DOMINATION NUMBER OF GRAPHS WITH MINIMUM
DEGREE FIVE

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Abstract

We prove that for every graph $G$ on $n$ vertices and with minimum degree five, the domination number $\gamma(G)$ cannot exceed $n/3$. The proof combines an algorithmic approach and the discharging method. Using the same technique, we provide a shorter proof for the known upper bound $4n/11$ on the domination number of graphs of minimum degree four.

Keywords: dominating set, domination number, discharging method.

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1. Introduction

In this paper we study the minimum dominating sets in graphs of given order $n$ and minimum degree $\delta$. For the case of $\delta = 5$, we improve the previous best upper bound $0.344n$ by proving that the domination number $\gamma$ is at most $n/3$. For graphs of $\delta = 4$, the relation $\gamma \leq 4n/11$ was proved by Sohn and Xudong [22] in 2009. Using a different approach, we provide a simpler proof for this theorem.

Standard definitions. In a simple graph $G$, the vertex set is denoted by $V(G)$ and the edge set by $E(G)$. For a vertex $v \in V(G)$, its closed neighborhood $N[v]$ contains $v$ and its neighbors. For a set $S \subseteq V(G)$, we use the analogous notation $N[S] = \bigcup_{v \in S} N[v]$. The degree of a vertex $v$ is denoted by $d(v)$, while $\delta(G)$ and $\Delta(G)$, respectively, stand for the minimum and maximum vertex degree in $G$. A set $D \subseteq V(G)$ is a dominating set if $N[D] = V(G)$. The minimum cardinality of a dominating set is the domination number $\gamma(G)$ of the graph. An earlier general survey on domination theory is [11], while two new directions were initiated recently in [6] and [5].
General upper bounds on $\gamma(G)$ in terms of the order and minimum degree. The first general upper bound on $\gamma(G)$ in terms of the order $n$ and the minimum degree $\delta$ was given by Arnautov [2] and, independently, by Payan [20]:

$$\gamma(G) \leq \frac{n}{\delta + 1} \sum_{j=1}^{\delta + 1} \frac{1}{j}.$$  

We remark that a bit stronger general results were later published by Clark et al. [9] and Biró et al. [3]. On the other hand, already (1) implies the upper bound

$$\gamma(G) \leq n \left(\frac{1 + \ln(\delta + 1)}{\delta + 1}\right).$$  

It was proved by Alon [1] that (2) is asymptotically sharp when $\delta \to \infty$.

Upper bounds for graphs of small minimum degrees. There are several ways to show that $\gamma(G) \leq n/2$ holds if $\delta(G) = 1$ (see [19] for the first proof). Blank [4], and later independently McCuaig and Shepherd [18] proved that $\gamma(G) \leq 2n/5$ is true if $G$ is connected, $\delta(G) = 2$, and $n \geq 8$. For graphs $G$ with $\delta(G) = 3$, Reed [21] proved the famous result that $\gamma(G) \leq 3n/8$. He also presented a connected cubic graph on 8 vertices for which the upper bound is tight.

In the same paper [21], Reed provided the conjecture that the upper bound can be improved to $\lceil n/3 \rceil$ once the connected cubic graph has an appropriately large order. It was disproved by Kostochka and Stodolsky [14] by constructing an infinite sequence of connected cubic graphs such that all of them have $\gamma(G) \geq \left(\frac{1}{3} + \frac{1}{69}\right) n$. Later, in [15], the same authors proved that $\gamma(G) \leq \frac{4}{11} n = \left(\frac{1}{3} + \frac{1}{33}\right) n$ holds for every connected cubic graph of order $n > 8$. However, it seems a challenging and difficult problem to close the small gap between $\frac{1}{3} + \frac{1}{69}$ and $\frac{1}{3} + \frac{1}{33}$.

For graphs of minimum degree 4, the best known upper bound is $\gamma(G) \leq \frac{4}{11} n$ that was established by Sohn and Xudong [22]. For the case of $\delta(G) = 5$, Xing, Sun, and Chen [23] proved $\gamma(G) \leq \frac{5}{11} n$ which was improved to $\gamma(G) \leq \frac{2671}{1766} n < 0.344 n$ by the authors of [7]. It was also shown in [7] that for graphs of minimum degree 6, the domination number is strictly smaller than $n/3$. Note that similar upper bounds involving the girth and other parameters of the graph can be found in many papers, e.g. in [10, 12, 16, 17], while results for plane triangulations and maximal outerplanar graphs were established in [13] and [8].

Our approach. In the seminal paper [21] of Reed, the upper bound $3n/8$ was proved by considering a vertex-disjoint path cover with specific properties. Later,

\[\text{There are seven small graphs, the cycle } C_4 \text{ and six graphs with } n = 7 \text{ and } \delta = 2, \text{ which do not satisfy } \gamma(G) \leq 2n/5.\]
the same method (with updated conditions and thorough analysis) was used in [15, 22, 23] to establish results on cubic graphs and on graphs of minimum degree 4 and 5. In [7], we introduced a different algorithmic method that resulted in improvement for all cases with $5 \leq \delta \leq 50$. Here, we combine the latter approach with a discharging process. This allows us to prove that already graphs of minimum degree 5 satisfy $\gamma(G) \leq n/3$.

**Residual graph.** Given a graph $G$ and a set $D \subseteq V(G)$, the residual graph $G_D$ is obtained from $G$ by assigning colors to the vertices and deleting some edges according to the following definitions.

