ANTIMAGIC LABELING OF SOME BIREGULAR BIPARTITE GRAPHS

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Abstract

An antimagic labeling of a graph $G = (V,E)$ is a one-to-one mapping from $E$ to $\{1, 2, \ldots, |E|\}$ such that distinct vertices receive different label sums from the edges incident to them. $G$ is called antimagic if it admits an antimagic labeling. It was conjectured that every connected graph other than $K_2$ is antimagic. The conjecture remains open though it was verified for several classes of graphs such as regular graphs. A bipartite graph is called $(k,k')$-biregular, if each vertex of one of its parts has the degree $k$, while each vertex of the other parts has the degree $k'$. This paper shows the following results. (1) Each connected $(2,k)$-biregular $(k \geq 3)$ bipartite graph is antimagic; (2) Each $(k,pk)$-biregular $(k \geq 3, p \geq 2)$ bipartite graph is antimagic; (3) Each $(k,k^2 + y)$-biregular $(k \geq 3, y \geq 1)$ bipartite graph is antimagic.

Keywords: antimagic labeling, bipartite, biregular.

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1. Introduction

Let $G = (V, E)$ be a graph. Suppose $f$ is a one-to-one mapping from $E$ to $\{1, 2, \ldots, |E|\}$. For each vertex $v$ in $V$, the vertex sum $\varphi_f(v)$ at $v$ under $f$ is defined as $\varphi_f(v) = \sum_{e \in E(v)} f(e)$, where $E(v)$ is the set of edges incident to $v$. If $\varphi_f(u) \neq \varphi_f(v)$ for any vertex pair $u, v \in V$, then $f$ is called an antimagic labeling of $G$. A graph $G$ is called antimagic if $G$ admits an antimagic labeling. The antimagic labeling of graphs was introduced by Hartsfield and Ringel [8] in 1989 (also in [9]), who verified the antimagicness of paths, 2-regular graphs and complete graphs. Moreover, they put forth the following conjecture.

**Conjecture 1** [9]. Every connected graph other than $K_2$ is antimagic.

The conjecture has received much attention, but remains open. It was proved by Alon et al. [1] that there is an absolute constant $c$ such that graphs with minimum degree $\delta(G) \geq c \log |V|$ are antimagic, and graphs with maximum degree at least $|V| - 2$ and complete bipartite graphs except $K_2$ are antimagic. And then graphs of large linear size were shown to be antimagic [6]. For regular graphs, the antimagicnesses of $k$-regular ($k \geq 2$) bipartite graphs [3], cubic graphs [12], odd degree regular graphs [4], and finally even regular graphs [2] were verified, respectively. For more results on antimagic labeling such as those about trees, one can refer to [5, 10, 11, 13, 14, 17] and the survey of Gallian [7].

A bipartite graph is called $(k, k')$-biregular, if each vertex in one of its two parts has the degree $k$, while each vertex in the other part has the degree $k'$. This paper shows the following results. (1) Each connected $(2, k)$-biregular ($k \geq 3$) bipartite graph is antimagic; (2) Each $(k, pk)$-biregular ($k \geq 3, p \geq 2$) bipartite graph is antimagic; (3) Each $(k, k^2 + y)$-biregular ($k \geq 3, y \geq 1$) bipartite graph is antimagic. The first result is shown in Section 2, where we treat each connected $(2, k)$-biregular ($k \geq 3$) bipartite graph as the subdivision graph of a connected $k$-regular graph. A subdivision graph $G_s$ of a graph $G$, is obtained from $G$ by replacing each edge with a path of length two. The second and the third results are shown in Section 3, based on an extended version of Hall’s matching theorem [15, 16].

2. Connected $(2, k)$-Biregular ($k \geq 3$) Bipartite Graph

With respect to a given labeling, two vertices are in conflict if they have a common vertex sum. When we have labeled a subset of the edges, we call the resulting sum at each vertex a partial vertex-sum. For short, we denote by $[i, j]$ the integer set $\{i, i + 1, \ldots, j\}$ for integers $i$ and $j$ (where $i < j$).

**Theorem 2.** The subdivision graph $G_s$ of every connected $k$-regular ($k \geq 3$) graph $G$ is antimagic.
Proof. Choose an arbitrary vertex $v^*$ in $G$ as a root. Let $\alpha$ be the longest distance of a vertex from $v^*$ in $G$. Suppose $i \in [1, \alpha]$. Denote by $V_i$ the sets of vertices at distance exactly $i$ from $v^*$, by $G[V_i]$ the subgraph induced by $V_i$, and by $G[V_{i-1}; V_i]$ (here we suppose $V_0 = \{v^*\}$) the induced bipartite subgraph with parts $V_{i-1}$ and $V_i$, respectively. For $v \in V_i$, let $\sigma(v)$ be an arbitrary edge in $G[V_{i-1}; V_i]$ which is incident to $v$. Let $\sigma(V_i) = \{\sigma(v) \mid v \in V_i\}$ and $G[\sigma[V_{i-1}; V_i]] = G[V_{i-1}; V_i] \setminus \sigma(V_i)$.

Now subdivide $G$ into $G_s$. Then every vertex in $V_i$ is at distance exactly $2i$ from $v^*$ in $G_s$. Denote by $S_i, U_i$ and $W_i$ the newly added vertex sets on the edges of $G[V_i], G_\sigma[V_{i-1}; V_i]$ and $\sigma(V_i)$, respectively, when subdividing $G$ into $G_s$. Let $X = \bigcup_{i=1}^\alpha X_i$ for $X = V, S, U, W$. For a vertex $v \in V_i$, let $w(v)$ be the vertex in $W_i$ which is adjacent to $v$. For every vertex $x \in (S_i \cup U_i \cup W_i)$, let $e^x$ and $\sigma^x$ be the two edges incident to $x$. If $x \in (U_i \cup W_i)$, we suppose $e^x$ is incident to some vertex in $V_i$, while $\sigma^x$ is incident to some vertex in $V_{i-1}$. For $X = S, U, W$, let $E^X = \{e^x \mid x \in X_i\}, \overline{E}^X_i = \{\sigma^x \mid x \in X_i\}$ and $E^X_i = E^X \setminus \overline{E}^X_i$.

