ANTIMAGIC LABELING OF SOME BIREGULAR
BIPARTITE GRAPHS

KECAI DENG

School of Mathematical Sciences
Huaqiao University
Quanzhou 362000, Fujian, P.R. China
e-mail: kecaideng@126.com

AND

YUNFEI LI

School of Accounting and Finance
Xiamen University Tan Kah Kee College
Zhangzhou 363000, Fujian, P.R. China
e-mail: lyfdkc@xujc.com

Abstract

An antimagic labeling of a graph $G = (V,E)$ is a one-to-one mapping
from $E$ to $\{1, 2, \ldots, |E|\}$ such that distinct vertices receive different label
sums from the edges incident to them. $G$ is called antimagic if it admits
an antimagic labeling. It was conjectured that every connected graph other
than $K_2$ is antimagic. The conjecture remains open though it was verified
for several classes of graphs such as regular graphs. A bipartite graph is
called $(k,k')$-biregular, if each vertex of one of its parts has the degree $k$,
while each vertex of the other parts has the degree $k'$. This paper shows
the following results. (1) Each connected $(2,k)$-biregular ($k \geq 3$) bipartite
graph is antimagic; (2) Each $(k,pk)$-biregular ($k \geq 3, p \geq 2$) bipartite graph
is antimagic; (3) Each $(k,k^2 + y)$-biregular ($k \geq 3, y \geq 1$) bipartite graph is
antimagic.

Keywords: antimagic labeling, bipartite, biregular.

2010 Mathematics Subject Classification: 05C69.

1Corresponding author.
1. Introduction

Let $G = (V,E)$ be a graph. Suppose $f$ is a one-to-one mapping from $E$ to \{1,2,\ldots,|E|\}. For each vertex $v$ in $V$, the vertex sum $\varphi_f(v)$ at $v$ under $f$ is defined as $\varphi_f(v) = \sum_{e \in E(v)} f(e)$, where $E(v)$ is the set of edges incident to $v$. If $\varphi_f(u) \neq \varphi_f(v)$ for any vertex pair $u,v \in V$, then $f$ is called an antimagic labeling of $G$. A graph $G$ is called antimagic if $G$ admits an antimagic labeling. The antimagic labeling of graphs was introduced by Hartsfield and Ringel [8] in 1989 (also in [9]), who verified the antimagicness of paths, 2-regular graphs and complete graphs. Moreover, they put forth the following conjecture.

**Conjecture 1** [9]. Every connected graph other than $K_2$ is antimagic.

The conjecture has received much attention, but remains open. It was proved by Alon et al. [1] that there is an absolute constant $c$ such that graphs with minimum degree $\delta(G) \geq c \log |V|$ are antimagic, and graphs with maximum degree at least $|V| - 2$ and complete bipartite graphs except $K_2$ are antimagic. And then graphs of large linear size were shown to be antimagic [6]. For regular graphs, the antimagicnesses of $k$-regular ($k \geq 3$) bipartite graphs [3], cubic graphs [12], odd degree regular graphs [4], and finally even regular graphs [2] were verified, respectively. For more results on antimagic labeling such as those about trees, one can refer to [5, 10, 11, 13, 14, 17] and the survey of Gallian [7].

A bipartite graph is called $(k,k')$-biregular, if each vertex in one of its two parts has the degree $k$, while each vertex in the other part has the degree $k'$. This paper shows the following results. (1) Each connected $(2,k)$-biregular ($k \geq 3$) bipartite graph is antimagic; (2) Each $(k,pk)$-biregular ($k \geq 3$, $p \geq 2$) bipartite graph is antimagic; (3) Each $(k,k^2+y)$-biregular ($k \geq 3$, $y \geq 1$) bipartite graph is antimagic. The first result is shown in Section 2, where we treat each connected $(2,k)$-biregular ($k \geq 3$) bipartite graph as the subdivision graph of a connected $k$-regular graph. A subdivision graph $G_s$ of a graph $G$, is obtained from $G$ by replacing each edge with a path of length two. The second and the third results are shown in Section 3, based on an extended version of Hall’s matching theorem [15, 16].

2. Connected $(2,k)$-Biregular ($k \geq 3$) Bipartite Graph

With respect to a given labeling, two vertices are in conflict if they have a common vertex sum. When we have labeled a subset of the edges, we call the resulting sum at each vertex a partial vertex-sum. For short, we denote by $[i,j]$ the integer set \{i, i+1, \ldots, j\} for integers $i$ and $j$ (where $i < j$).

**Theorem 2.** The subdivision graph $G_s$ of every connected $k$-regular ($k \geq 3$) graph $G$ is antimagic.
Proof. Choose an arbitrary vertex $v^*$ in $G$ as a root. Let $\alpha$ be the longest distance of a vertex from $v^*$ in $G$. Suppose $i \in [1, \alpha]$. Denote by $V_i$ the sets of vertices at distance exactly $i$ from $v^*$, by $G[V_i]$ the subgraph induced by $V_i$, and by $G[V_{i-1}; V_i]$ (here we suppose $V_0 = \{v^*\}$) the induced bipartite subgraph with parts $V_{i-1}$ and $V_i$, respectively. For $v \in V_i$, let $\sigma(v)$ be an arbitrary edge in $G[V_{i-1}; V_i]$ which is incident to $v$. Let $\sigma(V_i) = \{\sigma(v) \mid v \in V_i\}$ and $G_\sigma[V_{i-1}; V_i] = G[V_{i-1}; V_i] \setminus \sigma(V_i)$.

Now subdivide $G$ into $G_s$. Then every vertex in $V_i$ is at distance exactly $2i$ from $v^*$ in $G_s$. Denote by $S_i, U_i$ and $W_i$ the newly added vertex sets on the edges of $G[V_i], G_\sigma[V_{i-1}; V_i]$ and $\sigma(V_i)$, respectively, when subdividing $G$ into $G_s$. Let $X = \bigcup_{i=1}^{\alpha} X_i$ for $X = V, S, U, W$. For a vertex $v \in V_i$, let $w(v)$ be the vertex in $W_i$ which is adjacent to $v$. For every vertex $x \in (S_i \cup U_i \cup W_i)$, let $x^+$ and $x^-$ be the two edges incident to $x$. If $x \in (U_i \cup W_i)$, we suppose $x^+$ is incident to some vertex in $V_i$, while $x^-$ is incident to some vertex in $V_{i-1}$. For $X = S, U, W$, let $E_X^i = \{x^+ \mid x \in X_i\}, \ \overline{E}_X^i = \{x^- \mid x \in X_i\}$ and $E^{XY}_i = E_X^i \cup \overline{E}_Y^i$.

