjacent in $G$ are also adjacent in $H$. For $v \in V(G)$, the degree of $v$ in $G$ is denoted by $d_G(v)$. A graph $G$ is said to be claw-free if there is no induced subgraph of $G$ isomorphic to $K_{1,3}$.

For a family of connected graphs $\mathcal{F}$, a spanning subgraph $H$ of a graph $G$ is called an $\mathcal{F}$-factor of $G$ if each component of $H$ is isomorphic to some graph in $\mathcal{F}$. A spanning subgraph $H$ of a graph $G$ is called a $P_{\geq k}$-factor of $G$ if every component of $H$ is isomorphic to a path of order at least $k$. For example, a $P_{\geq 3}$-factor means a graph factor in which every component is a path of order at least
A graph $G$ is called a $P_{\geq k}$-factor covered graph if there is a $P_{\geq k}$-factor of $G$ covering $e$ for any $e \in E(G)$.

Since Tutte proposed the well known Tutte 1-factor theorem [15], there are many results on graph factors [2, 3, 8, 9, 16] and $P_{\geq k}$-factors in claw-free graphs and cubic graphs [4, 12, 13]. More results on graph factors can be found in the survey papers and books in [2, 14, 18]. We use $\omega(G)$, $i(G)$ to denote the number of components and isolated vertices of a graph $G$, respectively. For a subset $X \subseteq V(G)$, $G - X$ denotes the graph obtained from $G$ by deleting all the vertices of $X$. Akiyama, Avis and Era [1] proved the following theorem, which is a criterion for a graph to have a $P_{\geq 2}$-factor.

**Theorem 1** (Akiyama et al. [2]). A graph $G$ has a $P_{\geq 2}$-factor if and only if $i(G - X) \leq 2|X|$ for all $X \subseteq V(G)$.

Kaneko [10] introduced the concept of a sun and gave a characterization for a graph with a $P_{\geq 3}$-factor. It is perhaps the first characterization of graphs which have a path factor not including $P_2$. Recently, Kano et al. [11] presented a simpler proof for Kaneko’s theorem [10].

A graph $H$ is called factor-critical if $H - \{v\}$ has a 1-factor for each $v \in V(H)$. Let $H$ be a factor-critical graph and $V(H) = \{v_1, v_2, \ldots, v_n\}$. By adding new vertices $\{u_1, u_2, \ldots, u_n\}$ together with new edges $\{v_iu_i : 1 \leq i \leq n\}$ to $H$, the resulting graph is called a sun. Note that, according to Kaneko [10], we regard $K_1$ and $K_2$ also as a sun, respectively. Usually, the suns other than $K_1$ are called big suns. It is called a sun component of $G - X$ if the component of $G - X$ is isomorphic to a sun. We denote by $\text{sun}(G - X)$ the number of sun components in $G - X$.

**Theorem 2** (Kaneko [10]). A graph $G$ has a $P_{\geq 3}$-factor if and only if $\text{sun}(G - X) \leq 2|X|$ for all $X \subseteq V(G)$.

Zhang and Zhou [19] proposed the concept of path-factor covered graph, which is a generalization of matching cover. They also obtained a characterization for $P_{\geq 2}$-factor and $P_{\geq 3}$-factor covered graphs, respectively.

**Theorem 3** (Zhang et al. [19]). Let $G$ be a connected graph. Then $G$ is a $P_{\geq 2}$-factor covered graph if and only if $i(G - S) \leq 2|S| - \varepsilon(S)$ for all $S \subseteq V(G)$, where $\varepsilon(S)$ is defined by

$$
\varepsilon(S) = \begin{cases} 
2 & \text{if } S \neq \emptyset \text{ and } S \text{ is not an independent set}, \\
1 & \text{if } S \neq \emptyset, \text{ } S \text{ is an independent set and there exists} \\
a \text{ component of } G - S \text{ with at least two vertices}, \\
0 & \text{otherwise.}
\end{cases}
$$
Theorem 4 (Zhang et al. [19]). Let $G$ be a connected graph. Then $G$ is a $P_{\geq 3}$-factor covered graph if and only if $\text{sun}(G - S) \leq 2|S| - \varepsilon(S)$ for all $S \subseteq V(G)$, where $\varepsilon(S)$ is defined by

$$\varepsilon(S) = \begin{cases} 2 & \text{if } S \neq \emptyset \text{ and } S \text{ is not an independent set,} \\ 1 & \text{if } S \neq \emptyset, \text{ S is an independent set and there exists a} \\ & \text{non-sun component of } G - S, \\ 0 & \text{otherwise.} \end{cases}$$

For a connected graph $G$, its toughness, denoted by $t(G)$, was first introduced by Chvátal [7] as follows. If $G$ is complete, then $t(G) = +\infty$; otherwise,

$$t(G) = \min \left\{ \frac{|S|}{\omega(G - S)} : S \subseteq V(G), \omega(G - S) \geq 2 \right\}. $$

Bazgan, Benhamdine, Li and Woźniak [5] showed a toughness condition for the existence of a $P_{\geq 3}$-factor in a graph.