- A vertex $v$ is white if $v \notin N[D]$.
- A vertex $v$ is blue if $v \in N[D]$ and $N[v] \nsubseteq N[D]$.
- A vertex $v$ is red if $N[v] \subseteq N[D]$.
- $G_D$ contains only those edges from $G$ that are incident to at least one white vertex.

In $G_D$, we refer to the set of white, blue, and red vertices, respectively, by the notations $W$, $B$, and $R$. It is clear by definitions that $D \subseteq R$ and $W \cup B \cup R = V(G)$ hold. The white-degree $d_W(v)$ of a vertex $v$ is the number of its white neighbors in $G_D$. Analogously, we sometimes refer to the blue-degree $d_B(v)$ of a vertex. The maximum of white-degrees over the sets of white and blue vertices, respectively, are denoted by $\Delta_W(W)$ and $\Delta_W(B)$.

**Observation 1.** Let $G$ be a graph and $D \subseteq V(G)$. The following statements are true for the residual graph $G_D$.

(i) If $v \in W$, then $G_D$ contains all edges which are incident with $v$ in $G$ and, in particular, $N[v] \cap R = \emptyset$ and $d_W(v) + d_B(v) = d(v)$ hold.

(ii) If $v \in B$, then $d_W(v) = |W \cap N[v]| < d(v)$ and $d_B(v) = 0$.

(iii) If $v \in R$, then $v$ is an isolated vertex in $G_D$.

(iv) If $\delta(G) = d$ and $v$ is a white vertex with $d_W(v) = \ell < d$, then $d_B(v) \geq d - \ell$ holds in $G_D$.

(v) $D$ is a dominating set of $G$ if and only if $R = V(G)$ (or equivalently, $W = \emptyset$) in $G_D$.

(vi) If $D \subseteq D' \subseteq V(G)$ and a vertex $v$ is red in $G_D$, it remains red in $G_{D'}$; if $v$ is blue in $G_D$, then it is either blue or red in $G_{D'}$.

**Structure of the paper.** In the next section we prove the improved upper bound $n/3$ on the domination number of graphs with minimum degree 5. In Section 3 we consider graphs of minimum degree 4 and show an alternative proof for the theorem $\gamma \leq 4n/11$. 
2. Graphs of Minimum Degree 5

**Theorem 2.** For every graph $G$ on $n$ vertices and with minimum degree 5, the domination number satisfies $\gamma(G) \leq \frac{2}{3}n$.

**Proof.** Consider a graph $G$ and a subset $D$ of the vertex set $V = V(G)$. Let $W$, $B$, and $R$ denote the set of white, blue, and red vertices respectively, in the residual graph $G_D$. Further, for the sets of blue vertices that have at least 5 white neighbors, or exactly 4, 3, 2, 1 white neighbors, we use the notations $B_5$, $B_4$, $B_3$, $B_2$, and $B_1$ respectively. A vertex is a blue leaf if it belongs to $B_1$. In the proof, a residual graph $G_D$ is associated with the following value:

$$f(G_D) = 35|W| + 23|B_5| + 21|B_4| + 19|B_3| + 17|B_2| + 14|B_1|.$$ 

By Observation 1(v), $f(G_D)$ equals zero if and only if $D$ is a dominating set in $G$. If $G$ and $D$ are fixed and $A$ is a subset of $V \setminus D$, we define

$$s(A) = f(G_D) - f(G_{D\cup A})$$

that is the decrease in the value of $f$ when $D$ is extended by the vertices of $A$. We define the following property for $G_D$:

**Property 1.** There exists a nonempty set $A \subseteq V \setminus D$ such that $s(A) \geq 105|A|$.

Our goal is to prove that every graph $G$ with $\delta(G) = 5$ and every $D \subseteq V$ with $f(G_D) > 0$ satisfy Property 1. Once we do it, Theorem 2 will follow easily. In the continuation, we suppose that a graph $G$ with minimum degree 5 and a set $D$ with $f(G_D) > 0$ do not satisfy Property 1 and prove, by a series of claims, that this assumption leads to a contradiction.

**Claim A.** In $G_D$, every white vertex $v$ has at most two white neighbors, and every blue vertex $u$ has at most three white neighbors.

**Proof.** First suppose that there is vertex $v \in W$ with $d_W(v) \geq 6$. Choosing $A = \{v\}$, the white vertex $v$ becomes red in $G_{D\cup A}$ that decreases $f$ by 35. The white neighbors of $v$ become blue or red which decreases $f$ by at least $6 \cdot (35 - 23)$. Hence, we have $s(A) \geq 35 + 72 = 107 > 105|A|$ complying with Property 1. This contradicts our assumption on $G_D$ and implies that $\Delta_W(W) \leq 5$.

Now, suppose that $\Delta_W(W) = 5$ in $G_D$. Let $v$ be a white vertex with $d_W(v) = 5$ and consider $A = \{v\}$. In $G_{D\cup A}$, the vertex $v$ becomes red and its white neighbors become blue (or red). Since each neighbor $u$ had at most 5 white neighbors in $G_D$ and at least one of them, namely $v$, becomes red, $u$ may have at most 4 white neighbors in $G_{D\cup A}$. Therefore, $s(A) \geq 35 + 5 \cdot (35 - 21) = 105|A|$ holds which is a contradiction again.