Respect to a labeling $f$ on $E(G_s)$, if $v \in V_i$, we denote the partial sum at $v$ (omitting the label on $e^{w(v)}$) by $p(v) = \sum_{e \in E(v) \setminus \{e^{w(v)}\}} f(e) = \varphi_f(v) - f(e^{w(v)})$. Let $p(v^*) = \varphi_f(v^*) - f(e^*)$ where $e^*$ is the edge in $E(v^*)$ which receives the greatest label among $E(v^*)$.

Note that $V(G_s) = V \cup S \cup U \cup W \cup \{v^*\}$. To show $G_s$ is antimagic, we will construct a labeling $f$ which satisfies the following conditions.

(1) The vertex sums in $X_i$ are all odd and pairwise different, for $X \in \{S, U, W\}$ and $i \in [1, \alpha]$.

(2) The vertex sums in $V_i$ are all even and pairwise different for $i \in [1, \alpha]$.

(3) The vertex sums in $(S_i \cup U_i \cup W_i)$ are smaller than those in $(S_{i-1} \cup U_{i-1} \cup W_{i-1})$ for $i \in [2, \alpha]$.

(4) The vertex sums in $S_i$ are smaller than those in $U_i$, while the later ones are smaller than those in $W_i$ for $i \in [1, \alpha]$.

(5) The vertex sums in $V_i$ are smaller than those in $V_{i-1}$ for $i \in [2, \alpha]$.

(6) The vertex sum at $v^*$ is greater than those in $V_i$ and those in $W_i$.

Conditions (1) and (2) make sure there is no conflict between $V$ and $(S \cup U \cup W)$. Conditions (1), (3), (4) make sure there is no conflict inside $(S \cup U \cup W)$. Conditions (2) and (5) make sure there is no conflict inside $V$. Conditions (3), (4), (5) and (6) make sure there is no conflict between $v^*$ and any other vertex in $G_s$. So these conditions imply that $f$ is antimagic.

Note that $E(G_s) = \bigcup_{i=1}^{\alpha} (E^S_i \cup E^U_i \cup E^W_i)$. We will label $E(G_s)$ in the order $E^S_\alpha$, $(E^S_\alpha \cup E^U_\alpha)$, $E^S_{\alpha-1}$, $(E^U_{\alpha-1} \cup E^W_{\alpha-1})$, $E^S_1$, $(E^U_1 \cup E^W_1)$, using the smallest unused labels on each edge set when we come to it. This label assignment immediately implies that (3) holds, and that the vertex sums in $S_i$ are smaller than those in $(U_i \cup W_i)$ for $i \in [1, \alpha]$.

Suppose $i \in [1, \alpha]$ in the following. Note that $|E^S_i| = 2|X_i|$, for $X = S, U, W$. Antimagic LABELING OF SOME BIREGULAR BIPARTITE GRAPHS 3
(I) The labeling of $E^S$. We first label $E^S_i$ arbitrarily using the $|S_i|$ odd labels from the $2|S_i|$ assigned labels for $E^S_i$. Secondly let $f(\pi^s) = f(\alpha^s) + 1$ for each $s \in S_i$. Then the vertex sums in $S_i$ are odd and pairwise different.

(II) The labeling of $(E^U \cup E^W)$. If $|U_i|$ is odd, then $i \in [2, \alpha]$, since $U_1$ is an empty set. We will label $(E^U_i \cup E^W_i)$ in the order $E^U_i$, $E^U_i$, $E^W_i$ using the smallest unused assigned labels on each edge subset when we come to it. This sub-assignment (based on our global assignment), gives that $p(v) < p(v')$ for arbitrary $v \in V_i$ and $v' \in V_{i-1}$, which implies $\varphi_f(v) = p(v) + f(wv(v)) < p(v') + f(wv(v')) = \varphi_f(v')$, since $f(wv(v)) < f(wv(v'))$ by our global assignment. So (5) holds for those $i$ with $|U_i|$ being odd. It gives that the vertex sums in $U_i$ are smaller than those in $W_i$. So (4) holds for those $i$ with $|U_i|$ being odd. We first label $E^U_i$ arbitrarily using its assigned labels. Secondly let $f(\pi^u) = f(\alpha^u) + |U_i|$ for each $u \in U_i$. This gives that the vertex sums in $U_i$ are odd and pairwise different. Third, suppose $V_i = \{v_1, v_2, \ldots, v_{|V_i|}\}$ where $p(v_1) \leq p(v_2) \leq \cdots \leq p(v_{|V_i|})$. For $r \in [1, |V_i|]$, label $\alpha^{w_{(v_r)}}$ with the $r$-th smallest label among the odd (even) assigned labels for $E^W_i$, when $p(v_r)$ is odd (even). This implies that the vertex sums in $V_i$ are even and pairwise different. So (2) holds for those $i$ with $|U_i|$ being odd. Fourth, let $f(\pi^w) = f(\alpha^w) + 1$ when $f(\alpha^w)$ is odd, while $f(\pi^w) = f(\alpha^w) - 1$ when $f(\alpha^w)$ is even. This implies that vertex sums in $W_i$ are odd and pairwise different. So (1) holds for those $i$ with $|U_i|$ being odd.