Theorem 5 (Bazgan, Benhamdine, Li and Woźniak [5]). Let $G$ be a graph with at least three vertices. If $t(G) \geq 1$, then $G$ includes a $P_{\geq 3}$-factor.

For a connected graph $G$, its isolated toughness, denoted by $I(G)$, was first introduced by Yang, Ma and Liu [17] as follows. If $G$ is complete, then $I(G) = +\infty$; otherwise,

$$I(G) = \min \left\{ \frac{|S|}{i(G - S)} : S \subseteq V(G), i(G - S) \geq 2 \right\}. $$


Theorem 6 (Zhou and Wu [20]). A graph $G$ is a $P_{\geq 3}$-factor covered graph if one the following holds.

(i) $G$ is a connected graph with at least three vertices and $t(G) > 2/3$;
(ii) $G$ is a connected graph with at least three vertices and $I(G) > 5/3$;
(iii) $G$ is a $k$-regular graph with $k \geq 2$.

In this paper, we proceed to investigate $P_{\geq k}$-factor covered graphs. We respectively obtain two special classes of $P_{\geq 2}$-factor covered graphs and $P_{\geq 3}$-factor covered graphs. Our main results will be shown in Sections 2 and 3, respectively.
2. \(P_{\geq 2}\)-Factor Covered Graphs

In this section, we mainly obtain two special classes of \(P_{\geq 2}\)-factor covered graphs. First, we will give a sufficient condition for a connected claw-free graph to be a \(P_{\geq 2}\)-factor covered graph as following. Note that the result in Theorem 7 is sharp in the sense that there exists a connected claw-free graph of minimum degree 1, which is not a \(P_{\geq 2}\)-factor covered graph. An example can be seen in Figure 1.

\[
\begin{array}{c}
\text{Figure 1. A connected claw-free graph of minimum degree 1 that does not contain any } \\
P_{\geq 2}\text{-factor covering } e = x_2x_3.
\end{array}
\]

**Theorem 7.** Let \(G\) be a connected claw-free graph of minimum degree at least 2. Then \(G\) is a \(P_{\geq 2}\)-factor covered graph.

**Proof.** Suppose \(G\) is not a \(P_{\geq 2}\)-factor covered graph. Then by Theorem 3, there exists a subset \(S \subseteq V(G)\) such that \(i(G - S) > 2|S| - \varepsilon(S)\). In terms of the integrality of \(i(G - S)\), we obtain that \(i(G - S) \geq 2|S| - \varepsilon(S) + 1\). We will distinguish two cases below to show that \(G\) is a \(P_{\geq 2}\)-factor covered graph.

**Case 1.** \(|S| \leq 1\). If \(S = \emptyset\), then \(\varepsilon(S) = |S| = 0\) by the definition of \(\varepsilon(S)\). It follows easily that

\[
i(G) = i(G - S) \geq 2|S| - \varepsilon(S) + 1 = 1.
\]

On the other hand, \(i(G) \leq \omega(G) = 1\) since \(G\) is a connected graph. Combining the results above, we obtain \(i(G) = 1\), which contradicts the connectivity of \(G\).

If \(|S| = 1\), let \(S = \{s\}\). We obtain \(\varepsilon(S) \leq 1\) by the definition of \(\varepsilon(S)\). If \(\varepsilon(S) = 0\), then

\[
\omega(G - S) \geq i(G - S) \geq 2|S| - \varepsilon(S) + 1 = 3.
\]