If $\Delta_W(W) \leq 4$ and $\Delta_W(B) \geq 6$, let $v$ be a blue vertex with $d_W(v) \geq 6$ and define $A = \{v\}$ again. In $G_D$, the vertex $v$ belongs to $B_5$, while we have $v \in R$ in
$G_{D\cup A}$ which causes a decrease of 23 in the value of $f$. Each white neighbor $u$ of $v$ has at most four white neighbors in $G_D$ and, therefore, $u \in B_3 \cup B_2 \cup B_1 \cup R$ in $G_{D\cup A}$. Hence, we have $s(A) \geq 23 + 6(35 - 21) = 107 > 105 |A|$, a contradiction to our assumption. Note that in the continuation, where we suppose $\Delta_W(B) \leq 5$, if a blue vertex loses $\ell$ white neighbors in a step, it causes a decrease of at least $2\ell$ in the value of $f$.

Assume that $\Delta_W(W) = 4$ and $\Delta_W(B) \leq 5$ and let $v$ be a white vertex with $d_W(v) = 4$ in $G_D$. Set $A = \{v\}$ and consider the decrease $s(A)$. As $v$ turns to be red, this contributes by 35 to $s(A)$. The four white neighbors become blue (or red) and each of them has at most 3 white neighbors in $G_{D\cup A}$. Hence, the contribution to $s(A)$ is at least $4(35 - 19)$. Further, we have $d_W(u) \leq 4$ for each white vertex $u$ from $N[v]$. This implies, by Observation 1(iv), that $u$ has at least one blue neighbor in $G_D$ the white-degree of which is smaller in $G_{D\cup A}$ than in $G_D$.

Even if some blue vertices from $N[N[v]]$ have more than one neighbor from $N[v]$, it remains true that the sum of the white-degrees over $B \cap N[N[v]]$ decreases by at least $d_W(v) + 1 = 5$. We may conclude $s(A) \geq 35 + 4\cdot(35 - 19) + 5 \cdot 2 = 109 > 105 |A|$. Assume that $\Delta_W(W) \leq 3$ and $\Delta_W(B) = 5$ hold in $G_D$ and $v$ is a blue vertex with $d_W(v) = 5$. Let $A = \{v\}$ and consider the decrease $s(A)$. Since $v$ belongs to $B_5$ in $G_D$ and to $R$ in $G_{D\cup A}$, this change contributes by 23 to $s(A)$. The five white neighbors of $u$ become blue or red and belong to $B_3 \cup B_2 \cup B_1 \cup R$ in $G_{D\cup A}$. The contribution to $s(A)$ is not smaller than $5(35 - 19)$. By Observation 1(iv) and by $\Delta_W(W) \leq 3$, each white vertex has at least two blue neighbors in $G_D$. That is, each white neighbor has at least one blue neighbor that is different from $v$. As the five white vertices from $N(v)$ turn blue (or red) in $G_{D\cup A}$, the sum of the white-degrees over $B \cap N[N[v]\setminus\{v\}]$ decreases by at least 5. We infer that $s(A) \geq 23 + 5(35 - 19) + 5 \cdot 2 = 113 > 105 |A|$ which is a contradiction again.

The next case which we consider is $\Delta_W(W) = 3$ and $\Delta_W(B) \leq 4$. Let $v$ be a white vertex with $d_W(v) = 3$ and estimate the value of $s(A)$ for $A = \{v\}$. When $D$ is replaced by $D \cup A$, vertex $v$ is recolored red, the three white neighbors of $v$ become blue or red and belong to $B_2 \cup B_1 \cup R$ in $G_{D\cup A}$. Additionally, each of the three white neighbors and also $v$ itself has at least two blue neighbors. The decrease in their white-degrees contributes to $s(A)$ by at least $4 \cdot 2 \cdot 2$. Consequently, we have $s(A) \geq 35 + 3(35 - 17) + 16 = 105 |A|$ that is a contradiction.

The last case is when $\Delta_W(W) \leq 2$ and $\Delta_W(B) = 4$. We assume that $v$ is a vertex from $B_4$ in $G_D$. Let $A = \{v\}$ and observe that $v$ is recolored red and the white neighbors of $v$ belong to $B_2 \cup B_1 \cup R$ in $G_{D\cup A}$. Since now we have $\Delta_W(W) \leq 2$ in $G_D$, each white vertex has at least three blue neighbors. Therefore, each white neighbor of $v$ has at least two blue neighbors which are different from $v$. We conclude that $s(A) \geq 21 + 4(35 - 17) + 4 \cdot 2 \cdot 2 = 109 > 105 |A|$, this contradiction finishes the proof of Claim A. 

From now on we may suppose that $\Delta_W(W) \leq 2$ and $\Delta_W(B) \leq 3$ holds in the
counterexample $G_D$. This implies that the graph $G_D[W]$, which is induced by the white vertices of $G_D$, contains only paths and cycles as components. Before performing a discharging, we prove some further properties of $G_D$.

**Claim B.** In $G_D[W]$, each component is a path $P_1, P_2$ or a cycle $C_4, C_5, C_7$ or $C_{10}$.

**Proof.** First, suppose that $P_j : v_1 \cdots v_j$ is a path component on $j \geq 3$ vertices in $G_D[W]$. Let us choose $A = \{v_2\}$. In $G_{D \cup A}$ not only $v_2$ but also $v_1$ becomes red, while $v_3$ turns to be either a blue leaf or a red vertex. These changes contribute to $s(A)$ by at least $2 \cdot 35 + (35 - 14)$. By Observation 1(iv), $v_1, v_2, v_3$ and $v_3$, respectively, have at least 4, 3, 3 blue neighbors in $G_D$. The decrease in their white-degrees contributes to $s(A)$ by at least 20. We may infer that $s(A) \geq 70 + 21 + 20 = 111 > 105 \left| A \right|$, a contradiction to our assumption.