If $|U_i|$ is even ($|U_i|$ may equal to 0), then $i \in [1, \alpha]$. We will label edges in $E^U_i$ using the smallest $(2|U_i| + 1)$ assigned labels for $E^U_i \cup E^W_i$ except the $(|U_i| + 1)$-th smallest one (denoted by $\xi_{|U_i|+1}$). We first label the edges of $E^U_i$ arbitrarily using the $|U_i|$ smallest assigned labels. This gives that $p(v) < p(v')$ for arbitrary $v \in V_i$ and $v' \in V_{i-1}$. And then, if $i \neq 1$, one has $\varphi_f(v) = p(v) + f(wv(v)) < p(v') + f(wv(v')) = \varphi_f(v')$ for arbitrary $v \in V_i$ and $v' \in V_{i-1}$, since $f(wv(v)) < f(wv(v'))$ by our global assignment. So (5) also holds for those $i$ ($i \neq 1$) with $|U_i|$ being even. Secondly let $f(\pi^u) = f(\alpha^u) + |U_i| + 1$ for each $u \in U_i$. This implies that the vertex sums in $U_i$ are odd and pairwise different. It also implies that the vertex sums in $U_i$ are smaller than those in $W_i$, since any pair of the rest assigned labels left for $W_i$ has a sum greater than any vertex sum in $U_i$. So (4) also holds for those $i$ with $|U_i|$ being even. Note that, $\xi_{|U_i|+1}$ and $(\xi_{|U_i|+1} + |U_i| + 1)$ have distinct parity, and so far, they are the smallest two unused assigned labels for $W_i$. Third, suppose $|V_i| = \{v_1, v_2, \ldots, v_{|V_i|}\}$ where $p(v_1) \leq p(v_2) \leq \cdots \leq p(v_{|V_i|})$. For $r \in [1, |V_i|]$, label $\alpha^{w_{(v_r)}}$ with the $r$-th smallest label among the rest odd (even) assigned labels, if $p(v_r)$ is odd (even). This implies that the vertex sums in $V_i$ are even and pairwise different. So (2) also holds for those $i$ with $|U_i|$ being even. And note that either $\xi_{|U_i|+1}$ or $(\xi_{|U_i|+1} + |U_i| + 1)$ is assigned to $w(v_1)$ by our labeling way. Fourth, let $f(\pi^{w_{(v_1)}}) = \xi_{|U_i|+1}$ if $f(\alpha^{w_{(v_1)}}) = \xi_{|U_i|+1} + |U_i| + 1$, while $f(\pi^{w_{(v_1)}}) = \xi_{|U_i|+1} + |U_i| + 1$ if
And for the following corollary.

And for each $|E(v)| + 1$ if $f(e)$ is odd, while $f(E(v)) = f(E(v)) - 1$ if $f(E)$ is even. This implies that vertex sums in $W_i$ are odd and pairwise different. So (1) also holds for those $i$ with $|U_i|$ being even.

For (6), note that the process of the labeling of $E(v^*) = E^W_1$ is discussed in the case when $|U_i|$ is even (since $U_1 = \emptyset$ and $|U_1| = 0$). Recall that, $|E^W_1| = 2k$ and $E^W_i$ are assigned with the greatest $2k$ labels, i.e., those labels in $L_{2k} = \{ |E(G_s)| - 1, \ldots, |E(G_s)| - 2k + 1 \}$. More precisely, $E^W_i = E(v^*)$ are assigned with the labels in $\{ i_1, i_2, \ldots, i_k \} \subseteq L_{2k}$ where either $i_j = |E(G_s)| - 2j + 1$ or $i_j = |E(G_s)| - 2j + 2$ for $j = 1, 2, \ldots, k$. So $p(v^*) \geq p(v_1) + 1 + 3 \cdot \ldots \cdot (2k - 3) > p(v_1) + 3$ for arbitrary $v_1 \in V_1$ (recall that $k \geq 3$). Then $\varphi_f(v^*) = p(v^*) + f(e^*) \geq p(v^*) + |E(G_s)| - 1 > p(v_1) + |E(G_s)| - 2 > p(v_1) + |E(G_s)| \geq p(v_1) + f(v_1, w(v_1)) = \varphi_f(v_1)$ for each $v_1 \in V_1$. On the other hand, $\varphi_f(v^*) \geq (|E(G_s)| - 1) + (|E(G_s)| - 3) + (|E(G_s)| - 5) = 3|E(G_s)| - 9$, since $k \geq 3$. Thus, each vertex in $W_1$ receives a sum at most $2|E(G_s)| - 1$. So $\varphi_f(v^*) \geq 3|E(G_s)| - 9 > 2|E(G_s)| - 1 \geq \varphi_f(w_1)$ for each $w_1 \in W_1$ (one has $|E(G_s)| \geq 12$, because $k \geq 3$). So (6) holds.

Thus, $G_s$ is antimagic. This completes our proof.

It is interesting to consider the case when $G$ is $k$-regular ($k \geq 3$) but disconnected. In the proof of Theorem 2, suppose $G$ has $m$ edges. Then $G_s$ has $m$ 2-vertices. Note that the total sum of all the vertex sums is even, since each label contributes to the total sum twice. Thus, each 2-vertex contributes an odd value to the total sum, while each $k$-vertex other than $v^*$ contributes an even value, under our labeling way in the proof of Theorem 2. Thus, $\varphi_f(v^*)$ is odd if and only if $m$ is odd.

**Theorem 3.** Let $G$ be an disconnected $k$-regular ($k \geq 3$) graph, which has at most one connected component with an odd number of edges. Then $G_s$ is antimagic.

**Proof.** Suppose $G$ consists of the connected components $H_1, H_2, \ldots, H_\beta$ ($\beta \geq 2$), where $H_i$ has an even number of edges for each $i \in [1, \beta - 1]$. We can label $E(G_s)$ in the order $E((H_1)_s), E((H_2)_s), \ldots, E((H_\beta)_s)$ using the smallest unused labels on each edge set when we come to it. Next, we label each connected component of $G_s$ in the same way to that in Theorem 2, choosing a root for each component of $G$. Then there is no conflict among each $(H_i)_s$ for $i \in [1, \beta]$. Each 2-vertex receives an odd sum, while each $k$-vertex other than the root of $(H_\beta)_s$ receives an even sum. Each 2-vertex in $(H_i)_s$ receives a smaller sum than each 2-vertex in $(H_j)_s$, while each $k$-vertex in $(H_i)_s$ receives a smaller sum than each $k$-vertex in $(H_j)_s$, whenever $i < j \leq \beta$ holds. And the root vertex in $(H_\beta)_s$ receives a greater sum than those of any other vertex in $G_s$. So we obtain an antimagic labeling.

