A NOTE ON THE FAIR DOMINATION NUMBER IN OUTERPLANAR GRAPHS

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Abstract

For $k \geq 1$, a $k$-fair dominating set (or just $k$FD-set), in a graph $G$ is a dominating set $S$ such that $|N(v) \cap S| = k$ for every vertex $v \in V - S$. The $k$-fair domination number of $G$, denoted by $fd_k(G)$, is the minimum cardinality of a $k$FD-set. A fair dominating set, abbreviated FD-set, is a $k$FD-set for some integer $k \geq 1$. The fair domination number, denoted by $fd(G)$, of $G$ that is not the empty graph, is the minimum cardinality of an FD-set in $G$. In this paper, we present a new sharp upper bound for the fair domination number of an outerplanar graph.

Keywords: fair domination, outerplanar graph, unicyclic graph.

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1. Introduction

For notation and graph theory terminology not given here, we follow [13]. Specifically, let $G$ be a simple graph with vertex set $V(G) = V$ of order $|V| = n$ and let $v$ be a vertex in $V$. The open neighborhood of $v$ is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and the closed neighborhood of $v$ is $N_G[v] = \{v\} \cup N_G(v)$. If the graph $G$ is
clear from the context, then we simply write \( N(v) \) rather than \( N_G(v) \). The degree of a vertex \( v \), is \( \text{deg}(v) = |N(v)| \). A vertex of degree one is called a leaf and its neighbor a support vertex. A strong support vertex is a support vertex adjacent to at least two leaves, and a weak support vertex is a support vertex adjacent to precisely one leaf. For a set \( S \subseteq V \), its open neighborhood is the set \( N(S) = \bigcup_{v \in S} N(v) \), and its closed neighborhood is the set \( N[S] = N(S) \cup S \). The distance \( d(u, v) \) between two vertices \( u \) and \( v \) in a graph \( G \) is the minimum number of edges of a path from \( u \) to \( v \). A graph \( G \) of order at least three is 2-connected if the deletion of any vertex does not disconnect the graph. A cut-vertex in a connected graph is a vertex whose removal disconnect the graph. A maximal connected subgraph without a cut-vertex is called a block. A graph \( G \) is outerplanar if it can be embedded in the plane such that all vertices lie on the boundary of its exterior region. A graph \( G \) is Hamiltonian if there is a spanning cycle in \( G \). For a subset \( S \) of vertices of \( G \), we denote by \( G[S] \) the subgraph of \( G \) induced by \( S \).

A subset \( S \subseteq V \) is a dominating set of \( G \) if every vertex not in \( S \) is adjacent to a vertex in \( S \). The domination number of \( G \), denoted by \( \gamma(G) \), is the minimum cardinality of a dominating set of \( G \). A vertex \( v \) is said to be dominated by a set \( S \) if \( N[v] \cap S \neq \emptyset \).

Caro et al. [1] studied the concept of fair domination in graphs. For \( k \geq 1 \), a \( k \)-fair dominating set, abbreviated \( k \)-FD-set, in \( G \) is a dominating set \( S \) such that \( |N(v) \cap D| = k \) for every vertex \( v \in V - D \). The \( k \)-fair domination number of \( G \), denoted by \( fd_k(G) \), is the minimum cardinality of a \( k \)-FD-set. A \( k \)-FD-set of \( G \) of cardinality \( fd_k(G) \) is called a \( fd_k(G) \)-set. A fair dominating set, abbreviated FD-set, in \( G \) is a \( k \)-FD-set for some integer \( k \geq 1 \). The fair domination number, denoted by \( fd(G) \), of a graph \( G \) that is not the empty graph is the minimum cardinality of an FD-set in \( G \). An FD-set of \( G \) of cardinality \( fd(G) \) is called a \( fd(G) \)-set. The concept of fair domination in graphs was further studied in [9, 10, 11]. There is a close relation between the fair domination number and variant, namely perfect domination number of a graph. A perfect dominating set in a graph \( G \) is a dominating set \( S \) such that every vertex in \( V(G) - S \) is adjacent to exactly one vertex in \( S \). Hence a 1FD-set is precisely a perfect dominating set. The concept of perfect domination was introduced by Cockayne et al. in [4], and Fellows et al. [8] with a different terminology which they called semiperfect domination. This concept was further studied, see for example, [2, 3, 5, 6, 12].