We now prove that no cycle of length 3$k$ occurs in $G_D[W]$. Assuming that a cycle $C_{3k} : v_1 \cdots v_{3k}v_1$ exists, all vertices of it can be dominated by the $k$-element set $A = \bigcup_{i=1}^{k} \{v_{3i}\}$. Then, in $G_{D \cup A}$, all the $3k$ vertices are red and, by Observation 1(iv), the sum of the white-degrees of the blue neighbors decreases by at least $3 \cdot 3k$. Consequently, we get the contradiction $w(A) \geq 35 \cdot 3k + 2 \cdot 3 \cdot (3k + 2) = 123k > 105 \left| A \right|$.

Similarly, if we suppose the existence of a cycle $C_{3k+2} : v_1 \cdots v_{3k+2}v_1$ with $k \geq 2$ and define $A = \left( \bigcup_{i=1}^{k} \{v_{3i}\} \right) \cup \{v_{3k+2}\}$, the set $A$ dominates all vertices. Since $k \geq 2$, the relation $s(A) \geq 35 \cdot (3k + 2) + 2 \cdot 3 \cdot (3k + 2) = 123k + 82 > 105(k + 1) = 105 \left| A \right|$ clearly holds and gives the contradiction.

In the last case, consider a cycle $C_{3k+1} : v_1 \cdots v_{3k+1}v_1$ with $k \geq 4$ and set $A = \left( \bigcup_{i=1}^{k} \{v_{3i}\} \right) \cup \{v_{3k+1}\}$. In $G_{D \cup A}$, every vertex from the cycle is red and, as before, one can prove that $s(A) \geq 35 \cdot (3k + 1) + 2 \cdot 3 \cdot (3k + 1) = 123k + 41 > 105(k + 1) = 105 \left| A \right|$. This contradiction finishes the proof of Claim B. \hfill \Box

For $i = 0, 1, 2$, we will use the notation $W_i$ for the set of white vertices having exactly $i$ white neighbors in $G_D$. Note that $W_0$ consists of the vertices of the components of $G_D[W]$ which are isomorphic to $P_1$, while $W_1$ and $W_2$, respectively, contain the vertices from the $P_2$-components and the cycles of $G_D[W]$.

**Claim C.** No vertex from $B_3$ is adjacent to a vertex from $W_0$ in $G_D$.

**Proof.** In contrary, suppose that a vertex $v \in B_3$ has a neighbor $u$ from $W_0$. Let $A = \{v\}$ and denote by $u_1$ and $u_2$ the further two white neighbors of $v$. In $G_{D \cup A}$, we have $v, u \in R$ and $u_1, u_2 \in B_2 \cup B_1 \cup R$. This contributes to $s(A)$ by at least $19 + 35 + 2(35 - 17) = 90$. By Observation 1(iv), the neighbors $u, u_1$ and $u_2$ have, respectively, at least 4, 2, 2 blue neighbors which are different from $v$. As follows, $s(A) \geq 90 + 2 \cdot 8 = 106 > 105 \left| A \right|$ must be true but this contradicts our assumption on $G_D$. \hfill \Box

We call a vertex from $B_2$ special, if it is adjacent to a vertex from $W_0$. 


Claim D. No special vertex is adjacent to two vertices from $W_0$.

**Proof.** Suppose that a vertex $v \in B_2$ is adjacent to two vertices, say $u_1$ and $u_2$ from $W_0$. Then, we set $A = \{v\}$ and observe that all the three vertices $v$, $u_1$ and $u_2$ are red in $G_{D^{\cup}A}$. By Claim C, all the blue neighbors of $u_1$ and $u_2$ are from $B_2 \cup B_1$ in $G_D$ and, therefore, when the white-degree of these neighbors decreases by $\ell$, the value of $f$ falls by at least $(17 - 14)\ell = 3\ell$. Since, by Observation 1(iv), each of $u_1$ and $u_2$ has at least four blue neighbors, we have $s(A) \geq 17 + 2 \cdot 35 + 3 \cdot 8 = 111 > 105 |A|$. This contradiction proves the claim. □

Claim E. No special vertex is adjacent to a vertex from a $C_4$ or $C_7$.

**Proof.** Suppose first that a special vertex $v \in B_2$ is adjacent to $u_1$ which is from a 4-cycle component $C_4$: $u_1u_2u_3u_4u_1$ in $G_D$. The other neighbor of $v$ is $u_0$ which is from $W_0$. Let $A = \{v, u_3\}$ and observe that all the six vertices $v$, $u_0$, $u_1$, $u_2$, $u_3$ and $u_4$ are red in $G_{D^{\cup}A}$. In $G_D$, the white vertex $u_0$ has at least four blue neighbors which are different from $v$ and, by Claim C, each of them belongs to $B_2 \cup B_1$; $u_1$ has at least two neighbors from $(B_3 \cup B_2 \cup B_1) \setminus \{v\}$; each of $u_2$, $u_3$ and $u_4$ has at least three neighbors from $(B_3 \cup B_2 \cup B_1) \setminus \{v\}$. Therefore, $s(A) \geq 17 + 5 \cdot 35 + 4 \cdot 3 + 11 \cdot 2 = 226 > 105 |A|$, a contradiction.