Since $m = \frac{n^2}{2}$, for each $k$-regular graph with $n$ vertices and $m$ edges, we have the following corollary.
Corollary 4. Let $G$ be an disconnected $k$-regular ($k \geq 3$) graph. Then $G_s$ is antimagic if one of the following holds.

1. $k = 4t$ ($t \geq 1$);
2. $k$ is even and at most one of the connected components of $G$ has an odd number of vertices;
3. At most one of the connected components of $G$ has a number of vertices which is not a multiple of 4.

3. $(k, pk)$-Biregular ($k \geq 3, p \geq 2$) Bipartite Graph

For a bipartite graph $G(A, B)$, a complete $p$-claw matching $CM$ from $A$ to $B$ is a set of edges of $G$ that induce a subgraph $G[CM]$ such that each vertex of $A$ in $G$ is also a vertex in $G[CM]$ and each component of $G[CM]$ is a copy of $K_{1, p}$ where the vertex of degree $p$ is in $A$, while the vertices of degree 1 are in $B$. For $A_0 \subseteq A$, denote by $N(A_0)$ the set of vertices in $B$ each of which has a neighbor in $A_0$. Let $E_1, E_2, \ldots, E_k \subseteq E(G)$ be disjoint edge sets. If $E_1 \cup E_2 \cup \cdots \cup E_k = E(G)$, then we say $G$ decomposes into $E_1, E_2, \ldots, E_k$.

Lemma 5 (An extended version of Hall’s theorem, [15, 16]). A bipartite graph $G[A, B]$ admits a complete $p$-claw matching from $A$ to $B$, if and only if $p|A_0| \leq |N(A_0)|$ for every subset $A_0$ of $A$.

Lemma 6. Let $G[A, B]$ be a $(k, pk)$-biregular ($k \geq 3, p \geq 2$) bipartite graph where the degree of each vertex in $A$ is $kp$, while each vertex in $B$ has degree $k$. Then $G$ decomposes into $k$ complete $p$-claw matchings from $A$ to $B$.

Proof. Let $A_0 \subseteq A$. Let $G[A_0, N(A_0)]$ be the graph induced by $A_0 \cup N(A_0)$. Then each vertex of $A_0$ in $G[A_0, N(A_0)]$ has the degree $kp$, while each vertex of $N(A_0)$ in $G[A_0, N(A_0)]$ has the degree at most $k$. So there are exactly $kp|A_0|$ edges in $G[A_0, N(A_0)]$. On the other hand, suppose $|N(A_0)| < p|A_0|$. Then the number of edges in $G[A_0, N(A_0)]$ is less than $k \cdot p|A_0|$, a contradiction. So $|N(A_0)| \geq p|A_0|$. By Lemma 5, there exists a complete $p$-claw matching $CM_1$ from $A$ to $B$ in $G[A, B]$. Then $G_1 = G[A, B] - CM_1$ is a $(k - 1, p(k - 1))$-biregular bipartite graph. So we can use Lemma 5 repeatedly until we obtain a $(1, p)$-biregular bipartite graph $G_{k-1}$ which is also a complete $p$-claw matching from $A$ to $B$. Thus, $G[A, B]$ decomposes into $k$ complete $p$-claw matchings from $A$ to $B$.

Lemma 7. Let $I = [i + 1, i + 2q]$. Then, there exist partitions $P_1$ (when $q$ is odd) and $\{P_2, P_3, P_4\}$ (when $q$ is even) of $I$, such that under $P_1$, $j \in [1, 4]$, $I$ is divided into $q$ parts where each part has 2 integers, integers in $[i + (x - 1)q + 1, i + xq]$ ($x \in [1, 2]$) are distinct parts and the following conditions are satisfied.
Under $P_1$, the $q$ parts have distinct sums which attain all the values in $[(2i + 2q + 1) - (q - 1)/2, (2i + 2q + 1) + (q - 1)/2]$;

(2) Under $P_2$, $q/2$ parts have distinct sums which attain all the values in $[(2i + 2q + 1) - (q/2 - 1), 2i + 2q + 1]$, while the other $q/2$ parts have distinct sums which attain all the values in $[2i + 2q + 1, (2i + 2q + 1) + (q/2 - 1)]$;

(3) Under $P_3$, the $(q - 1)$ parts have distinct sums which attain all the values in $[(2i + 2q + 2) - (q/2 - 1), (2i + 2q + 2) + (q/2 - 1)]$ and the other part has the sum $2i + q + 2$;

(4) Under $P_4$, the $(q - 1)$ parts have distinct sums which attain all the values in $[(2i + 2q) - (q/2 - 1), (2i + 2q) + (q/2 - 1)]$ and the other part has the sum $2i + 2q$.

**Proof.** It is sufficient to show the case when $i = 0$.

(1) If $q$ is odd, let $\{2j - 1, -j + (3q + 1)/2 + 1\}$ be in the same partition for $j \in [1, (q + 1)/2]$, and let $\{2j, -j + 2q + 1\}$ be in the same partition for $j \in [1, (q - 1)/2]$, which is the desired partition $P_1$.

(2) If $q$ is even, let $\{2j, -j + 3q/2 + 1\}$ be in the same partition and let $\{2j - 1, -j + 2q + 1\}$ be in the same partition for $j \in [1, q/2]$, which is the desired partition $P_2$.

(3) If $q$ is even, let $\{2j, -j + 3q/2 + 2\}$ be in the same partition for $j \in [1, q/2]$, let $\{2j + 1, -j + 2q + 1\}$ be in the same partition for $j \in [1, q/2 - 1]$, and let $\{1, q + 1\}$ be in the same partition, which is the desired partition $P_3$.

(4) If $q$ is even, let $\{2j - 1, -j + 3q/2 + 1\}$ be in the same partition for $j \in [1, q/2]$, let $\{2j, -j + 2q\}$ be in the same partition for $j \in [1, q/2 - 1]$, and let $\{q, 2q\}$ be in the same partition, which is the desired partition $P_4$. 