Therefore, if either \(\varepsilon(S)\) is 0 or 1, then there are at least three components of \(G - \{s\}\). It follows easily that there exists a claw with center vertex \(s\) in \(G\), a contradiction.
Case 2. \(|S| \geq 2\). Let \(|S| = k \geq 2\) and \(S = \{s_1, s_2, \ldots, s_k\}\). By the definition of \(\varepsilon(S)\), we have \(\varepsilon(S) \leq 2\). It follows easily that
\[
i(G - S) \geq 2|S| - \varepsilon(S) + 1 = 2|S| - 1.
\]
Let \(i(G - S) = m \geq 2k - 1\) and \(\{x_1, x_2, \ldots, x_m\}\) be the set of isolated vertices of \(G - S\). Since the minimum degree of \(G\) is at least two, we immediately obtain the number of edges incident with the vertices in \(\{x_1, x_2, \ldots, x_m\}\) is at least \(2m\). Because \(G\) does not have multiple edges and
\[
\frac{2m}{|S|} = \frac{2m}{k} \geq \frac{2(2k - 1)}{k} = 4 - \frac{2}{k} \geq 3,
\]
there must exist a vertex \(s_i \in S\) adjacent to at least three vertices in \(\{x_1, x_2, \ldots, x_m\}\) by pigeonhole principle. It follows easily that there exists a claw with center vertex \(s_i\) in \(G\), a contradiction.

Combining Case 1 and Case 2, Theorem 7 is proved.

Next, we study the relationship between isolated toughness and \(P_{\geq 2}\)-factor covered graphs, and obtain an isolated toughness condition for the existence of \(P_{\geq 2}\)-factor covered graphs. The example in Figure 2 shows the sharpness of the results in Theorem 8 in the sense that there exists a connected graph with \(I(G) = 2/3\), which is not a \(P_{\geq 2}\)-factor covered graph.

![Figure 2](image-url)
\[ i(G - S) > 2|S| - \varepsilon(S). \] Then, by the integrality of \( i(G - S) \), we obtain that 
\[ i(G - S) \geq 2|S| - \varepsilon(S) + 1. \]

**Case 1.** \(|S| \leq 1. \) If \(|S| = 0\), by the definition of \( \varepsilon(S) \), we have \( S = \emptyset \) and 
\( \varepsilon(S) = 0. \) It follows immediately that
\[ i(G) = i(G - S) \geq 2|S| - \varepsilon(S) + 1 = 1, \]
which contradicts the connectivity of \( G. \)

Thus we may assume \(|S| = 1, \) we have \( \varepsilon(S) \leq 1 \) by the definition of \( \varepsilon(S). \) It follows easily that
\[ i(G - S) \geq 2|S| - \varepsilon(S) + 1 \geq 2|S|. \]
By the definition of \( I(G), \) we have that
\[ I(G) \leq \frac{|S|}{i(G - S)} \leq \frac{1}{2}, \]
which contradicts \( I(G) > 2/3. \)

**Case 2.** \(|S| \geq 2. \) In this case, it follows from the definition of \( \varepsilon(S) \) that 
\( \varepsilon(S) \leq 2, \) which implies that
\[ i(G - S) \geq 2|S| - \varepsilon(S) + 1 \geq 2|S| - 1 \geq 3. \]
Thus we immediately obtain
\[ |S| \leq \frac{i(G - S) + 1}{2}. \]
By the definition of \( I(G), \) we have
\[ I(G) \leq \frac{|S|}{i(G - S)} \leq \frac{i(G - S) + 1}{2i(G - S)} \leq \frac{1}{2} + \frac{1}{2i(G - S)} \leq \frac{1}{2} + \frac{1}{6} = \frac{2}{3}, \]
which contradicts \( I(G) > 2/3. \)

This completes the proof of Theorem 8.

### 3. \( P_{\geq 3} \)-Factor Covered Graphs

In this section, we mainly obtain two special classes of \( P_{\geq 3} \)-factor covered graphs. First, we give a minimum degree condition for a connected claw-free graph to be a \( P_{\geq 3} \)-factor covered graph as following. Note that the results in Theorem 9 is also sharp in the sense that there exists a connected claw-free graph of minimum degree 2, which is not a \( P_{\geq 3} \)-factor covered graph. It is shown by the example in Figure 3.
Figure 3. A connected claw-free graph of minimum degree 2 that does not contain any $P_{\geq 3}$-factor covering $e = x_2x_3$.

**Theorem 9.** Let $G$ be a connected claw-free graph of minimum degree at least 3. Then $G$ is a $P_{\geq 3}$-factor covered graph.

**Proof.** Suppose $G$ is not a $P_{\geq 3}$-factor covered graph. Then by Theorem 4, there exists a subset $S \subseteq V(G)$ such that $\text{sun}(G - S) > 2|S| - \varepsilon(S)$. In terms of the integrality of $\text{sun}(G - S)$, we obtain that $\text{sun}(G - S) \geq 2|S| - \varepsilon(S) + 1$. We will distinguish two cases below to show that $G$ is a $P_{\geq 3}$-factor covered graph.