The argumentation is similar if we suppose that a special vertex $v$ is adjacent to $u_0$ from $W_0$ and to a vertex $u_1$ from the 7-cycle $u_1 \cdots u_7u_1$. Here we set $A = \{v, u_3, u_6\}$ and observe that $s(A) \geq 17 + 8 \cdot 35 + 4 \cdot 3 + 20 \cdot 2 = 349 > 105 |A|$ that contradicts our assumption on $G_D$. □

Claim F. If $v_1$ and $v_2$ are two adjacent vertices from $W_1$, then at most one of them may have a special blue neighbor.

**Proof.** Assume to the contrary that $v_1$ is adjacent to the special vertex $u_1$, and $v_2$ is adjacent to the special vertex $u_2$. Denote the other neighbors of $u_1$ and $u_2$ by $x_1$ and $x_2$, respectively. Hence, $v_1, v_2 \in W_1$, $u_1, u_2 \in B_2$ and $x_1, x_2 \in W_0$ hold in $G_D$. Consider the set $A = \{u_1, u_2\}$ and observe that all the six vertices become red in $G_{D^{\cup}A}$. Further, for $i = 1, 2$, vertex $x_i$ has at least four neighbors from $(B_2 \cup B_1) \setminus \{u_i\}$ and $v_i$ has at least three neighbors from $(B_3 \cup B_2 \cup B_1) \setminus \{u_i\}$. Thus, $s(A) \geq 2 \cdot 17 + 4 \cdot 35 + 8 \cdot 3 + 6 \cdot 2 = 210 = 105 |A|$ and this contradiction proves the claim. □

Having Claims A–F in hand, we are ready to prove that every $G_D$ (where $D$ is not a dominating set) satisfies Property 1. The last step of this proof is based on a discharging.

**Discharging.** First, we assign charges to the (non-red) vertices of $G_D$ so that every white vertex gets 35, and every vertex from $B_3$, $B_2$, and $B_1$ gets 19, 17, and 14, respectively. Note that the sum of the charges equals $f(G_D)$. Then, every blue vertex, except the special ones, distributes its charge equally among the white neighbors. The exact rules are the following.
• Every vertex from $B_3$ gives $19/3$ to each white neighbor.
• Every non-special vertex from $B_2$ gives $17/2$ to each white neighbor.
• Every special vertex gives 14 to its neighbor from $W_0$, and gives 3 to the other neighbor.
• Every vertex from $B_1$ gives 14 to its neighbor.

After the discharging, every vertex from a $P_1$-component of $G_D$ has a charge of at least $35 + 5 \cdot 14 = 105$. By Claim F, every $P_2$-component has at least four non-special blue neighbors and, therefore, its charge is at least $2 \cdot 35 + 4 \cdot 3 + 4 \cdot 19/3 = 321/3$. By Claim E, every $C_4$-component has at least $4 \cdot 35 + 12 \cdot 19/3 = 216$ and every $C_7$-component has at least $7 \cdot 35 + 21 \cdot 19/3 = 378$ as a charge. Finally, every $C_5$-component has $5 \cdot 35 + 15 \cdot 3 = 220$, and every $C_{10}$-component has $10 \cdot 35 + 30 \cdot 3 = 440$ after the discharging. Let the number of $P_1$, $P_2$, $C_4$, $C_5$, $C_7$, and $C_{10}$-components of $G_D[W]$ be denoted by $p_1$, $p_2$, $c_4$, $c_5$, $c_7$, and $c_{10}$, respectively, and let $A$ be a minimum dominating set in $G_D[W]$. Then,

$$|A| = p_1 + p_2 + 2c_4 + 2c_5 + 3c_7 + 4c_{10}.$$  

As $D \cup A$ is a dominating set in the graph $G$, we have $f(G_{D \cup A}) = 0$. Thus, $s(A) = f(G_D)$, and the discharging shows the following lower bound:

$$s(A) = f(G_D) \geq 105p_1 + \frac{321}{3}p_2 + 216c_4 + 220c_5 + 378c_7 + 440c_{10} \geq 105(p_1 + p_2 + 2c_4 + 2c_5 + 3c_7 + 4c_{10}) = 105|A|.$$  

As it contradicts our assumption on $G_D$, we infer that every graph $G$ with minimum degree 5 and every $D \subseteq V(G)$ with $f(G_D) > 0$ satisfy Property 1.

To finish the proof of Theorem 2, we first observe that $f(G_0) = 35n$. Then, by Property 1, there exists a nonempty set $A_1$ such that $f(G_{A_1}) \leq f(G_0) - 105|A_1|$. Applying this iteratively, at the end we obtain a dominating set $D = A_1 \cup \cdots \cup A_j$ such that

$$f(G_D) = 0 \leq f(G_0) - 105|D| = 35n - 105|D|,$$

and we may conclude

$$\gamma(G) \leq |D| \leq \frac{35n}{105} = \frac{n}{3}.$$  

In a graph $G$, a set $X \subseteq V(G)$ is a 2-packing, if any two distinct vertices from $X$ are at a distance of at least 3. The proof of Theorem 2 directly corresponds to an algorithm that outputs a dominating set of cardinality at most $n/3$. If $G$ is 5-regular and $X$ is a 2-packing in it, we may start the algorithmic process with choosing the vertices of $X$ one by one. Hence, we conclude the following.