Among other results, Caro et al. [1] proved that \( fd(G) < 17n/19 \) for any maximal outerplanar graph \( G \) of order \( n \), and among open problems posed by Caro et al. [1], one asks to find \( fd(G) \) for other families of graphs.

In this paper, we study fair domination in outerplanar graphs. We present a new sharp upper bound for the fair domination number of outerplanar graphs.

We call a block \( K \) in an outerplanar graph \( G \) a strong-block if \( K \) contains
at least three vertices. We call a vertex \( w \) in a strong-block \( K \) of an outerplanar graph \( G \) a special cut-vertex if \( w \) belongs to a shortest path from \( K \) to a strong-block \( K' \neq K \). We call a strong-block \( K \) in an outerplanar graph \( G \) a leaf-block if \( K \) contains exactly one special cut-vertex. We denote by \( r(G) \) the number of strong-blocks of a graph \( G \). The following is straightforward.

**Observation 1.** Every outerplanar graph with at least two strong-blocks contains at least two leaf-blocks.

We make use of the following.

**Observation 2** (Caro et al. [1]). Every 1FD-set in a graph contains all its strong support vertices.

**Theorem 3** (Leydolda et al. [14]). An outerplanar graph \( G \) is Hamiltonian if and only if it is 2-connected.

**Theorem 4** (Hajian et al. [9]). If \( G \) is a unicyclic graph of order \( n \), then \( fd(G) \leq (n + 1)/2 \).

## 2. Main Result

**Theorem 5.** If \( G \) is an outerplanar graph of order \( n \) and size \( m \) with \( r \geq 1 \) strong-blocks, then \( fd(G) \leq (4m - 3n + 3)/2 - r \). This bound is sharp.

**Proof.** Let \( G \) be an outerplanar graph of order \( n \) and size \( m \) with \( r \geq 1 \) strong-blocks. We prove that \( fd(G) \leq (4m - 3n + 3)/2 - r \). The result follows from Theorem 4 if \( G \) is a unicyclic graph. Thus assume that \( G \) is not a unicyclic graph. Suppose to the contrary that \( fd(G) > (4m - 3n + 3)/2 - r \). Assume that \( G \) has the minimum order, and among all such graphs, we may assume that the size of \( G \) is as minimum as possible. Let \( K_1, K_2, \ldots, K_r \) be the \( r \) strong-blocks of \( G \). By Theorem 3, \( K_j \) is Hamiltonian, for \( 1 \leq j \leq r \). Let \( C' = c_0^j c_1^j \cdots c_{l_j}^j c_0^j \) be a Hamiltonian cycle for \( K_i \), for \( 1 \leq i \leq r \). We proceed with the following Claims 1 and 2.

**Claim 1.** For any \( 1 \leq i \leq r \), if \( c_j^i \) is a vertex of \( C' \), for some \( j \in \{0, 1, \ldots, l_i\} \), such that \( deg_G (c_j^i) = 2 \), then \( deg_G (c_{j+1}^i) \geq 2 \) and \( deg_G (c_{j-1}^i) \geq 3 \), where the calculations in \( j + 1 \) and \( j - 1 \) are taken modulo \( l_i \).

**Proof.** Assume that \( deg_G (c_j^i) = 2 \) for some \( j \in \{0, 1, \ldots, l_i\} \). Suppose that \( deg_G (c_{j+1}^i) = 2 \). Let \( G' = G - c_j^i c_{j+1}^i \). Clearly \( r - 1 \leq r(G') \leq r \). By the choice of \( G \), \( fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') \leq (4m - 3n + 3)/2 - r - 1 \). Let \( S' \) be a \( fd_1(G') \)-set. If \( |S' \cap \{c_j^i, c_{j+1}^i\}| \in \{0, 2\}, \)
then $S'$ is a 1FD-set for $G$ of cardinality at most $(4m - 3n + 3)/2 - r - 1$, and so $fd_1(G) \leq (4m - 3n + 3)/2 - r - 1$, a contradiction. Thus $|S' \cap \{c_j', c_{j+1}'\}| = 1$.

Assume that $c_j' \in S'$. Then $c_{j+1}' \notin S'$, and $c_{j+2}' \in S'$, since $S'$ is a dominating set. Thus $\{c_{j+1}'\} \cup S'$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. Next assume that $c_{j+1}' \in S'$.