Respect to a labeling $f$ on $E(G_s)$, if $v \in V_i$, we denote the partial sum at $v$ (omitting the label on $e^w(v)$) by $p(v) = \sum_{e \in E(v) \setminus \{e^w(v)\}} f(e) = \varphi_f(v) - f(e^w(v))$. Let $p(v^*) = \varphi_f(v^*) - f(e^*)$ where $e^*$ is the edge in $E(v^*)$ which receives the greatest label among $E(v^*)$.

Note that $V(G_s) = V \cup S \cup U \cup W \cup \{v^*\}$. To show $G_s$ is antimagic, we will construct a labeling $f$ which satisfies the following conditions.

1. The vertex sums in $X_i$ are all odd and pairwise different, for $X \in \{S, U, W\}$ and $i \in [1, \alpha]$.
2. The vertex sums in $V_i$ are all even and pairwise different for $i \in [1, \alpha]$.
3. The vertex sums in $(S_i \cup U_i \cup W_i)$ are smaller than those in $(S_{i-1} \cup U_{i-1} \cup W_{i-1})$ for $i \in [2, \alpha]$.
4. The vertex sums in $S_i$ are smaller than those in $U_i$, while the later ones are smaller than those in $W_i$ for $i \in [1, \alpha]$.
5. The vertex sums in $V_i$ are smaller than those in $V_{i-1}$ for $i \in [2, \alpha]$.
6. The vertex sum at $v^*$ is greater than those in $V_i$ and those in $W_1$.

Conditions (1) and (2) make sure there is no conflict between $V$ and $(S \cup U \cup W)$. Conditions (1), (3), (4) make sure there is no conflict inside $(S \cup U \cup W)$. Conditions (2) and (5) make sure there is no conflict inside $V$. Conditions (3), (4), (5) and (6) make sure there is no conflict between $v^*$ and any other vertex in $G_s$. So these conditions imply that $f$ is antimagic.

Note that $E(G_s) = \bigcup_{i=1}^\alpha (E^S_i \cup E^U_i \cup E^W_i)$. We will label $E(G_s)$ in the order $E^S_\alpha, (E^U_\alpha \cup E^W_\alpha), E^S_{\alpha-1}, (E^U_{\alpha-1} \cup E^W_{\alpha-1}), \ldots, E^S_1, (E^U_1 \cup E^W_1)$, using the smallest unused labels on each edge set when we come to it. This label assignment immediately implies that (3) holds, and that the vertex sums in $S_i$ are smaller than those in $(U_i \cup W_i)$ for $i \in [1, \alpha]$.

Suppose $i \in [1, \alpha]$ in the following. Note that $|E^X_i| = 2|X_i|$, for $X = S, U, W$. Antimagic Labeling of Some Biregular Bipartite Graphs
(I) The labeling of $E^S$. We first label $E^S_i$ arbitrarily using the $|S_i|$ odd labels from the $2|S_i|$ assigned labels for $E^S_i$. Secondly let $f(\bar{\pi}^s) = f(\pi^s) + 1$ for each $s \in S_i$. Then the vertex sums in $S_i$ are odd and pairwise different.

(II) The labeling of $(E^U \cup E^W)$. If $|U_i|$ is odd, then $i \in [2, \alpha]$, since $U_1$ is an empty set. We will label $(E^U_i \cup E^W_i)$ in the order $E^U_i$, $E^W_i$, using the smallest unused assigned labels on each edge subset when we come to it. This sub-assignment (based on our global assignment), gives that $p(v) < p(v')$ for arbitrary $v \in V_i$ and $v' \in V_{i - 1}$, which implies $\varphi_f(v) = p(v) + f(vw(v)) < p(v') + f(v'w(v')) = \varphi_f(v')$, since $f(vw(v)) < f(v'w(v'))$ by our global assignment. So (5) holds for those $i$ with $|U_i|$ being odd. It gives that the vertex sums in $U_i$ are smaller than those in $W_i$. So (4) holds for those $i$ with $|U_i|$ being odd. We first label $E^U_i$ arbitrarily using its assigned labels. Secondly let $f(\bar{\pi}^u) = f(\pi^u) + |U_i|$ for each $u \in U_i$. This gives that the vertex sums in $U_i$ are odd and pairwise different. Third, suppose $V_i = \{v_1, v_2, \ldots, v_{|V_i|}\}$ where $p(v_1) \leq p(v_2) \leq \cdots \leq p(v_{|V_i|})$. For $r \in [1, |V_i|]$, label $\pi^w(v_r)$ with the $r$-th smallest label among the odd (even) assigned labels for $E^W_i$, when $p(v_r)$ is odd (even). This implies that the vertex sums in $V_i$ are even and pairwise different. So (2) holds for those $i$ with $|U_i|$ being odd. Fourth, let $f(\bar{\pi}^w) = f(\pi^w) + 1$ when $f(\pi^w)$ is odd, while $f(\bar{\pi}^w) = f(\pi^w) - 1$ when $f(\pi^w)$ is even. This implies that vertex sums in $W_i$ are odd and pairwise different. So (1) holds for those $i$ with $|U_i|$ being odd.