**Case 1.** $|S| \leq 1$. If $S = \emptyset$, then $\varepsilon(S) = |S| = 0$ by the definition of $\varepsilon(S)$. It follows easily that

$$\text{sun}(G) = \text{sun}(G - S) \geq 2|S| - \varepsilon(S) + 1 = 1.$$  

On the other hand, $\text{sun}(G) \leq \omega(G) = 1$ since $G$ is a connected graph. Combining the results above, we obtain that $G$ is a big sun, which contradicts the minimum degree of $G$.

If $|S| = 1$, let $S = \{s\}$. We obtain $\varepsilon(S) \leq 1$ by the definition of $\varepsilon(S)$. If $\varepsilon(S) = 0$, then

$$\omega(G - S) \geq \text{sun}(G - S) \geq 2|S| - \varepsilon(S) + 1 = 3.$$  

Otherwise $\varepsilon(S) = 1$, then there exists a non-sun component of $G - S$ and thus

$$\omega(G - S) \geq \text{sun}(G - S) + 1 \geq 2|S| - \varepsilon(S) + 1 + 1 = 3.$$  

Therefore, if either $\varepsilon(S)$ is 0 or 1, then there are at least three components of $G - \{s\}$. It follows easily that there exists a claw with center vertex $s$ in $G$, a contradiction.

This completes the proof of Case 1.

**Case 2.** $|S| \geq 2$. Let $|S| = k \geq 2$ and $S = \{s_1, s_2, \ldots, s_k\}$. By the definition of $\varepsilon(S)$, we have $\varepsilon(S) \leq 2$. It follows easily that

$$\text{sun}(G - S) \geq 2|S| - \varepsilon(S) + 1 \geq 2|S| - 1.$$
Let \( \text{sun}(G - S) = m \geq 2k - 1 \) and \( \{C_1, C_2, \ldots, C_m\} \) be the set of sun components of \( G - S \). For any sun component \( C_i \) of \( G - S \), let \( L(C_i) \subseteq V(C_i) \) be the set of vertices with exactly one neighbour vertex in \( C_i \), and \( L(C_i) = V(C_i) \) if \( C_i \cong K_1 \), where \( 1 \leq i \leq m \). Let \( E(S, V(C_i) \setminus L(C_i)) \) be the set of edges in graph \( G \) between vertex \( a \) and \( b \) for any \( a \in S, b \in V(C_i) \setminus L(C_i) \) for \( 1 \leq i \leq m \). Then we construct a bipartite multigraph \( H \) from \( G \) by deleting all edges of \( E(G[S]) \cup \bigcup_{i=1}^{m} E(S, V(C_i) \setminus L(C_i)) \) and all vertices of \( V(G) \setminus S \cup \bigcup_{i=1}^{m} V(C_i) \) and contracting each \( C_i \) to a vertex \( c_i \) for \( 1 \leq i \leq m \).

**Claim 1.** For any vertex \( u, v \in V(H) \), there are at most two edges between \( u \) and \( v \) in \( H \).

**Proof.** Without loss of generality, we assume \( u = s_1 \) and \( v = c_1 \). Suppose there are three edges between \( u \) and \( v \) in \( H \). Then there are three vertices in \( L(C_1) \) corresponding to the vertex \( c_1 \), denoted by \( \{c_1^1, c_1^2, c_1^3\} \). By the definition of big sun, \( c_i^1 c_i^j \notin E(G) \) for any \( 1 \leq i < j \leq 3 \), which implies a claw with center vertex \( u \) in \( G \). This is a contradiction.

Since the minimum degree of \( G \) is at least three, it is clear that \( d_H(c_i) \geq 3 \) for \( 1 \leq i \leq m \). Trivially,

\[
|E(H)| \geq 3m \geq 3(2k - 1) = 6k - 3.
\]

By pigeonhole principle and

\[
\frac{|E(H)|}{|S|} \geq \frac{3m}{k} \geq \frac{6k - 3}{k} = 6 - \frac{3}{k} > 4,
\]

there must exist a vertex \( s_i \in S \) incident with at least five edges in \( E(H) \). According to Claim 1 and pigeonhole principle, there exists at least three vertices, denoted by \( \{c_1, c_2, c_3\} \), adjacent to \( s_i \). Since \( \{s_i, c_1, c_2, c_3\} \) induces a claw in \( H \), it follows easily that there exists a claw with center vertex \( s_i \) in \( G \), a contradiction. This completes the proof of Case 2.