**Lemma 8.** Let $I = [i + 1, i + zq]$ $(z \geq 3)$. Then, there exist partitions $P_1$ (when $z$ is even or $q$ is odd) and $\{P_2, P_3\}$ (when $z$ is odd and $q$ is even) of $I$, such that under $P_j$, $j \in [1, 3]$, $I$ is departed into $q$ parts where each part has $z$ integers, integers in $[i + (x - 1)q + 1, i + xq]$ $(x \in [1, z])$ are in distinct parts and the following conditions are satisfied.

(1) Under $P_1$, the $q$ parts have the same sum $(2i + zq + 1)z/2$;

(2) Under $P_2$, $q/2$ parts have the same sum $(2i + zq + 1)z/2 + 1/2$ and the other $q/2$ parts have the same sum $(2i + zq + 1)z/2 - 1/2$;

(3) Under $P_3$, $(q - 1)$ parts have the same sum $(2i + zq + 1)z/2 + 3/2$ and the other part has the sum $(2i + zq + 1)z/2 - 3q/2 + 3/2$.

**Proof.** It is sufficient to show the case when $i = 0$.

(1) If $z$ is even, let $\{(j - 1)q + l | j \in [1, z/2]\} \cup \{jq - 1 | j \in [z/2 + 1, z]\}$ be in the partition for $l \in [1, q]$, which is the desired partition $P_1$ and (1) holds in this case.
If $z$ is odd (then $(z - 3)$ is even) and $q$ is odd, we first assign the $(z - 3)q$ integers in $[2q + 1, (z - 1)q]$ to the $q$ parts (suppose $I_1, I_2, \ldots, I_q$ are the $q$ parts) such that these $q$ parts receive the same partial sum $(zq + q + 1)(z - 3)/2$. We can do this since $(z - 3)$ is even. Second, assign $[(z - 1)q + l]$ to $I_l$ for $l \in [1, q]$ such that the $q$ parts have distinct partial sums and attain all values in $[(zq + q + 1)(z - 3)/2 + (z - 1)q + 1, (zq + q + 1)(z - 3)/2 + zq]$. Third, partition $[1, 2q]$ into $q$ parts (denoted by $I_{1}', I_{2}', \ldots, I_{q}'$) which have distinct sums which attain all the values in $[(2q + 1) - (q - 1)/2, (2q + 1) + (q - 1)/2]$. We can do this owing to the partition in Lemma 7(1). Then assign $I_{i}'$ to $I_l$ if the sum of $I_{i}'$ equals to $[(2q + 1) + (q - 1)/2 - l + 1]$ for $l \in [1, q]$. Then the final sum of $I_l$ equals to $[(2q + 1) - (q - 1)/2, (2q + 1) + (q - 1)/2]$ for each $l \in [1, q]$. So (1) also holds in this case.

(2) If $z$ is odd and $q$ is even, we first partition $[2q + 1, zq]$ into $q$ parts $I_1, I_2, \ldots, I_q$ which have distinct partial sums and attain all values in $[(zq + q + 1)(z - 3)/2 + (z - 1)q + 1, (zq + q + 1)(z - 3)/2 + zq]$. We can do this owing to the discussion in (1). Then partition $[1, 2q]$ into $q$ parts (denoted by $I_{1}', I_{2}', \ldots, I_{q}'$) such that $q/2$ parts have distinct sums which attain all the values in $[(2q + 1) - (q/2 - 1), 2q + 1]$, while the other $q/2$ parts have distinct sums which attain all the values in $[2q + 1, (2q + 1) + (q/2 - 1)]$. We can do this owing to the partition in Lemma 7(2). Denote by $I_{q/2, 1}'$ and $I_{q/2, 2}'$ the two parts each of which admits the sum $(2q + 1)$. Then assign $I_{i}'$ to $I_l$ if the sum of $I_{i}'$ equals to $[(2q + 1) + (q/2 - 1) - l + 1]$ for $l \in [1, q/2 - 1]$. Assign $I_{q/2, 1}'$ to $I_{q/2}$, while assign $I_{q/2, 2}'$ to $I_{q/2 + 1}$. And assign $I_{i}'$ to $I_l$ if the sum of $I_{i}'$ equals to $(2q + 1) + (q/2 - 1) - l + 2$ for $l \in [q/2 + 2, q]$. Then for $l \in [1, q/2 - 1]$ the final sum of $I_l$ equals to $[(zq + q + 1)(z - 3)/2 + (z - 1)q + l] + [(2q + 1) + (q/2 - 1) - l + 1] = (zq + 1)z/2 - 1/2$. The final sum of $I_{q/2}$ equals to $[(zq + q + 1)(z - 3)/2 + (z - 1)q + q/2] + [(2q + 1)] = (zq + 1)z/2 - 1/2$, while the final sum of $I_{q/2 + 1}$ equals to $[(zq + q + 1)(z - 3)/2 + (z - 1)q + q/2 + 1] + [(2q + 1)] = (zq + 1)z/2 + 1/2$. Thus, for $l \in [q/2 + 2, q]$ the final sum of $I_l$ equals to $[(zq + q + 1)(z - 3)/2 + (z - 1)q + l] + [(2q + 1) + (q/2 - 1) - l + 2] = (zq + 1)z/2 + 1/2$. So (2) holds.