If $|U_i|$ is even ($|U_i|$ may equal to 0), then $i \in [1, \alpha]$. We will label edges in $E^U_i$ using the smallest $(2|U_i| + 1)$ assigned labels for $E^U_i \cup E^W_i$ except the $(|U_i| + 1)$-th smallest one (denoted by $\xi_{|U_i|+1}$). We first label the edges of $E^U_i$ arbitrarily using the $|U_i|$ smallest assigned labels. This gives that $p(v) < p(v')$ for arbitrary $v \in V_i$ and $v' \in V_{i - 1}$. And then, if $i \neq 1$, one has $\varphi_f(v) = p(v) + f(vw(v)) < p(v') + f(v'w(v')) = \varphi_f(v')$ for arbitrary $v \in V_i$ and $v' \in V_{i - 1}$, since $f(vw(v)) < f(v'w(v'))$ by our global assignment. So (5) also holds for those $i$ ($i \neq 1$) with $|U_i|$ being even. Secondly let $f(\bar{\pi}^w) = f(\pi^w) + |U_i| + 1$ for each $u \in U_i$. This implies that the vertex sums in $U_i$ are odd and pairwise different. It also implies that the vertex sums in $U_i$ are smaller than those in $W_i$, since any pair of the rest assigned labels left for $W_i$ has a sum greater than any vertex sum in $U_i$. So (4) also holds for those $i$ with $|U_i|$ being even. Note that, $\xi_{|U_i|+1}$ and $(\xi_{|U_i|+1} + |U_i| + 1)$ have distinct parity, and so far, they are the smallest two unused assigned labels for $W_i$. Third, suppose $|V_i| = \{v_1, v_2, \ldots, v_{|V_i|}\}$ where $p(v_1) \leq p(v_2) \leq \cdots \leq p(v_{|V_i|})$. For $r \in [1, |V_i|]$, label $\pi^w(v_r)$ with the $r$-th smallest label among the rest odd (even) assigned labels, if $p(v_r)$ is odd (even). This implies that the vertex sums in $V_i$ are even and pairwise different. So (2) also holds for those $i$ with $|U_i|$ being even. And note that either $\xi_{|U_i|+1}$ or $(\xi_{|U_i|+1} + |U_i| + 1)$ is assigned to $w(v_1)$ by our labeling way. Fourth, let $f(\bar{\pi}^w(v_1)) = \xi_{|U_i|+1}$ if $f(\pi^w(v_1)) = \xi_{|U_i|+1} + |U_i| + 1$, while $f(\bar{\pi}^w(v_1)) = \xi_{|U_i|+1} + |U_i| + 1$ if
f(\(v_i\)) = \(\xi_{|U_i|+1}\), so that \(\{f(\xi_{|U_i|+1}), f(\xi_{|U_i|+1} + |U_i| + 1)\}\). And for \(r \in [2, |V_i|]\), let \(f(\tau_{|U_i|}) = f(\xi_{|U_i|}) + 1\) if \(f(\xi_{|U_i|})\) is odd, while \(f(\tau_{|U_i|}) = f(\xi_{|U_i|}) - 1\) if \(f(\xi_{|U_i|})\) is even. This implies that vertex sums in \(W_i\) are odd and pairwise different. So (1) also holds for those \(i\) with \(|U_i|\) being even.

For (6), note that the process of the labeling of \(E(v^*) = E^W_1\) is discussed in the case when \(|U_1|\) is even (since \(U_1 = \emptyset\) and \(|U_1| = 0\)). Recall that, \(|E^W_1| = 2k\) and \(E^W_1\) are assigned with the greatest 2\(k\) labels, i.e., those labels in \(L_{2k} = \{|E(G_s)| - 1, \ldots, |E(G_s)| - 2k + 1\}\). More precisely, \(E^W_1 = E(v^*)\) are assigned with the labels in \(|E(G_s)| - 2j + 1\) or \(|E(G_s)| - 2j + 2\) for \(j = 1, 2, \ldots, k\). So \(p(v^*) \geq p(v_1) + 3 + \cdots + (2k - 3) > p(v_1) + 3\) for arbitrary \(v_1 \in V_1\) (recall that \(k \geq 3\)). Then \(\varphi_f(v^*) = p(v^*) + f(e^*) \geq \varphi_f(v_1)\) for each \(v_1 \in V_1\). On the other hand, \(\varphi_f(v^*) \geq |E(G_s)| - 3 + \cdots + |E(G_s)| - 9,\) since \(k \geq 3\). Thus, each vertex in \(W_1\) receives a sum at most \(2|E(G_s)| - 1\). So \(\varphi_f(v^*) \geq 3|E(G_s)| - 9 > 2|E(G_s)| - 1 \geq \varphi_f(w_1)\) for each \(w_1 \in W_1\) (one has \(3|E(G_s)| \geq 12\), because \(k \geq 3\)). So (6) holds.

Thus, \(G_s\) is antimagic. This completes our proof.

It is interesting to consider the case when \(G\) is \(k\)-regular \((k \geq 3)\) but disconnected. In the proof of Theorem 2, suppose \(G\) has \(m\) edges. Then \(G_s\) has \(m\) 2-vertices. Note that the total sum of all the vertex sums is even, since each label contributes to the total sum twice. Thus, each 2-vertex contributes an odd value to the total sum, while each \(k\)-vertex other than \(v^*\) contributes an even value, under our labeling way in the proof of Theorem 2. Thus, \(\varphi_f(v^*)\) is odd if and only if \(m\) is odd.

**Theorem 3.** Let \(G\) be an disconnected \(k\)-regular \((k \geq 3)\) graph, which has at most one connected component with an odd number of edges. Then \(G_s\) is antimagic.

**Proof.** Suppose \(G\) consists of the connected components \(H_1, H_2, \ldots, H_\beta\) \((\beta \geq 2)\), where \(H_i\) has an even number of edges for each \(i \in [1, \beta - 1]\). We can label \(E(G_s)\) in the order \(E((H_1)_s), E((H_2)_s), \ldots, E((H_\beta)_s)\) using the smallest unused labels on each edge set when we come to it. Next, we label each connected component of \(G_s\) in the same way to that in Theorem 2, choosing a root for each component of \(G\). Then there is no conflict among each \((H_i)_s\) for \(i \in [1, \beta]\). Each 2-vertex receives an odd sum, while each \(k\)-vertex other than the root of \((H_\beta)_s\) receives an even sum. Each 2-vertex in \((H_1)_s\) receives a smaller sum than each 2-vertex in \((H_\beta)_s\), while each \(k\)-vertex in \((H_1)_s\) receives a smaller sum than each \(k\)-vertex in \((H_j)_s\), whenever \(i < j \leq \beta\) holds. And the root vertex in \((H_\beta)_s\) receives a greater sum than those of any other vertex in \(G_s\). So we obtain an antimagic labeling.