Then $c_j' \notin S'$ and $c_{j+1}' \in S'$. Thus $\{c'_j\} \cup S'$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r$. So $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. Hence $\deg_G(c_{j+1}') \geq 3$. Similarly, $\deg_G(c_{j-1}') \geq 3$.

Claim 2. If $c_j'$ is a vertex of $C'$, for some $j \in \{0, 1, \ldots, l_1\}$, such that $\deg_G(c_j') = 2$, then non of $c_{j+1}'$ and $c_{j-1}'$ is a support vertex of $G$.

Proof. Assume that $\deg_G(c_j') = 2$ for some $j \in \{0, 1, \ldots, l_1\}$. Suppose that $c_{j+1}'$ is a support vertex of $G$. Let $G' = G - c_{j+1}'$. Clearly $r - 1 \leq r(G') \leq r$.

By the choice of $G$, $fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') \leq (4m - 1 - 3n + 3)/2 - r - 1 = (4m - 3n + 3)/2 - r - 1$. Let $S'$ be a $fd_1(G')$-set. By Observation 2, $c_{j+1}' \in S'$, since $c_{j+1}'$ is a strong support vertex of $G'$. If $c_{j-1}' \notin S'$, then $S'$ is a 1FD-set for $G$ of cardinality at most $(4m - 3n + 3)/2 - r + 1$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r - 1$, a contradiction. Thus $c_{j-1}' \in S'$ and so $\{c_j', c_{j+1}'\} \cup S'$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r$, and so $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. Hence $c_{j+1}'$ is not a support vertex of $G$. Similarly, $c_{j-1}'$ is not a support vertex of $G$.

We consider the following cases.

Case 1. $r = 1$. First assume that $V(G) = \{c_0', c_1', \ldots, c_{l_1}'\}$ and so $n = l_1 + 1$.

By Claim 1, at least $[n/2]$ vertices of $C^1$ are of degree at least 3. Now, we can easily see that $m = \frac{1}{2} \sum_{v \in V(G)} \deg(v) \geq n + [n/2]/2$. (Since $\delta(G) \geq 2$ and at least $[n/2]$ vertices of $G$ are of degree at least 3, we have $\sum_{v \in V(G)} \deg(v) \geq 2n + [n/2]$.) Thus $m \geq n + [n/2]/2$. If $n$ is even, then $n \leq (4m - 3n)/2$ and if $n$ is odd, then $n \leq (4m - 3n - 1)/2$. We thus obtain that $n \leq (4m - 3n + 3)/2 - 1$. Now $V(G)$ is a 1FD-set in $G$ of cardinality $n$, and thus $fd_1(G) \leq (4m - 3n + 3)/2 - 1$, a contradiction. We deduce that $V(G) \neq \{c_0', c_1', \ldots, c_{l_1}'\}$. Since $r = 1$, there is a vertex of degree one in $G$. Let $v_d$ be a leaf of $G$ such that $d(v_d, C^1)$ is maximum. Let $v_0v_1 \cdots v_d$ be the shortest path from $v_d$ to a vertex $v_0 \in C^1$.

Clearly, $\{v_0, v_1, \ldots, v_d\} \cap V(C^1) = \{v_0\}$.

Assume that $d \geq 2$. Suppose that $\deg_G(v_{d-1}) = 2$. Let $G' = G - \{v_d, v_{d-1}\}$. Clearly $r(G') = r$.

By the choice of $G$, $fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') = (4m - 2 - 3(n - 2) + 3)/2 - 1 = (4m - 3n + 3)/2 - 2$. Let $S'$ be a $fd_1(G')$-set. If $v_{d-2} \notin S'$, then $S' \cup \{v_d\}$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - 1$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - 1$, a contradiction. Thus $v_{d-2} \in S'$. Then $S' \cup \{v_{d-1}\}$ is a 1FD-set in $G$ of cardinality at most
(4m − 3n + 3)/2 − 1 and so \(fd_1(G) \leq (4m − 3n + 3)/2 − 1\), a contradiction. Thus assume that \(\deg_G(v_{i-1}) \geq 3\). Clearly any vertex of \(N_G(v_{i-1}) − \{v_{i-2}\}\) is a leaf. Let \(G'\) be obtained from \(G\) by removing all leaves adjacent to \(v_{i-1}\). Clearly \(r(G') = r\). By the choice of \(G\), \(fd_1(G') \leq (4m(G') − 3n(G') + 3)/2 − r(G') ≤ (4(m − 2) − 3(n − 2) + 3)/2 − 1 = (4m − 3n + 3)/2 − 2\). Let \(S'\) be a \(fd_1(G')\)-set. If \(v_{i-1} \in S'\), then \(S'\) is a \(1FD\)-set in \(G\) of cardinality at most \((4m − 3n + 3)/2 − 2\) and so \(fd_1(G) \leq (4m − 3n + 3)/2 − 2\), a contradiction. Thus assume that \(v_{i-1} \notin S'\). Then \(v_{i-2} \in S'\). Now \(S' \cup \{v_{i-1}\}\) is a \(1FD\)-set in \(G\) of cardinality at most \((4m − 3n + 3)/2 − 1\) and so \(fd_1(G) \leq (4m − 3n + 3)/2 − 1\), a contradiction.