**Corollary 1.** If $G$ is a 5-regular graph on $n$ vertices and $X \subseteq V(G)$ is a 2-packing in $G$, then $X$ can be extended to a dominating set $D$ of cardinality at most $n/3$. 
3. Graphs of Minimum Degree 4

In this section, we apply the previous approach for graphs of minimum degree four and get a shorter alternative proof for the following theorem which was first proved by Sohn and Xudong [22] in 2009.

**Theorem 3.** For every graph \( G \) on \( n \) vertices and with minimum degree 4, the domination number satisfies \( \gamma(G) \leq \frac{4n}{11} \).

**Proof.** Consider a graph \( G \) of minimum degree 4 and let \( D \) be a subset of \( V = V(G) \). Let \( W, B, \) and \( R \) denote the set of white, blue, and red vertices in \( G_D \). The set of blue vertices that have at least 4 white neighbors is denoted by \( B_4 \) while, for \( i = 1, 2, 3 \), \( B_i \) stands for the set of blue vertices that have exactly \( i \) white neighbors. In the proof, a residual graph \( G_D \) is associated with the following value:

\[
g(G_D) = 16|W| + 10|B_4| + 9|B_3| + 8|B_2| + 7|B_1|.
\]

For a set \( A \subseteq V \setminus D \), we use the notation

\[
s(A) = g(G_D) - g(G_{D \cup A})
\]

and define the following property for \( G_D \):

**Property 2.** There exists a nonempty set \( A \subseteq V \setminus D \) such that \( s(A) \geq 44|A| \).

We now suppose for a contradiction that a residual graph \( G_D \) with \( \delta(G) = 4 \) and \( g(G_D) > 0 \) does not satisfy Property 2. We prove several claims for \( G_D \) and then get the final contradiction via performing a discharging.

**Claim G.** \( \Delta_W(W) \leq 2 \) and \( \Delta_W(B) \leq 3 \) hold.

**Proof.** All the following cases can be excluded.

**Case 1.** \( \Delta_W(W) \geq 5 \). Choose a white vertex \( v \) with \( d_W(v) \geq 5 \) and let \( A = \{v\} \). In \( G_{D \cup A} \), the white vertex \( v \) becomes red and its white neighbors become blue or red. This gives \( s(A) \geq 16 + 5 \cdot (16 - 10) = 46 > 44|A| \) which contradicts our assumption that \( G_D \) does not satisfy Property 2.

**Case 2.** \( \Delta_W(W) = 4 \). Consider a white vertex \( v \) with \( d_W(v) = 4 \) and set \( A = \{v\} \). In \( G_{D \cup A} \), the vertex \( v \) becomes red and its white neighbors become blue or red. Since each white neighbor \( u \) had at most four white neighbors in \( G_D \), \( u \) may have at most three white neighbors in \( G_{D \cup A} \). Therefore, \( s(A) \geq 16 + 4 \cdot (16 - 9) = 44|A| \), a contradiction.

**Case 3.** \( \Delta_W(W) \leq 3 \) and \( \Delta_W(B) \geq 5 \). Let \( v \) be a blue vertex with \( d_W(v) \geq 5 \) and define \( A = \{v\} \) again. In \( G_D \), the vertex \( v \) belongs to \( B_4 \), while we have
$v \in R$ in $G_{DUA}$. Further, since $\Delta_W(W) \leq 3$, each white neighbor $u$ of $v$ has at most three white neighbors in $G_D$ and $u \in B_3 \cup B_2 \cup B_1 \cup R$ in $G_{DUA}$. As follows, $s(A) \geq 10 + 5(16 - 9) = 45 > 44 |A|$ that is a contradiction to our assumption.

**Case 4.** $\Delta_W(W) = 3$ and $\Delta_W(B) \leq 4$. First remark that, by the condition $\Delta_W(B) \leq 4$, if a blue vertex loses $\ell$ white neighbors in a step, then $g(G_D)$ decreases by at least $\ell$. Select a white vertex $v$ with $d_W(v) = 3$ and let $A = \{v\}$. In $G_{DUA}$, vertex $v$ becomes red and its three white neighbors become blue or red having at most 2 white neighbors. By Observation 1(iv), each of $v$ and its white neighbors has at least one blue neighbor in $G_D$. Thus, we get $s(A) \geq 16 + 3(16 - 8) + 4 \cdot 1 = 44 |A|$ which is a contradiction.

**Case 5.** $\Delta_W(W) \leq 2$ and $\Delta_W(B) = 4$. Here, we choose a vertex $v$ from $B_4$ and define $A = \{v\}$. First, observe that $v$ belongs to $B_4$ in $G_D$ and to $R$ in $G_{DUA}$. In $G_D$, $v$ has four white neighbors which become blue or red and belong to $B_1 \cup B_2 \cup R$ in $G_{DUA}$. By Observation 1(iv) and $\Delta_W(W) \leq 2$, each white neighbor has at least one blue neighbor which is different from $v$. Therefore, $s(A) \geq 10 + 4(16 - 8) + 4 \cdot 1 = 46 > 44 |A|$ that is a contradiction again. This finishes the proof of the claim.

In the continuation, we suppose that $\Delta_W(W) \leq 2$ and $\Delta_W(B) \leq 3$ hold in the counterexample $G_D$ and, therefore, the graph $G_D[W]$, which is induced by the white vertices of $G_D$, consists of components which are paths and cycles. We prove some further properties for $G_D$.