Since \(m = \frac{nk}{2}\), for each \(k\)-regular graph with \(n\) vertices and \(m\) edges, we have the following corollary.
Corollary 4. Let $G$ be an disconnected $k$-regular ($k \geq 3$) graph. Then $G_s$ is antimagic if one of the following holds.

(1) $k = 4t$ ($t \geq 1$);

(2) $k$ is even and at most one of the connected components of $G$ has an odd number of vertices;

(3) At most one of the connected components of $G$ has a number of vertices which is not a multiple of 4.

3. $(k, pk)$-Biregular ($k \geq 3, p \geq 2$) Bipartite Graph

For a bipartite graph $G(A, B)$, a complete $p$-claw matching $CM$ from $A$ to $B$ is a set of edges of $G$ that induce a subgraph $G[CM]$ such that each vertex of $A$ in $G$ is also a vertex in $G[CM]$ and each component of $G[CM]$ is a copy of $K_{1,p}$ where the vertex of degree $p$ is in $A$, while the vertices of degree 1 are in $B$. For $A_0 \subseteq A$, denote by $N(A_0)$ the set of vertices in $B$ each of which has a neighbor in $A_0$. Let $E_1, E_2, \ldots, E_k \subseteq E(G)$ be disjoint edge sets. If $E_1 \cup E_2 \cup \cdots \cup E_k = E(G)$, then we say $G$ decomposes into $E_1, E_2, \ldots, E_k$.

Lemma 5 (An extended version of Hall’s theorem, [15, 16]). A bipartite graph $G[A, B]$ admits a complete $p$-claw matching from $A$ to $B$, if and only if $p|A_0| \leq |N(A_0)|$ for every subset $A_0$ of $A$.

Lemma 6. Let $G[A, B]$ be a $(k, pk)$-biregular ($k \geq 3, p \geq 2$) bipartite graph where the degree of each vertex in $A$ is $kp$, while each vertex in $B$ has degree $k$. Then $G$ decomposes into $k$ complete $p$-claw matchings from $A$ to $B$.

Proof. Let $A_0 \subseteq A$. Let $G[A_0, N(A_0)]$ be the graph induced by $A_0 \cup N(A_0)$. Then each vertex of $A_0$ in $G[A_0, N(A_0)]$ has the degree $kp$, while each vertex of $N(A_0)$ in $G[A_0, N(A_0)]$ has the degree at most $k$. So there are exactly $kp|A_0|$ edges in $G[A_0, N(A_0)]$. On the other hand, suppose $|N(A_0)| < p|A_0|$. Then the number of edges in $G[A_0, N(A_0)]$ is less than $k \cdot p|A_0|$, a contradiction. So $|N(A_0)| \geq p|A_0|$. By Lemma 5, there exists a complete $p$-claw matching $CM_1$ from $A$ to $B$ in $G[A, B]$. Then $G_1 = G[A, B] - CM_1$ is a $(k - 1, p(k - 1))$-biregular bipartite graph. So we can use Lemma 5 repeatedly until we obtain a $(1, p)$-biregular bipartite graph $G_{k-1}$ which is also a complete $p$-claw matching from $A$ to $B$. Thus, $G[A, B]$ decomposes into $k$ complete $p$-claw matchings from $A$ to $B$.

Lemma 7. Let $I = [i + 1, i + 2q]$. Then, there exist partitions $P_1$ (when $q$ is odd) and $\{P_2, P_3, P_4\}$ (when $q$ is even) of $I$, such that under $P_1$, $j \in [1, 4]$, $I$ is divided into $q$ parts where each part has 2 integers, integers in $[i + (x - 1)q + 1, i + xq]$ ($x \in [1, 2]$) are in distinct parts and the following conditions are satisfied.
(1) Under $P_1$, the $q$ parts have distinct sums which attain all the values in $[(2i + 2q + 1) - (q - 1)/2, (2i + 2q + 1) + (q - 1)/2]$;

(2) Under $P_2$, $q/2$ parts have distinct sums which attain all the values in $[(2i + 2q + 1) - (q/2 - 1), 2i + 2q + 1]$, while the other $q/2$ parts have distinct sums which attain all the values in $[2i + 2q + 1, (2i + 2q + 1) + (q/2 - 1)]$;

(3) Under $P_3$, the $(q - 1)$ parts have distinct sums which attain all the values in $[(2i + 2q + 2) - (q/2 - 1), (2i + 2q + 2) + (q/2 - 1)]$ and the other part has the sum $2i + q + 2$;

(4) Under $P_4$, the $(q - 1)$ parts have distinct sums which attain all the values in $[(2i + 2q) - (q/2 - 1), (2i + 2q) + (q/2 - 1)]$ and the other part has the sum $2i + 3q$.

**Proof.** It is sufficient to show the case when $i = 0$.

(1) If $q$ is odd, let $\{2j - 1, -j + (3q + 1)/2 + 1\}$ be in the same partition for $j \in [1, (q + 1)/2]$, and let $\{2j, -j + 2q + 1\}$ be in the same partition for $j \in [1, (q - 1)/2]$, which is the desired partition $P_1$.

(2) If $q$ is even, let $\{2j, -j + 3q/2 + 1\}$ be in the same partition and let $\{2j - 1, -j + 2q + 1\}$ be in the same partition for $j \in [1, q/2]$, which is the desired partition $P_2$.

(3) If $q$ is even, let $\{2j, -j + 3q/2 + 2\}$ be in the same partition for $j \in [1, q/2]$, let $\{2j + 1, -j + 2q + 1\}$ be in the same partition for $j \in [1, q/2 - 1]$, and let $\{1, q + 1\}$ be in the same partition, which is the desired partition $P_3$.