We next assume that \(d = 1\). Let \(D_1 = \{c_j | \deg_G(c_j) = 2\}\) and \(D_2 = \{c_j | \deg_G(c_j) \geq 3\}\). Clearly \(|D_1| = 2\) and \(|D_2| = l_1 + 1\). Thus assume that \(|D_2| \geq 1\). By Claims 1 and 2, \(|D_1| \leq |D_3|\). Observe that \(m = \frac{1}{2} \sum_{v \in V(G)} \deg(v) \geq n + |D_3|/2\). Clearly \(n \geq l_1 + 1 + |D_2|\). Thus

\[
(4m − 3n + 3)/2 − 1 \geq (4(n + |D_3|)/2 − 3n + 3)/2 − 1 \\
\geq (l_1 + 1 + |D_2| + 2|D_3| + 3)/2 − 1 \\
\geq (l_1 + 1 + |D_1| + |D_2| + |D_3| + 3)/2 − 1 \\
= l_1 + 3/2 > l_1 + 1.
\]

Evidently, \(\{c_1, \ldots, c_l\}\) is a \(fd_1(G)\)-set of cardinality \(l_1 + 1\). Thus \(fd_1(G) < (4m − 3n + 3)/2 − r\), a contradiction.

**Case 2.** \(r \geq 2\). By Observation 1, \(G\) has at least two leaf-blocks. Let \(K_i\) be a leaf-block of \(G\), where \(i \in \{1, 2, \ldots, r\}\). By relabeling of the vertices of \(C^i\) we may assume that \(c_0\) is a special cut-vertex of \(G\). Let \(G'\) be the graph obtained by removal of all edges \(c_0c_j\), with \(c_j \in \{c_1, \ldots, c_l\}\). Clearly \(G'\) has two components. Let \(G_1'\) be the component of \(G'\) containing \(c_0\), and \(G_2'\) be the component of \(G'\) containing \(c_0\). Clearly, \(\{c_1, c_2, \ldots, c_l\} \subseteq V(G_1')\). We consider the following subcases.

**Subcase 2.1.** \(V(G_1') = \{c_1, c_2, \ldots, c_l\}\). Let \(G_1'^* = G[V(G_1') \cup \{c_0\}]\). Clearly \(n(G_1'^*) = l_i + 1\). By Claim 1, at least \(l_i/2\) vertices of \(C^i − c_0\) are of degree at least 3.

Assume that \(l_i\) is even. Thus at least \(l_i/2\) vertices of \(C^i − c_0\) are of degree at least 3. Now, we can easily see that \(m(G_1'^*) = \frac{1}{2} \sum_{v \in V(G_1')} \deg(v) \geq l_i + 1 + l_i/4\). Let \(G_2'^* = G[V(G_2') \cup \{c_1, c_2\}]\) \(− \{c_1, c_1\}\). Clearly \(n = n(G_2'^*) + l_i − 2\), \(m = m(G_2'^*) + m(G_1'^*) − 2\) and \(r(G_2'^*) = r − 1\). By the choice of \(G\), \(fd_1(G_2'^*) \leq (4m(G_2'^*) − 3n(G_2'^*) + 3)/2 − r(G_2'^*)\). Let \(S''\) be a \(fd_1(G_2'^*)\)-set. By Observation 2, \(c_0 \in S''\), since \(c_0\) is a strong support vertex of \(G_2'^*\). Then \(S'' \cup \{c_1, c_2, \ldots, c_l\}\) is
a 1FD-set for \( G \) of cardinality \( |S''| + l_i \). On the other hand

\[
\frac{(4m - 3n + 3)}{2} - r \\
\geq (4(m(G^*_2) + m(G^*_1) - 2) - 3(n(G^*_2) + n(G^*_1) - 3) + 3)/2 - r \\
= (4m(G^*_2) - 3n(G^*_2) + 3)/2 - r(G^*_2) + (4m(G^*_1) - 3(l_i + 1) + 1)/2 - 1 \\
\geq |S''| + (4(l_i + 1 + l_i/4) - 3l_i - 2)/2 - 1 = |S''| + l_i.
\]

Thus \( fd_1(G) \leq (4m - 3n + 3)/2 - r \), a contradiction.