(3) If $z$ is odd and $q$ is even, we first partition $[2q + 1, zq]$ into $q$ parts $I_1, I_2, \ldots, I_q$ which have distinct partial sums and attain all values in $[(zq + q + 1)(z - 3)/2 + (z - 1)q + 1, (zq + q + 1)(z - 3)/2 + zq]$. We can do this owing to the discussion in (1). Then partition $[1, 2q]$ into $q$ parts (denoted by $I_{1}', I_{2}', \ldots, I_{q}'$) such that the $(q - 1)$ parts have distinct sums which attain all the values in $[(2q + 2) - (q/2 - 1), (2q + 2) + (q/2 - 1)]$ and the other part has the sum $(q + 2)$. We can do this owing to the partition in Lemma 7(3). Denote by $I_{i}'$ the part with the sum $(q + 2)$. Then assign $I_{i}'$ to $I_1$, and assign $I_{i}'$ to $I_l$ if the sum of $I_{i}'$ equals to $[(2q + 2) + (q/2 - 1) - l + 2]$ for $l \in [2, q]$. Then the final sum of $I_l$ equals to $[(zq + q + 1)(z - 3)/2 + (z - 1)q + l] + [q + 2] = (zq + 1)z/2 - 3q/2 + 3/2$, and
for \( l \in [2, q] \), the final sum of \( I_l \) equals to 
\[
\left( (qz + q + 1)(z - 3)/2 + (z - 1)q + q + l \right) + 
\left( (2q + 2) + (q/2 - 1) - l + 2 \right) = (qz + 1)z/2 + 3/2.
\]
So (3) holds.

**Theorem 9.** Every \((k, pk)\)-biregular \((k \geq 3, p \geq 2)\) bipartite graph is antimagic.

**Proof.** Let \( G[A, B] \) be a \((k, pk)\)-biregular \((k \geq 3, p \geq 2)\) bipartite graph, where each vertex in \( A \) has the degree \( pk \), while each vertex in \( B \) has the degree \( k \).

Suppose \(|A| = n \geq k \) and \(|B| = pn \). Let \( A = \{a_1, a_2, \ldots, a_n\} \) and \( B = \{b_1, b_2, \ldots, b_{pn}\} \). By Lemma 6, \( G \) decomposes into \( k \) complete \( p \)-claw matchings \( CM_1, CM_2, \ldots, CM_k \) from \( A \) to \( B \). Denote by \( CM_i(V_0) \) \((i \in [1, k])\) the edges in \( CM_i \) which are incident to some vertex in \( V_0 \) for \( V_0 \subseteq V(G) \).

**Step 1.** Label \( \left( \bigcup_{i=1}^{k-1} CM_i \right) \) with \([1, (k-1)pn]\).

First, label \( CM_{k-1} \) with \([(k-2)pn + 1, (k-1)pn], \) i.e., \([(k-2)pn + 1, (k-2)pn + pn] \) such that the following conditions are satisfied.

1. (1.1) Within \( CM_{k-1} \), vertices in \( A \) have the same partial sum \([(2k-3)pn + 1]/p/2 \) if \( p \) is even or \( n \) is odd. We can do this owing to the partition in Lemma 8(1).

2. (1.2) Within \( CM_{k-1} \), \( n/2 \) vertices in \( A \) have the same partial sum \([(2k-3)pn + 1]/p/2 + 1/2 \) and the other \( n/2 \) vertices in \( A \) have the same partial sum \([(2k-3)pn + 1]/p/2 - 1/2 \) if \( p \) is odd and \( n \) is even. We can do this owing to the partition in Lemma 8(2).

Second, based on the labeling to \( CM_{k-1} \), for each \( i \in [1, k-2] \), label \( CM_i \) with \([(i-1)pn + 1, ipn]\) such that the following conditions are satisfied.

3. (1.3) Within \( \left( \bigcup_{i=1}^{k-1} CM_i \right) \), the vertices in \( B \) have the same partial sum \([(k-1)pn + 1]/(k-1)/2 \) if \((k-1)\) even or \( pn \) is odd. We can do this owing to the partition in Lemma 8(1).

4. (1.4) Within \( \left( \bigcup_{i=1}^{k-1} CM_i \right) \), \( pn - 1 \) vertices in \( B \) have the same partial sum \([(k-1)pn + 1]/(k-1)/2 + 3/2 \) while the other vertex (denoted by \( b_0 \)) has the partial sum \([(k-1)np + 1]/(k-1)/2 + 3/2 - 3pn/2 \) if \((k-1)\) odd and \( pn \) is even. We can do this owing to the partition in Lemma 8(3).

Note that, (1.3) implies the vertices in \( B \) will receive distinct final vertex sums, when \((k-1)\) is even or \( pn \) is odd, if we label the rest edges \( CM_k \) using the rest labels \([(k-1)pn + 1, kpn] \). Thus in (1.4), the partial sum of \( b_0 \) is at least \( 3pn/2 \) smaller than those of the vertices in \((B \setminus \{b_0\})\). So the final vertex sum of \( b_0 \) will still be smaller than those of the vertices in \((B \setminus \{b_0\})\), if we label \( CM_k \) with \([(k-1)pn + 1, kpn] \). Hence, the final vertex sums of in \((B \setminus \{b_0\}) \) will be pairwise different. That is, all vertices in \( B \) will also receive distinct final vertex sums when \((k-1)\) is odd and \( pn \) is even.

**Step 2.** Label \( CM_k \) with \([(k-1)pn + 1, kpn], \) i.e., \([(k-1)pn + 1, (k-1)pn + pn] \).

Suppose \( f_1(a_1) \leq f_1(a_2) \leq \cdots \leq f_1(a_n) \) where \( f_1(a_i) \) is the partial vertex sum of \( a_i \) within \( \left( \bigcup_{i=1}^{k-1} CM_i \right) \) for \( j \in [1, n] \).
If $p$ is odd (then $(p-1)$ is even) or $n$ is odd, let $\sigma(a)$ be an edge in $CM_k(a)$ for each $a \in A$. Label $|CM_k \setminus \bigcup_{a \in A} \{\sigma(a)\}|$ with $[(k-1)pn + 1, (k-1)pn + (p-1)n]$ such that, within $[CM_k \setminus \bigcup_{a \in A} \{\sigma(a)\}]$, the vertices in $A$ have the same partial sum $[(2k-1)pn - n + 1](p-1)/2$. We can do this owing to the partition in Lemma 8(1). Next label $\sigma(a_t)$ with $(kpm - n + j)$ for $j \in [1, n]$. Then the vertex sums in $A$ are pairwise different.