(4) If $q$ is even, let $\{2j - 1, -j + 3q/2 + 1\}$ be in the same partition for $j \in [1, q/2]$, let $\{2j, -j + 2q\}$ be in the same partition for $j \in [1, q/2 - 1]$, and let $\{q, 2q\}$ be in the same partition, which is the desired partition $P_4$. ■

**Lemma 8.** Let $I = [i + 1, i + zq]$ $(z \geq 3)$. Then, there exist partitions $P_1$ (when $z$ is even or $q$ is odd) and $P_2, P_3$ (when $z$ is odd and $q$ is even) of $I$, such that under $P_j$, $j \in [1, 3]$, $I$ is departed into $q$ parts where each part has $z$ integers, integers in $i + (x - 1)q + 1, i + xq$ $(x \in [1, z])$ are in distinct parts and the following conditions are satisfied.

(1) Under $P_1$, the $q$ parts have the same sum $(2i + zq + 1)z/2$;

(2) Under $P_2$, $q/2$ parts have the same sum $(2i + zq + 1)z/2 + 1/2$ and the other $q/2$ parts have the same sum $(2i + zq + 1)z/2 - 1/2$;

(3) Under $P_3$, $(q - 1)$ parts have the same sum $(2i + zq + 1)z/2 + 3/2$ and the other part has the sum $(2i + zq + 1)z/2 - 3q/2 + 3/2$.

**Proof.** It is sufficient to show the case when $i = 0$.

(1) If $z$ is even, let $\{(j - 1)q + l | j \in [1, z/2]\} \cup \{jq - l + 1 | j \in [z/2 + 1, z]\}$ be in the partition for $l \in [1, q]$, which is the desired partition $P_1$ and (1) holds in this case.
If $z$ is odd (then $(z - 3)$ is even) and $q$ is odd, we first assign the $(z - 3)q$
integers in $[2q + 1, (z - 1)q]$ to the $q$ parts (suppose $I_1, I_2, \ldots, I_q$ are the $q$ parts)
such that these $q$ parts receive the same partial sum $(zq + q + 1)(z - 3)/2$. We
can do this since $(z - 3)$ is even. Second, assign $[(z - 1)q + \ell]$ to $I_l$ for $l \in [1, q]$ such that the $q$
parts have distinct partial sums and attain all values in
$[(qz + q + 1)(z - 3)/2 + (z - 1)q + 1, (qz + q + 1)(z - 3)/2 + zq]$. Third, partition
$[1, 2q]$ into $q$ parts (denoted by $I_1', I_2', \ldots, I_q'$) which have distinct sums which
attain all the values in $[(2q + 1) - (q - 1)/2, (2q + 1) + (q - 1)/2]$. We can do
this owing to the partition in Lemma 7(1). Then assign $I_l'$ to $I_l$ if the sum of $I_l'$
equals to $[(2q + 1) + (q - 1)/2 - l + 1]$ for $l \in [1, q]$. Then the final sum of $I_l$
equals to $[(2q + 1) + (q - 1)/2 - l + 1]$ + $[(z - 1)q + \ell] + [q + 2] = (qz + 1)z/2 - 3q/2 + 3/2$, and
for each $l \in [1, q]$. So (1) also holds in this case.

(2) If $z$ is odd and $q$ is even, we first partition $[2q + 1, zq]$ into $q$
parts $I_1, I_2, \ldots, I_q$ which have distinct partial sums and attain all values in
$[(qz + q + 1)(z - 3)/2 + (z - 1)q + 1, (qz + q + 1)(z - 3)/2 + zq]$. We can do this
owing to the discussion in (1). Then partition $[1, 2q]$ into $q$ parts (denoted by
$I_1', I_2', \ldots, I_q'$) such that $q/2$ parts have distinct sums which attain all the values in
$[(2q + 1) - (q/2 - 1), 2q + 1]$, while the other $q/2$ parts have distinct sums which
attain all the values in $[2q + 1, (2q + 1) + (q/2 - 1)]$. We can do this
owing to the partition in Lemma 7(2). Denote by $I_{q/2,1}'$ and $I_{q/2,2}'$ the two parts
each of which admits the sum $(2q + 1)$. Then assign $I_{q/2,1}'$ to $I_l$ if the sum of $I_{q/2,1}'$
equals to $[(2q + 1) + (q/2 - 1) - l + 1]$ for $l \in [1, q/2 - 1]$. Assign $I_{q/2,2}'$ to
$I_{q/2}$, while assign $I_{q/2,2}'$ to $I_{q/2+1}$. And assign $I_{q/2}'$ to $I_l$ if the sum of $I_{q/2}'$
equals to $(2q + 1) + (q/2 - 1) - l + 2$ for $l \in [q/2 + 2, q]$. Then for $l \in [1, q/2 - 1]$
the final sum of $I_l$ equals to $[(qz + q + 1)(z - 3)/2] + [(z - 1)q + \ell] + [(2q + 1) +
(q/2 - 1) - l + 1] = (qz + 1)z/2 - 1/2$. The final sum of $I_{q/2}$ equals to
$[(qz + q + 1)(z - 3)/2] + [(z - 1)q + q/2] + [(2q + 1)] = (qz + 1)z/2 - 1/2$, while the
final sum of $I_{q/2+1}$ equals to $[(qz + q + 1)(z - 3)/2] + [(z - 1)q + q/2 + 1] + [(2q + 1) +
(z + 1)] = (qz + 1)z/2 + 1/2$. Thus, for $l \in [q/2 + 2, q]$ the final sum of $I_l$ equals to
$[(qz + q + 1)(z - 3)/2] + [(z - 1)q + l] + [(2q + 1) + (q/2 - 1) - l + 2] = (qz + 1)z/2 + 1/2$. So (2) holds.