Assume next that \( l_i \) is odd. Observe that at least \((l_i - 1)/2\) vertices of \( C^i - c_0^i \) are of degree at least 3. Now, we can easily see that \( m(G^*_1) = \frac{1}{2}\sum_{v \in V(G^*_1)} \deg(v) \geq l_i + 1 + (l_i - 1)/4 \). We show that \( m(G^*_1) = l_i + 1 + (l_i - 1)/4 \). Suppose that \( m(G^*_1) > l_i + 1 + (l_i - 1)/4 \). Then \( m(G^*_1) \geq l_i + 1 + (l_i - 1)/4 + 1/4 \). Let \( G^*_2 = G'[G^*_1 \cup \{c_i, c_i^1\}] \). Clearly \( n = n(G^*_2) + l_i - 2 \), \( m = m(G^*_2) + m(G^*_1) - 2 \) and \( r(G^*_2) = r - 1 \). By the choice of \( G \), \( fd_1(G^*_2) \leq (4m(G^*_2) - 3n(G^*_2) + 3)/2 - r(G^*_2) \).

Let \( S'' \) be a \( fd_1(G^*_2) \)-set. By Observation 2, \( c_0^i \in S'' \), since \( c_0^i \) is a strong support vertex of \( G^*_2 \). Then \( S'' \cup \{c_1^i, c_2^i, \ldots, c_{l_i}^i\} \) is a 1FD-set for \( G \) of cardinality \( |S''| + l_i \). On the other hand

\[
\frac{(4m - 3n + 3)}{2} - r \\
\geq (4(m(G^*_2) + m(G^*_1) - 2) - 3(n(G^*_2) + n(G^*_1) - 3) + 3)/2 - r \\
= (4m(G^*_2) - 3n(G^*_2) + 3)/2 - r(G^*_2) + (4m(G^*_1) - 3(l_i + 1) + 1)/2 - 1 \\
\geq |S''| + (4(l_i + 1 + (l_i - 1)/4 + 1/4) - 3l_i - 2)/2 - 1 = |S''| + l_i.
\]

Thus \( fd_1(G) \leq (4m - 3n + 3)/2 - r \), a contradiction. We thus obtain that \( m(G^*_1) = l_i + 1 + (l_i - 1)/4 \). Note that \( |E(G^*_1) \cap E(C^i)| = l_i + 1 \). Hence \( |E(G^*_1) - E(C^i)| = (l_i - 1)/4 \). Since \((l_i - 1)/2\) vertices of \( C^i - c_0^i \) are of degree at least 3, we thus obtain that precisely \((l_i - 1)/2\) vertices of \( C^i - c_0^i \) are of degree 3, and so \((l_i + 1)/2\) vertices of \( C^i - c_0^i \) are of degree two. Now Claim 1 implies that \( \deg_G(c^i_j) = \deg_G(c^i_{j+1}) = 2 \).

Thus we obtain that \( \deg_{G(C)}(c^i_0) = 2 \). Let \( A_1 = \{c_j : \deg_G(c^i_j) = 2 \text{ for } 1 \leq j \leq l_i\} \) and \( A_2 = \{c^i_1, c^i_2, \ldots, c^i_{l_i}\} - A_1 \). Clearly \( |A_1| = (l_i + 1)/2 \) and \( |A_2| = (l_i - 1)/2 \). Note that \( |A_2| \) is even, since the number of odd vertices in every graph (here \( G^*_1 \) is even. Thus \( |A_1| \) is odd, since \( l_i \) is odd and \( |A_1| + |A_2| = l_i \). Then \( |A_1| \geq 3 \), since \( c^i_1, c^i_{l_i} \in A_1 \). Now Claim 1 implies that \( A_1 = \{c^i_1, c^i_3, \ldots, c^i_{l_i/2+1}\} \) and \( A_2 = \{c^i_2, c^i_4, \ldots, c^i_{l_i-1}\} \).