Combining Case 1 and Case 2, Theorem 9 is proved.

Next, we investigate the relationship between planar graphs and \( P_{\geq 3} \)-factor covered graphs, and obtain a connectivity condition for a planar graph to be a \( P_{\geq 3} \)-factor covered graphs as following. The example in Figure 4 shows the sharpness of the results in Theorem 11 in the sense that there exists a 2-connected planar graph, which is not a \( P_{\geq 3} \)-factor covered graph.
Figure 4. A 2-connected planar graph that does not contain any $P_{\geq 3}$-factor covering $e = x_3x_4$.

**Lemma 10** [6]. Let $G$ be a connected planar graph with at least three vertices. If $G$ does not contain triangles, then $|E(G)| \leq 2|G| - 4$.

**Theorem 11.** Let $G$ be a 3-connected planar graph. Then $G$ is a $P_{\geq 3}$-factor covered graph.

**Proof.** Suppose $G$ is not a $P_{\geq 3}$-factor covered graph. By Theorem 4, there exists a subset $S \subseteq V(G)$ such that $\text{sun}(G - S) > 2|S| - \varepsilon(S)$. According to the integrality of $\text{sun}(G - S)$, we obtain that $\text{sun}(G - S) \geq 2|S| - \varepsilon(S) + 1$. We distinguish three cases below to show that $G$ is a $P_{\geq 3}$-factor covered graph.

**Case 1.** $|S| = 0$. In this case, by the definition of $\varepsilon(S)$, we have $S = \emptyset$ and $\varepsilon(S) = 0$. Since $G$ is a connected graph, $\text{sun}(G) \leq \omega(G) = 1$. On the other hand, we obtain that

$$\text{sun}(G) = \text{sun}(G - S) \geq 2|S| - \varepsilon(S) + 1 = 1.$$  

It follows easily that $\text{sun}(G) = 1$, which is to say $G$ is a big sun. By the definition of sun, it contradicts the fact that $G$ is 3-connected. This completes the proof of Case 1.

**Case 2.** $|S| = 1$. In this case, we obtain $\varepsilon(S) \leq 1$ by the definition of $\varepsilon(S)$. It follows immediately that

$$\text{sun}(G - S) \geq 2|S| - \varepsilon(S) + 1 \geq 2.$$  

Let $S = \{x\} \subseteq V(G)$. Since $\omega(G - S) \geq \text{sun}(G - S) \geq 2$, $x$ is a cut-vertex of $G$, which contradicts the fact that $G$ is 3-connected. This completes the proof of Case 2.

**Case 3.** $|S| \geq 2$. In this case, we obtain $\varepsilon(S) \leq 2$ by the definition of $\varepsilon(S)$. It follows immediately that

$$\text{sun}(G - S) \geq 2|S| - \varepsilon(S) + 1 \geq 2|S| - 1.$$
Set $|S| = s$. We denote by $\text{Sun}(G-S)$ the set of sun components in $G-S$. Since $\text{sun}(G-S) \geq 2|S|-1$, let $C_1, C_2, \ldots, C_{2s-1}$ be $2s-1$ distinct sun components where $C_i \in \text{Sun}(G-S)$ for any $1 \leq i \leq 2s-1$. Then we construct a bipartite graph $H$ from $G$ by contracting each $C_i$ to a vertex $c_i$ for $1 \leq i \leq 2s-1$ and deleting all edges of $E(G[S])$ and all vertices of $V(G) \setminus (S \cup \bigcup_{i=1}^{2s-1} V(C_i))$.

Since $G$ is 3-connected, it is clear that $d_H(c_i) \geq 3$ for $1 \leq i \leq 2s-1$. Trivially,

$$|H| = s + (2s - 1) = 3s - 1 \geq 5,$$

and

$$|E(H)| \geq 3(2s - 1) = 6s - 3.$$

As $G$ is a 3-connected planar graph, it is easy to see that $H$ is also a connected planar graph. According to the fact that a bipartite graph does not contain any odd cycles, Lemma 10 implies that

$$6s - 3 \leq |E(H)| \leq 2|H| - 4 = 2(3s - 1) - 4 = 6s - 6,$$

which is a contradiction. This completes the proof of Case 3.

Combining Cases 1–3, Theorem 11 is proved.

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