(3) If $z$ is odd and $q$ is even, we first partition $[2q + 1, zq]$ into $q$
parts $I_1, I_2, \ldots, I_q$ which have distinct partial sums and attain all values in
$[(qz + q + 1)(z - 3)/2 + (z - 1)q + 1, (qz + q + 1)(z - 3)/2 + zq]$. We can do this owing to the
discussion in (1). Then partition $[1, 2q]$ into $q$ parts (denoted by $I_1', I_2', \ldots, I_q'$)
such that the $(q - 1)$ parts have distinct sums which attain all the values in
$[(2q + 2) - (q/2 - 1), (2q + 2) + (q/2 - 1)]$ and the other part has the sum $(q + 2)$.
We can do this owing to the partition in Lemma 7(3). Denote by $I_{q/2}'$ the part
with the sum $(q + 2)$. Then assign $I_{q/2}'$ to $I_1$, and assign $I_{q/2}'$ to $I_l$ if the sum of $I_{q/2}'$
equals to $[(2q + 2) + (q/2 - 1) - l + 2]$ for $l \in [2, q]$. Then the final sum of $I_l$
equals to $[(qz + q + 1)(z - 3)/2] + [(z - 1)q + \ell] + [q + 2] = (qz + 1)z/2 - 3q/2 + 3/2$, and
for \( l \in [2, q] \), the final sum of \( I_l \) equals to \([qz + q + 1](z - 3)/2 + [(z - 1)q + l] + [(2q + 2) + (q/2 - 1) - l + 2] = (qz + 1)z/2 + 3/2 \). So (3) holds.

**Theorem 9.** Every \((k, pk)\)-biregular \((k \geq 3, p \geq 2)\) bipartite graph is antimagic.

**Proof.** Let \( G[A, B] \) be a \((k, pk)\)-biregular \((k \geq 3, p \geq 2)\) bipartite graph, where each vertex in \( A \) has the degree \( pk \), while each vertex in \( B \) has the degree \( k \).

Suppose \(|A| = n (n \geq k)\) and \(|B| = pn\). Let \( A = \{a_1, a_2, \ldots, a_n\} \) and \( B = \{b_1, b_2, \ldots, b_{pn}\} \). By Lemma 6, \( G \) decomposes into \( k \) complete \( p \)-claw matchings \( CM_1, CM_2, \ldots, CM_k \) from \( A \) to \( B \). Denote by \( CM_i(V_0) \) \((i \in [1, k])\) the edges in \( CM_i \) which are incident to some vertex in \( V_0 \) for \( V_0 \subseteq V(G) \).

**Step 1.** Label \( (\bigcup_{i=1}^{k-1} CM_i) \) with \([1, (k - 1)pn]\).

First, label \( CM_{k-1} \) with \([(k - 2)pn + 1, (k - 1)pm] \), i.e., \([(k - 2)pn + 1, (k - 2)pn + pm + pn] \) such that the following conditions are satisfied.

(1.1) Within \( CM_{k-1} \), vertices in \( A \) have the same partial sum \([(2k - 3)pn + 1]p/2 \) if \( p \) is even or \( n \) is odd. We can do this owing to the partition in Lemma 8(1).

(1.2) Within \( CM_{k-1} \), \( n/2 \) vertices in \( A \) have the same partial sum \([(2k - 3)pn + 1]p/2 + 1/2 \) and the other \( n/2 \) vertices in \( A \) have the same partial sum \([(2k - 3)pn + 1]p/2 - 1/2 \) if \( p \) is odd and \( n \) is even. We can do this owing to the partition in Lemma 8(2).

Second, based on the labeling to \( CM_{k-1} \), for each \( i \in [1, k - 2] \), label \( CM_i \) with \([(i - 1)pn + 1, ipn] \), such that the following conditions are satisfied.

(1.3) Within \( (\bigcup_{i=1}^{k-1} CM_i) \), the vertices in \( B \) have the same partial sum \([(k - 1)pn + 1](k - 1)/2 \) if \( (k - 1) \) even or \( pn \) is odd. We can do this owing to the partition in Lemma 8(1).

(1.4) Within \( (\bigcup_{i=1}^{k-1} CM_i) \), \( (pn - 1) \) vertices in \( B \) have the same partial sum \([(k - 1)pn + 1](k - 1)/2 + 3/2 \) while the other vertex (denoted by \( b_0 \)) has the partial sum \([(k - 1)pn + 1](k - 1)/2 + 3/2 - 3pn/2 \) if \( (k - 1) \) is odd and \( pn \) is even. We can do this owing to the partition in Lemma 8(3).

Note that, (1.3) implies the vertices in \( B \) will receive distinct final vertex sums, when \((k - 1) \) is even or \( pn \) is odd, if we label the rest edges \( CM_k \) using the rest labels \([(k - 1)pn + 1, kp/n] \). Thus in (1.4), the partial sum of \( b_0 \) is at least \( 3pn/2 \) smaller than those of the vertices in \((B \setminus \{b_0\})\). So the final vertex sum of \( b_0 \) will still be smaller than those of the vertices in \((B \setminus \{b_0\})\), if we label \( CM_k \) with \([(k - 1)pn + 1, kp/n] \). Hence, the final vertex sums of \( b_0 \) will be pairwise different. That is, all vertices in \( B \) will also receive distinct final vertex sums when \((k - 1) \) is odd and \( pn \) is even.

**Step 2.** Label \( CM_k \) with \([(k - 1)pn + 1, kp/n] \), i.e., \([(k - 1)pn + 1, (k - 1)pn + pm] \).

Suppose \( f_1(a_1) \leq f_1(a_{i_2}) \leq \cdots \leq f_1(a_{i_t}) \) where \( f_1(a_{i_j}) \) is the partial vertex sum of \( a_{i_j} \) within \( (\bigcup_{i=1}^{k-1} CM_i) \) for \( j \in [1, n] \).
(2.1) If $p$ is odd (then $(p-1)$ is even) or $n$ is odd, let $\sigma(a)$ be an edge in $CM_k(a)$ for each $a \in A$. Label $[CM_k \setminus (\bigcup_{a \in A} \{\sigma(a)\})]$ with $[(k-1)pn+1, (k-1)pn+(p-1)n]$ such that, within $[CM_k \setminus (\bigcup_{a \in A} \{\sigma(a)\})]$, the vertices in $A$ have the same partial sum $[(2k-1)pn-n+1](p-1)/2$. We can do this owing to the partition in Lemma 8(1). Next label $\sigma(a_i)$ with $(kpn - n + j)$ for $j \in [1, n]$. Then the vertex sums in $A$ are pairwise different.