**Fact 1.** There are two adjacent vertices \( c^i_s, c^i_t \in A_2 \) such that \( |s - t| = 2 \).

**Proof.** Note that \( l_i \equiv 1 \pmod{4} \), since \( l_i \equiv l_i - 1 \equiv 4 \) is even. If \( l_i = 5 \), then \( c^i_2, c^i_4 \in A_2 \) are the desired vertices, since they are the only vertices of \( G^*_1 \) of degree three. Thus assume that \( l_i \geq 9 \). If \( \{c^i_{l_i+1}, c^i_{l_i+1+3}\} \cap N(c^i_{l_i+1-1}) \neq \emptyset \), then the desired pairs
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are $c_l^{li+1}$ and the vertex of $\{c_l^{li+1+1}, c_l^{li+1-3}\} \cap N(c_l^{li+1-1})$. Thus assume that
$\{c_l^{li+1+1}, c_l^{li+1-3}\} \cap N(c_l^{li+1-1}) = \emptyset$. Clearly there is a vertex $c_l^t \in A_2$ such that $c_l^t$ is adjacent to $c_l^{li+1-1}$. Without loss of generality, assume that $t < \frac{h+1}{2} - 3$. Since $G$ is an outerplanar graph, $|A_2 \cap \{c_l^t : t+2 \leq h \leq \frac{h+1}{2} - 3\}|$ is even. Furthermore, since $G$ is an outerplanar graph, any vertex of $A_2 \cap \{c_l^t : t+2 \leq h \leq \frac{h+1}{2} - 3\}$ is adjacent to a vertex of $A_2 \cap \{c_l^t : t+2 \leq h \leq \frac{h+1}{2} - 3\}$. Consequently, there are two pairs $c_{h_1}^t, c_{h_2}^t \in A_2 \cap \{c_l^t : t+2 \leq h \leq \frac{h+1}{2} - 3\}$ such that $c_{h_1}^t \in N(c_{h_2}^t)$ and $|h_1 - h_2| = 2$.

Let $c_l^t$ and $c_{l+2}^t$ be two adjacent vertices of $A_2$ according to Fact 1. Clearly, $\deg(c_{l+2}^t) = 2$. Let $G^* = G - c_l^{l+1} - c_{l+2}^t$. Clearly $n(G^*) = n, m(G^*) = m - 2$ and $r - 1 \leq r(G^*) \leq r$. By the choice of $G$, $fd_1(G^*) \leq (4m(G^*) - 3n(G^*) + 3)/2 - r(G^*) \leq (4m - 3n + 3)/2 - r - 3$. Let $S^*$ be a $fd_1(G^*)$-set. Since $c_{l+2}^t$ is a strong support vertex of $G^*$, by Observation 2, we have $c_{l+2}^t \in S^*$. If $c_{l-1}^t \notin S^*$, then $S^*$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r - 3$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r - 3$, a contradiction. Thus $c_{l-1}^t \in S^*$. Then $S^* \cup \{c_{l+1}^t, c_{l+2}^t\}$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r - 1$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r - 1$, a contradiction.

Subcase 2.2. $V(G_1') \neq \{c_1^*, c_2^*, \ldots, c_n^*\}$. Since $K_1$ is a leaf-block of $G$, $G_1' - C_i$ has some vertex of degree at most one. Let $v_d$ be a leaf of $G_1^*$ such that $d(v_d, C_i - c_i^*)$ is as maximum as possible, and the shortest path from $v_d$ to $C_i$ does not contain $c_0^*$. Let $v_0v_1 \cdots v_d$ be the shortest path from $v_d$ to a vertex $v_0 \in C_i$.

Suppose that $d \geq 2$. Assume that $\deg_{G_1}(v_{d-1}) = 2$. Let $G' = G - \{v_d, v_{d-1}\}$. Clearly $r(G') = r$. By the choice of $G$, $fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') = (4m - 2 - 3(n - 2) + 3)/2 - r = (4m - 3n + 3)/2 - r - 1$. Let $S'$ be a $fd_1(G')$-set. If $v_{d-2} \notin S'$, then $S' \cup \{v_d\}$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. Thus $v_{d-2} \in S'$. Then $S' \cup \{v_{d-1}\}$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction.