**Claim H.** In $G_D[W]$, each component is a path $P_1$, $P_2$ or a cycle $C_4$ or $C_7$.

**Proof.** Assume that there is a path component $P_j: v_1 \cdots v_j$ of order $j \geq 3$ in $G_D[W]$. We set $A = \{v_2\}$ and observe that both $v_1$ and $v_2$ become red and $v_3$ belongs to $B_1 \cup R$ in $G_{DUA}$. This contributes to $s(A)$ by at least $2 \cdot 16 + (16 - 7)$. By Observation 1(iv), $v_1$, $v_2$, and $v_3$, respectively, have at least 3, 2, 2 blue neighbors in $G_D$. The decrease in their white-degrees contributes to $s(A)$ by at least $7 \cdot 1$. Then, we get $s(A) \geq 32 + 9 + 7 = 48 > 44 |A|$, a contradiction.

Now, assume that a cycle $C_{3k}: v_1 \cdots v_{3k}v_1$ exists in $G_D[W]$ and set $A = \bigcup_{i=1}^{k} \{v_{3i}\}$. In $G_{DUA}$, all the $3k$ vertices of the cycle are recolored red and, by Observation 1(iv), the sum of the white-degrees of the blue vertices decreases by at least $2 \cdot 3k$. Consequently, we get the contradiction $w(A) \geq 16 \cdot 3k + 6k = 54k > 44 |A|$. A similar argumentation can be given if the cycle is $C_{3k+2}: v_1 \cdots v_{3k+2}v_1$, where $k \geq 1$, and $A = \left( \bigcup_{i=1}^{k} \{v_{3i}\} \right) \cup \{v_{3k+2}\}$. Here, $|A| = k + 1$ and we get $s(A) \geq 16 \cdot (3k + 2) + 2 \cdot (3k + 2) = 54k + 36 > 44k + 44 = 44 |A|$ that is a contradiction. For the case when the cycle is of order $3k + 1$, we suppose $k \geq 3$ and obtain a contradiction as follows. Let $C_{3k+1}: v_1 \cdots v_{3k+1}v_1$ and let $A$ be the $(k + 1)$-element dominating set $\left( \bigcup_{i=1}^{k} \{v_{3i}\} \right) \cup \{v_{3k+2}\}$. We get $s(A) \geq 44 |A|$. 


16 \cdot (3k + 1) + 2 \cdot (3k + 1) = 54k + 18 > 44k + 44 = 44 |A| \text{ since } k \geq 3 \text{ is supposed.}

This finishes the proof of Claim H. \hfill \Box

Claim I. No vertex from $B_3$ is adjacent to any vertices from $W_0$ in $G_D$.

Proof. Assume for a contradiction that a vertex $v \in B_3$ has a neighbor $u_0$ from $W_0$. Let $A = \{v\}$ and denote by $u_1$ and $u_2$ the further two white neighbors of $v$. In $G_{D\cup A}$, if $u_0 \in R$ and $u_1, u_2 \in B_2 \cup B_1 \cup R$. This change contributes to $s(A)$ by at least $9 + 16 + 2(16 - 8) = 41$. By Observation 1(iv), the neighbors $u_0, u_1$ and $u_2$ have, respectively, at least 3, 1, 1 blue neighbors which are different from $v$. Therefore, $s(A) \geq 41 + 5 \cdot 1 = 46 > 44 |A|$ that should be true but this contradicts our assumption on $G_D$. \hfill \Box

As follows, the vertices from $W_0$ may be adjacent only to some vertices from $B_2 \cup B_1$. We call a vertex from $B_2$ special, if it is adjacent to a vertex from $W_0$.

Claim J. No special vertex is adjacent to two vertices from $W_0$.

Proof. Suppose that a vertex $v \in B_2$ is adjacent to two vertices, say $u_1$ and $u_2$ from $W_0$. We set $A = \{v\}$ and observe that all the three vertices $v, u_1$ and $u_2$ are red in $G_{D\cup A}$. By Observation 1(iv), each of $u_1$ and $u_2$ has at least three blue neighbors different from $v$. This yields $s(A) \geq 8 + 2 \cdot 16 + 6 \cdot 1 = 46 > 44 |A|$ that contradicts our assumption on $G_D$. \hfill \Box

Claim K. No special vertex is adjacent to a vertex from a $C_4$ or $C_7$.

Proof. If a special vertex $v$ is adjacent to a vertex $u_0$ from $W_0$ and to a vertex $u_1$ from a 4-cycle component $C_4$: $u_1 u_2 u_3 u_4 u_1$ of $G_D[W]$, then we set $A = \{v, u_3\}$ and observe that $v, u_0, u_1, u_2, u_3$ and $u_4$ turn red in $G_{D\cup A}$. In $G_D$, the vertices $u_0, u_1, u_2, u_3$ and $u_4$, respectively, have at least 3, 1, 2, 2, 2 neighbors from $(B_3 \cup B_2 \cup B_1) \setminus \{v\}$. Thus, $s(A) \geq 8 + 5 \cdot 16 + 10 \cdot 1 = 98 > 44 |A|$, a contradiction. Similarly, if we suppose that a special vertex $v$ is adjacent to $u_0$ from $W_0$ and to a vertex $u_1$ from the 7-cycle $u_1 \cdots u_7 u_1$, we set $A = \{v, u_3, u_6\}$ and conclude that $s(A) \geq 8 + 8 \cdot 16 + 16 \cdot 1 = 152 > 44 |A|$ that contradicts our assumption on $G_D$. \hfill \Box

Claim L. If $v_1$ and $v_2$ are two adjacent vertices from $W_1$, then at most one of them may have a special blue neighbor.