(2.2) If $p$ is even (then $(p-2)$ is also even) and $n$ is even, let $\sigma_1(a)$ and $\sigma_2(a)$ be two distinct edges in $CM_k(a)$ for each $a \in A$. Label $[CM_k \setminus (\bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\})]$ using the labels in $[(k-1)pn+1, (k-1)pn+(p-2)n]$ such that, within $[CM_k \setminus (\bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\})]$, the vertices in $A$ have the same partial sum $[(2k-1)pn-2n+1](p-2)/2$. We can do this owing to the partition in Lemma 8(1). Then label $\bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\}$ with $[(kpn - 2n) + 1, (kpn - 2n) + 2n]$ such that $f(\sigma_1(a_i)) + f(\sigma_2(a_i)) = 2kpn - 5n/2 + j$ for $j \in [1, n - 1]$ while $f(\sigma_1(a_n)) + f(\sigma_2(a_n)) = 2kpn - n$. We can do this owing to the partition in Lemma 7(4). Then the vertex sums in $A$ are also pairwise different.

Recall that, owing to Step 1 (1.3) and (1.4), for each $b \in B$, one has

$$\varphi_f(b) \leq \frac{[(k-1)pn+1](k-1)}{2} + \frac{3}{2} + kpn.$$ 

On the other hand, owing to the labeling way in Step 1 (1.3) and (1.4), the labels assigned to $CM_i$ are those in $[(i-1)pn+1, ipn]$ for $i \in [1, k-2]$. Let $a \in A$. Then the sum of the labels in $CM_i(a)$ is at least $\sum_{j=1}^{p} [(i-1)pn+j]$ for $i \in [1, k-2]$.

So the sum of the labels in $\bigcup_{i=1}^{k-2} CM_i(a)$ is at least $\sum_{i=1}^{k-2} \sum_{j=1}^{p} [(i-1)pn+j]$. Recall that, owing to Step 1 (1.1) and (1.2), the sum of labels in $CM_{k-1}(a)$ is at least $[(2k-3)pn+1]/2 - 1/2$. Next recall that, owing to Step 2 (2.1), the sum of labels in $CM_k(a)$ is at least $[(2k-1)pn-n+1]/(p-1)/2 + (kpn-n+1)$. If $p$ is odd (then $(p-1)$ is even) or $n$ is odd, while owing to Step 2 (2.2), the sum of labels in $CM_k(a)$ is at least $[(2k-1)pn-2n+1]/(p-2)/2 + (2kpn-5n/2+1)$ if $p$ is even (then $(p-2)$ is also even) and $n$ is even. Thus, the later lower bound is $1/2$ smaller than the first lower bound. So

$$\varphi_f(a) \geq \sum_{i=1}^{k-2} \sum_{j=1}^{p} [(i-1)pn+j] + \left\{ \frac{[(2k-3)pn+1]p}{2} - \frac{1}{2} \right\}$$

$$+ \left\{ \frac{[(2k-1)pn-2n+1](p-2)}{2} + \left( 2kpn - \frac{5n}{2} + 1 \right) \right\}.$$ 

Then for each $a \in A$ and $b \in B$, one has

$$\varphi_f(a) - \varphi_f(b) \geq \frac{1}{2} \left[ \left( \frac{1}{2}k-1 \right) p^2 kn + (k-3)p^2 + k^2 \left( \frac{1}{2}p-1 \right)pn \right.$$ 

$$+ (p-1)(np+k) + (p^2-1)n + (p^2-3) > 0,$$
since $k \geq 3$ and $p \geq 2$.
Thus, we obtain an antimagic labeling. This completes our proof.

\begin{theorem}
Every $(k, k^2 + y)$-biregular $(k \geq 3, y \geq 1)$ bipartite graph is antimagic.
\end{theorem}

\begin{proof}
Let $G[A, B]$ be a $(k, k')$-biregular $(k' = k^2 + y)$ bipartite graph, where
each vertex in $A$ has the degree $k'$, while each vertex in $B$ has the degree $k$.
Suppose $|A| = k\eta$ and $|B| = k'\eta$ where $\eta$ may be not an integer. It is sufficient
to consider the case when $k' = kp + r$ for some integers $p$ and $r$ satisfying $p \geq k$
and $1 \leq r \leq k - 1$ (note that $r\eta$ is an integer since $k\eta$ and $k'\eta$ are integers).
Let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$. For $A_0 \subseteq A$, the graph
$G[A_0, N(A_0)]$ has $k'\eta$ edges, since each vertex of $A_0$ in $G[A_0, N(A_0)]$ has the
degree $k'$. On the other hand, suppose $|N(A_0)| < p|A_0|$. Then the number
of edges in $G[A_0, N(A_0)]$ is at most $k|N(A_0)| < pk|A_0| < k'|A_0|$, since each
vertex of $N(A_0)$ in $G[A_0, N(A_0)]$ has the degree at most $k$, a contradiction. So
$|N(A_0)| \geq p|A_0|$. So, by Lemma 5, $G$ admits a complete $p$-claw matching $CM$
from $A$ to $B$. Suppose $B = B_1 \cup B_2$ where $B_1 = V(CM) \cap B$ and $B_2 = B \setminus B_1$.
Then $|B_1| = k\eta$ and $|B_2| = r\eta$. Let $\sigma(b)$ be an edge incident to $b$ for each $b \in B_2$,
and let $\sigma(B_2) = \{\sigma(b) | b \in B_2\}.$

\begin{step} Label $(E(G) - CM - \sigma(B_2))$ with $[1, (k - 1)k'\eta].$
\end{step}

(1.1) If $(k - 1)$ is even or $k'\eta$ is odd, label $(E(G) - CM - \sigma(B_2))$ with $[1, (k - 1)k'\eta]$
such that, within $(E(G) - CM - \sigma(B_2))$, the vertices in $B$ have the same partial
sum $((k - 1)k'\eta + 1)(k - 1)/2$. We can do this owing to the partition in Lemma
8(1).