We deduce that $\deg_{G_1}(v_{d-1}) \geq 3$. Clearly any vertex of $N_{G_1}(v_{d-1}) - \{v_{d-2}\}$ is a leaf. Let $G'$ be obtained from $G$ by removing all leaves adjacent to $v_{d-1}$. Clearly $r(G') = r$. By the choice of $G$, $fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') \leq (4m - 2 - 3(n - 2) + 3)/2 - r = (4m - 3n + 3)/2 - r - 1$. Let $S'$ be a $fd_1(G')$-set. If $v_{d-1} \notin S'$, then $S'$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r - 1$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r - 1$, a contradiction. Thus $v_{d-1} \notin S'$. Then $v_{d-2} \in S'$. Now $S' \cup \{v_{d-1}\}$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction.

We thus assume that $d = 1$. Let $D_1 = \{c_j^t \mid \deg_{G_1}(c_j^t) = 2\}, D_2 = \{c_j^t \mid c_j^t$
is a support vertex of $G$} and $D_3 = \{c^j_i | \deg_G(c^j_i) \geq 3 \text{ and } c^j_i \text{ is not a support vertex of } G \}$. Clearly $|D_1| + |D_2| + |D_3| = l_i$. Observe that $|D_2| \geq 1$, since $d = 1$. Thus by Claims 1 and 2, $|D_1| \leq |D_3|$. Let $G^*_1 = G[G'_1 \cup \{c^0_i\}]$. Observe that $m(G^*_1) = \frac{1}{2} \sum_{v \in V(G'_1)} \deg(v) \geq n(G^*_1) + |D_3| / 2$. Then $n(G^*_1) \geq l_i + 1 + |D_2|$. Let $G^*_2 = \{\{G'_2 \cup \{c^1_i, c^2_i, \ldots, c^l_i\}\} + c^i_0\}$. Clearly $n = n(G^*_2) + n(G^*_1) - 3$, $m = m(G^*_2) + m(G^*_1) - 2$ and $r(G^*_2) = r - 1$. By the choice of $G$, $fd_1(G^*_2) \leq (4m(G^*_2) - 3n(G^*_2) + 3)/2 - r(G^*_2)$. Let $S''$ be a $fd_1(G^*_2)$-set. By Observation 2, $c^i_0 \in S''$, since $c^i_0$ is a strong support vertex of $G^*_2$. Then $S'' \cup \{c^1_i, c^2_i, \ldots, c^l_i\}$ is a 1FD-set for $G$ of cardinality $|S''| + l_i$. On the other hand

\[
(4m - 3n + 3)/2 - r \\
\geq (4(m(G^*_2) + m(G^*_1) - 2) - 3(n(G^*_2) + n(G^*_1) - 3) + 3)/2 - r \\
= (4m(G^*_2) - 3n(G^*_2) + 3)/2 - r(G^*_2) + (4m(G^*_1) - 3n(G^*_1) + 1)/2 - 1 \\
\geq |S''| + (4(n(G^*_1) + |D_3|/2) - 3n(G^*_1) + 1)/2 - 1 \\
= |S''| + (n(G^*_1) + 2|D_3| + 1)/2 - 1 \\
\geq |S''| + (l_i + 1 + |D_2| + 2|D_3| + 1)/2 - 1 \\
\geq (l_i + |D_2| + |D_3| + |D_1|)/2 \geq |S''| + l_i.
\]

Thus $fd_1(G) \leq |S''| + l_i \leq (4m - 3n + 3)/2 - r$, a contradiction.

To the sharpness, consider a cycle $C_5$.

\[\text{3. Concluding Remarks}\]

As it is noted, Caro et al. [1] proved that $fd(G) < 17n/19$ for any maximal outerplanar graph $G$ of order $n$. They also proved that $fd(G) \leq n - 2$ for any connected graph $G$ of order $n \geq 3$. It is worth-noting that the bound of Theorem 5 improves the bound $n - 2$ when $4m < 5n + 2r - 7$. It is also known that every maximal outerplanar graph $G$ of order at least 3 is 2-connected [7], and thus $r(G) = 1$. Therefore, the bound of Theorem 5 improves the bound $17n/19$ when $4m < \frac{9n}{19} - 1$. We have the following conjecture.

**Conjecture 6.** If $G$ is a graph of order $n$ and size $m$ with $r \geq 1$ strong-blocks, then $fd(G) \leq (4m - 3n + 3)/2 - r$.

\[\text{References}\]


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