If $p$ is even (then $(p-2)$ is also even) and $n$ is even, let $\sigma_1(a)$ and $\sigma_2(a)$ be two distinct edges in $CM_k(a)$ for each $a \in A$. Label $[CM_k \setminus \bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\}]$ using the labels in $[(k-1)pn + 1, (k-1)pn + (p-2)n]$ such that, within $[CM_k \setminus \bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\}]$, the vertices in $A$ have the same partial sum $[(2k-1)pn - 2n + 1](p-2)/2$. We can do this owing to the partition in Lemma 8(1). Then label $\bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\}$ with $[(kpm - 2n) + 1, (kpm - 2n) + 2n]$ such that $f(\sigma_1(a_t)) + f(\sigma_2(a_t)) = 2kpn - 5n/2 + j$ for $j \in [1, n - 1]$ while $f(\sigma_1(a_t)) + f(\sigma_2(a_t)) = 2kpn - n$. We can do this owing to the partition in Lemma 7(4). Then the vertex sums in $A$ are also pairwise different.

Recall that, owing to Step 1 (1.3) and (1.4), for each $b \in B$, one has

$$\varphi_f(b) \leq \frac{[(k - 1)pn + 1](k - 1)}{2} + \frac{3}{2} + kpn.$$

On the other hand, owing to the labeling way in Step 1 (1.3) and (1.4), the labels assigned to $CM_i$ are those in $[(i-1)pn+1, ipn]$ for $i \in [1, k-2]$. Let $a \in A$. Then the sum of the labels in $CM_i(a)$ is at least $\sum_{j=1}^{p}[(i-1)pn+j]$ for $i \in [1, k-2]$. So the sum of the labels in $\bigcup_{i=1}^{k-2} CM_i(a)$ is at least $\sum_{i=1}^{k-2} \sum_{j=1}^{p}[(i-1)pn+j]$. Recall that, owing to Step 1 (1.1) and (1.2), the sum of labels in $CM_{k-1}(a)$ is at least $[(2k-3)pn + 1]p/2 - 1/2$. Next recall that, owing to Step 2 (2.1), the sum of labels in $CM_k(a)$ is at least $[(2k-1)pn - n + 1](p-1)/2 + (kpn - n + 1)$ if $p$ is odd (then $(p-1)$ is even) or $n$ is odd, while owing to Step 2 (2.2), the sum of labels in $CM_k(a)$ is at least $[(2k-1)pn - 2n + 1](p-2)/2 + (2kpn - 5n/2 + 1)$ if $p$ is even (then $(p-2)$ is also even) and $n$ is even. Thus, the later lower bound is $1/2$ smaller than the first lower bound. So

$$\varphi_f(a) \geq \sum_{i=1}^{k-2} \sum_{j=1}^{p}[(i-1)pn+j] + \left\{ \frac{[(2k-3)pn + 1]p}{2} - \frac{1}{2} \right\} + \frac{[(2k-1)pn - 2n + 1](p-2)}{2} + \left(2kpn - \frac{5n}{2} + 1\right).$$

Then for each $a \in A$ and $b \in B$, one has

$$\varphi_f(a) - \varphi_f(b) \geq \frac{1}{2} \left[ \left(\frac{1}{2}k - 1\right)p^2kn + (k - 3)p^2 + k^2 \left(\frac{1}{2}p - 1\right)pn + (p - 1)(np + k) + (p^2 - 1)n + (p^2 - 3) \right] > 0,$$
since \( k \geq 3 \) and \( p \geq 2 \).

Thus, we obtain an antimagic labeling. This completes our proof. \( \square \)

**Theorem 10.** Every \((k,k^2 + y)\)-biregular \((k \geq 3, y \geq 1)\) bipartite graph is antimagic.

**Proof.** Let \( G[A,B] \) be a \((k,k')\)-biregular \((k' = k^2 + y)\) bipartite graph, where each vertex in \( A \) has the degree \( k' \), while each vertex in \( B \) has the degree \( k \). Suppose \(|A| = kn\) and \(|B| = k'\eta\) where \( \eta \) may be not an integer. It is sufficient to consider the case when \( k' = kp + r \) for some integers \( p \) and \( r \) satisfying \( p \geq k \) and \( 1 \leq r \leq k - 1 \) (note that \( r\eta \) is an integer since \( kn \) and \( k'\eta \) are integers).

Let \( A = \{a_1,a_2,\ldots,a_k\} \) and \( B = \{b_1,b_2,\ldots,b_{k'}\} \). For \( A_0 \subseteq A \), the graph \( G[A_0,N(A_0)] \) has \( k'\eta \) edges, since each vertex of \( A_0 \) in \( G[A_0,N(A_0)] \) has the degree \( k' \). On the other hand, suppose \(|N(A_0)| < p|A_0|\). Then the number of edges in \( G[A_0,N(A_0)] \) is at most \( k|N(A_0)| < pk|A_0| < k'|A_0|\), since each vertex of \( N(A_0) \) in \( G[A_0,N(A_0)] \) has the degree at most \( k \), a contradiction. So \(|N(A_0)| \geq p|A_0|\). So, by Lemma 5, \( G \) admits a complete \( p \)-claw matching \( CM \) from \( A \) to \( B \). Suppose \( B = B_1 \cup B_2 \) where \( B_1 = V(CM) \cap B \) and \( B_2 = B \setminus B_1 \). Then \(|B_1| = kp\eta \) and \(|B_2| = r\eta \). Let \( \sigma(b) \) be an edge incident to \( b \) for each \( b \in B_2 \), and let \( \sigma(B_2) = \{\sigma(b) | b \in B_2\} \).

**Step 1.** Label \((E(G) - CM - \sigma(B_2))\) with \([1, (k-1)k'\eta]\).

(1.1) If \((k-1)\) is even or \(k'\eta\) is odd, label \((E(G) - CM - \sigma(B_2))\) with \([1, (k-1)k'\eta]\) such that, within \((E(G) - CM - \sigma(B_2))\), the vertices in \( B \) have the same partial sum \(((k-1)k'\eta + 1)(k-1)/2\). We can do this owing to the partition in Lemma 8(1).