Proof. Assume to the contrary that $v_1 u_1, v_2 u_2 \in E(G)$ such that $u_1$ and $u_2$ are special vertices in $G_D$, and let $x_1$ and $x_2$ be the further white neighbors of $u_1$ and $u_2$. Hence, we have $v_1, v_2 \in W_1$, $u_1, u_2 \in B_2$, and $x_1, x_2 \in W_0$ in $G_D$. Consider the set $A = \{u_1, u_2\}$ and observe that all the six vertices $v_1, v_2, u_1, u_2, x_1, x_2$ become red in $G_{D\cup A}$. For $i = 1, 2$, by Claim I and Observation 1(iv), the vertex $x_i$ has at least three neighbors from $(B_2 \cup B_1) \setminus \{v\}$ and $v_i$ has at least two neighbors from $(B_3 \cup B_2 \cup B_1) \setminus \{v\}$. This implies the contradiction $s(A) \geq 2 \cdot 8 + 4 \cdot 16 + 10 \cdot 1 = 90 > 44 |A|$. \hfill \Box
**Discharging.** Applying Claims G–L, we now perform a discharging and prove that $G_D$ satisfies Property 1. We assign charges to the (non-red) vertices of $G_D$ so that every white vertex gets 16, and every vertex from $B_3$, $B_2$, and $B_1$ gets 9, 8, and 7, respectively. We remark that the sum of these charges equals $g(G_D)$. Then, every blue vertex, except the special ones, distributes its charge equally among the white neighbors as follows:

- Every vertex from $B_3$ gives 3 to each white neighbor.
- Every non-special vertex from $B_2$ gives 4 to each white neighbor.
- Every special vertex gives 7 to its neighbor from $W_0$, and gives 1 to the other neighbor.
- Every vertex from $B_1$ gives 7 to its neighbor.

After the discharging, every vertex from a $P_1$-component of $G_D[W]$ has a charge of at least $16 + 4 \times 7 = 44$. By Claim L, every $P_2$-component has at least three non-special blue neighbors and, therefore, its charge is at least $2 \cdot 16 + 3 \cdot 1 + 3 \cdot 3 = 44$. By Claim K, every $C_4$-component has at least $4 \cdot 16 + 8 \cdot 3 = 88$ and every $C_7$-component has at least $7 \cdot 16 + 14 \cdot 3 = 154$ as a charge. Let the number of $P_1$, $P_2$, $C_4$, and $C_7$-components of $G[W]$ be denoted by $p_1$, $p_2$, $c_4$, and $c_7$, respectively, and let $A$ be a minimum dominating set in $G[W]$. Then,

$$|A| = p_1 + p_2 + 2c_4 + 3c_7.$$

As $D \cup A$ is a dominating set in the graph $G$, we have $g(G_{D \cup A}) = 0$. Thus, $s(A) = g(G_D)$, and the discharging proves the following lower bound:

$$s(A) = g(G_D) \geq 44p_1 + 44p_2 + 88c_4 + 154c_7$$

$$\geq 44(p_1 + p_2 + 2c_4 + 3c_7) = 44|A|.$$

As it contradicts our assumption on $G_D$, we infer that every graph $G$ with minimum degree 4 and every $D \subseteq V(G)$ with $g(G_D) > 0$ satisfy Property 2.

To prove Theorem 3, we observe that $g(G_{\emptyset}) = 16n$ and, by Property 2, there exists a set $A_1$ such that $g(G_{A_1}) \leq g(G_{\emptyset}) - 44|A_1|$. As $G_{A_1}$ also satisfies Property 2, we may continue the process if $g(G_{A_1}) > 0$, and at the end we obtain a dominating set $D = A_1 \cup \cdots \cup A_j$ such that

$$g(G_D) = 0 \leq g(G_{\emptyset}) - 44|D| = 16n - 44|D|.$$

Consequently,

$$\gamma(G) \leq |D| \leq \frac{16}{44}n = \frac{4}{11}n$$

holds for every graph $G$ of minimum degree 4.
4. Concluding Remarks

Theorem 2 shows that $\gamma(G) \leq \frac{n}{3}$ holds for every graph with minimum degree at least 5. However, I do not believe that this upper bound is tight over the class of graphs with $\delta(G) \geq 5$. Examples with $\gamma/n > 1/4$ can possibly be found among larger graphs via computer search or large constructions, but it seems that $\delta(G) \geq 5$ and $n \leq 12$ together implies $\gamma(G) \leq n/4$ that is quite far from the proved $n/3$-upper bound.

Unfortunately, Theorem 3 does not seem sharp either. However, here we have 4-regular examples where the quotient $\gamma/n$ equals $1/3$ that is relatively close to the proved upper bound $4/11$. The smallest such 4-regular graph is $G = K_6 - M$ that is obtained from the complete graph $K_6$ by the deletion of a perfect matching. Then, we have $\gamma(G) = 2 = n/3$. One may guess that this is the sharp upper bound for graphs of minimum degree 4 or, at least, it is true under the following stronger condition:

Conjecture 1. There exists a constant $n_0$ such that for every connected 4-regular graph $G$ of order $n > n_0$, we have $\gamma(G) \leq \frac{n}{3}$.

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References


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