(1.2) If $(k - 1)$ is odd and $k'\eta$ is even, label $(E(G) - CM - \sigma(B_2))$ with $[1, (k - 1)k'\eta]$
such that, within $(E(G) - CM - \sigma(B_2))$, the vertices in $B$ have the same partial
sum $[(k - 1)k'\eta + 1]/2 + 3/2$ except one (denoted by $b_0$) which equals to
$[(k - 1)k'\eta + 1]/2 - 3k'\eta/2 + 3/2$. We can do this owing to the partition
in Lemma 8(3).

Note that (1.1) implies the final vertex sums in $B$ will be pairwise different
when $(k - 1)$ is even or $k'\eta$ is odd, if we label the rest edges $(CM \cup \sigma(B_2))$ with
the rest labels $[(k - 1)k'\eta + 1, kk'\eta]$. Then in (1.2), the partial sum of $b_0$ is at least
$3k'\eta/2$ smaller than those of the vertices in $(B \setminus \{b_0\})$. So the final vertex sum of
$b_0$ will be smaller than those of the vertices in $(B \setminus \{b_0\})$, if we label $(CM \cup \sigma(B_2))$
with $[(k - 1)k'\eta + 1, kk'\eta]$. Next, the final vertex sums of in $(B \setminus \{b_0\})$ will be
pairwise different. That is, vertices in $B$ will also receive distinct final vertex
sums, when $(k - 1)$ is odd and $k'\eta$ is even.

\begin{step} Label $\sigma(B_2)$ with $[(k - 1)k'\eta + 1, (k - 1)k'\eta + r\eta]$ arbitrarily.
\end{step}
Step 3. Label $CM$ with $[(k-1)k'+r\eta+1, kk'\eta]$, i.e., $[(kk'\eta-pk\eta)+1, (kk'\eta-pk\eta)+r\eta+1, kk'\eta]$. Suppose $f_1(a_{t_1}) \leq f_1(a_{t_2}) \leq \cdots \leq f_1(a_{t_{|\mathcal{B}|}})$, where $f_1(a_{t_j})$ is the partial vertex sum of $a_{t_j}$ after Steps 1 and 2, for $j \in [1, k\eta]$.

(3.1) If $p$ is odd (then $(p-1)$ is even) or $k\eta$ is odd, let $\sigma(a)$ be an edge in $CM(a)$ for each $a \in A$. Label $\left[CM \setminus \bigcup_{a \in A} \{\sigma(a)\}\right]$ with $[(kk'\eta-pk\eta)+1, (kk'\eta-pk\eta)+(p-1)k\eta]$ such that, within $\left[CM \setminus \left(\bigcup_{a \in A} \{\sigma(a)\}\right)\right]$, vertices in $A$ have the same vertex sum $[2(kk'\eta-pk\eta)+(p-1)k\eta+1](p-2)/2$. We can do this owing to the partition in Lemma 8(1). And label $\sigma(a_{t_j})$ with $(kk'\eta-k\eta+j)$ for $j \in [1, k\eta]$. Then vertex sums in $A$ are pairwise different.

(3.2) If $p$ is even (then $(p-2)$ is also even) and $k\eta$ is even, let $\sigma_1(a)$ and $\sigma_2(a)$ be two distinct edges in $CM(a)$ for each $a \in A$. Label $\left[CM \setminus \bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\}\right]$ with $[(kk'\eta-pk\eta)+1, (kk'\eta-pk\eta)+(p-2)k\eta]$ such that, within $\left[CM \setminus \left(\bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\}\right)\right]$, vertices in $A$ have the same vertex sum $[2(kk'\eta-pk\eta)+(p-2)k\eta+1](p-2)/2$. We can do this owing to the partition in Lemma 8(1).

Next, label $\left(\bigcup_{a \in A} \{\sigma_1(a), \sigma_2(a)\}\right)$ with $[(kk'\eta-2k\eta)+1, (kk'\eta-2k\eta)+2k\eta]$ such that $f(\sigma_1(a_{t_1})) + f(\sigma_2(a_{t_1})) = 2kk'\eta-5k\eta/2+j$ for $j \in [1, k\eta-1]$, while $f(\sigma_1(a_{t_{k\eta}})) + f(\sigma_2(a_{t_{k\eta}})) = 2kk'\eta-k\eta$. We can do this owing to the partition in Lemma 7(4). Then the vertex sums in $A$ are also pairwise different.

Recall that, owing to Step 1 (1.1) and (1.2), for each $b \in B$, one has

$$\varphi_f(b) \leq \frac{((k-1)k'+1)(k-1)}{2} + \frac{3}{2} + kk'p.$$  

On the other hand, let $a \in A$. Recall that, owing to Step 3 (3.1) the sum of the labels in $CM(a)$ is at least $\left\{2(kk'\eta-pk\eta)+(p-1)k\eta+1(p-1)/2\right\} + (kk'\eta-k\eta+1)$ if $p$ is odd (then $(p-1)$ is even) or $k\eta$ is odd. Then owing to Step 3 (3.2), the sum of the labels in $CM(a)$ is at least $\left\{2(kk'\eta-pk\eta)+(p-2)k\eta+1(p-2)/2\right\} + (2kk'\eta-5k\eta/2+1)$ if $p$ is even (then $(p-2)$ is also even) and $k\eta$ is even. And the later lower bound is 1/2 smaller than the first lower bound. So

$$\varphi_f(a) > \frac{2(kk'\eta-pk\eta)+(p-2)k\eta+1(p-2)}{2} + \left(\frac{2kk'\eta-5k\eta}{2} + 1\right).$$  

Then for each $a \in A$ and $b \in B$ one has

$$\varphi_f(a) - \varphi_f(b) > \frac{1}{2} [(p-k)kk'\eta + (k' - 2p - 1)pk\eta + (p^2 - 3)k\eta + (pk + k - k')\eta + (\eta - 1)k + (p - 2)] > 0,$$

since $p \geq k \geq 3$ and $2p + 1 < k' = pk + r \leq pk + k$.

Thus, we obtain an antimagic labeling. This completes our proof. \hfill \blacksquare
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References