(1.2) If \((k-1)\) is odd and \(k'\eta\) is even, label \((E(G) - CM - \sigma(B_2))\) with \([1, (k-1)k'\eta]\) such that, within \((E(G) - CM - \sigma(B_2))\), the vertices in \( B \) have the same partial sum \(((k-1)k'\eta + 1)(k-1)/2 + 3/2\) except one (denoted by \( b_0 \)) which equals to \(((k-1)k'\eta + 1)(k-1)/2 + 3k'\eta/2 + 3/2\). We can do this owing to the partition in Lemma 8(3).

Note that (1.1) implies the final vertex sums in \( B \) will be pairwise different when \((k-1)\) is even or \(k'\eta\) is odd, if we label the rest edges \((CM \cup \sigma(B_2))\) with the rest labels \([ (k-1)k'\eta + 1 \), \(kk'\eta]\). Then in (1.2), the partial sum of \( b_0 \) is at least \(3k'\eta/2\) smaller than those of the vertices in \((B \setminus \{b_0\})\). So the final vertex sum of \( b_0 \) will be smaller than those of the vertices in \((B \setminus \{b_0\})\), if we label \((CM \cup \sigma(B_2))\) with \([ (k-1)k'\eta + 1 \), \(kk'\eta]\). Next, the final vertex sums of in \((B \setminus \{b_0\})\) will be pairwise different. That is, vertices in \( B \) will also receive distinct final vertex sums, when \((k-1)\) is odd and \(k'\eta\) is even.

**Step 2.** Label \( \sigma(B_2) \) with \([ (k-1)k'\eta + 1 \), \((k-1)k'\eta + r\eta]\) arbitrarily.
Step 3. Label $CM$ with $[(k-1)k'\eta + r\eta + 1, kk'\eta]$, i.e., $[(kk'\eta - pk\eta) + 1, (kk'\eta - pk\eta) + pk\eta]$. Suppose $f_1(a_{t_1}) \leq f_1(a_{t_2}) \leq \cdots \leq f_1(a_{t_{\theta_k}})$, where $f_1(a_{t_j})$ is the partial vertex sum of $a_{t_j}$ after Steps 1 and 2, for $j \in [1, \theta_k]$.

(3.1) If $p$ is odd (then $(p-1)$ is even) or $k\eta$ is odd, let $\sigma(a)$ be an edge in $CM(a)$ for each $a \in A$. Label $[CM \setminus \bigcup_{a \in A} \{\sigma(a)\}]$ with $[(kk'\eta - pk\eta) + 1, (kk'\eta - pk\eta) + (p-1)k\eta]$ such that, within $[CM \setminus \bigcup_{a \in A} \{\sigma(a)\}]$, vertices in $A$ have the same vertex sum $2(kk'\eta - pk\eta) + (p-1)k\eta + 1/(p-2)/2$. We can do this owing to the partition in Lemma 8(1). And label $\{kk'\eta - k\eta + j\}$ for $j \in [1, k\eta]$. Then vertex sums in $A$ are pairwise different.

(3.2) If $p$ is even (then $(p-2)$ is also even) and $k\eta$ is even, let $\sigma_1(a)$ and $\sigma_2(a)$ be two distinct edges in $CM(a)$ for each $a \in A$. Label $[CM \setminus \bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\}]$ with $[(kk'\eta - pk\eta) + 1, (kk'\eta - pk\eta) + (p-2)k\eta]$ such that, within $[CM \setminus \bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\}]$, vertices in $A$ have the same vertex sum $2(kk'\eta - pk\eta) + (p-2)k\eta + 1/(p-2)/2$. We can do this owing to the partition in Lemma 8(1). Next, label $\bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\}$ with $[(kk'\eta - 2k\eta) + 1, (kk'\eta - 2k\eta) + 2k\eta]$ such that $f(\sigma_1(a_{t_{\theta_k}})) + f(\sigma_2(a_{t_{\theta_k}})) = 2kk'\eta - 5k\eta/2 + j$ for $j \in [1, k\eta - 1]$, while $f(\sigma_1(a_{t_{\theta_k}})) + f(\sigma_2(a_{t_{\theta_k}})) = 2kk'\eta - k\eta$. We can do this owing to the partition in Lemma 7(4). Then vertex sums in $A$ are also pairwise different.

Recall that, owing to Step 1 (1.1) and (1.2), for each $b \in B$, one has

$$\varphi_f(b) \leq \frac{((k-1)k'\eta + 1)(k-1)}{2} + \frac{3}{2} + kk'.p.$$

On the other hand, let $a \in A$. Recall that, owing to Step 3 (3.1) the sum of the labels in $CM(a)$ is at least $\{2(kk'\eta - pk\eta) + (p-1)k\eta + 1/(p-1)/2 + (kk'\eta - k\eta + 1)$ if $p$ is odd (then $(p-1)$ is even) or $k\eta$ is odd. Then owing to Step 3 (3.2), the sum of the labels in $CM(a)$ is at least $\{2(kk'\eta - pk\eta) + (p-2)k\eta + 1/(p-2)/2\} + (2kk'\eta - 5k\eta/2 + 1)$ if $p$ is even (then $(p-2)$ is also even) and $k\eta$ is even. And the later lower bound is $1/2$ smaller than the first lower bound. So

$$\varphi_f(a) > \frac{2(kk'\eta - pk\eta) + (p-2)k\eta + 1/(p-2)/2\} + \left(2kk'\eta - \frac{5k\eta}{2} + 1\right).$$

Then for each $a \in A$ and $b \in B$ one has

$$\varphi_f(a) - \varphi_f(b) > \frac{1}{2} \left\{(p-k)kk'\eta + (k' - 2p - 1)pk\eta + (p^2 - 3)k\eta + (pk + k - k')\eta + (\eta - 1)k + (p - 2)\right\} > 0,$$

since $p \geq k \geq 3$ and $2p + 1 < k' = pk + r \leq pk + k$.

Thus, we obtain an antimagic labeling. This completes our proof.
Acknowledgments

The authors would like to thank very much the anonymous referees for valuable suggestions, corrections and comments which results in a great improvement of the original manuscript. The first author is supported by NSFC (No. 11701195) and by the Scientific Research Funds of Huaqiao University (No. 16BS808).

References


Received 12 August 2019
Revised 1 June 2020
Accepted 1 June 2020