DECOMPOSING 10-REGULAR GRAPHS INTO PATHS OF LENGTH 5

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Abstract

Let $G$ be a 10-regular graph which does not contain any 4-cycles. In this paper, we prove that $G$ can be decomposed into paths of length 5, such that every vertex is a terminal of exactly two paths.

Keywords: 10-regular graph, decomposition, path.

2010 Mathematics Subject Classification: 05C38, 05C70.

1. Introduction

Graphs in this paper are simple. Let $G$ and $H$ be graphs. We say that $G$ has an $H$-decomposition $\mathcal{D} = \{H_1, H_2, \ldots, H_n\}$, if any two elements of $\mathcal{D}$ are edge-disjoint subgraphs of $G$, $H_i$ ($1 \leq i \leq n$) is isomorphic to $H$ and $E(G) = \bigcup_{i=1}^{n} H_i$. For convenience, we use $P_m$ and $C_m$ to denote the path and cycle with $m$ edges, respectively. For a positive integer $r$, an $r$-factor of $G$ is a spanning subgraph $F$ of $G$ such that $d_F(v) = r$ for each vertex $v$ of $G$. A decomposition $\mathcal{F}$ of $G$ is an $r$-factorization if every element of $\mathcal{F}$ is an $r$-factor, any two elements of $\mathcal{F}$ are edge-disjoint subgraphs of $G$, and $E(G)$ can be covered by $\mathcal{F}$.

Graham and Häggkvist [6] posed the following conjecture.

Conjecture 1 (Graham-Häggkvist [6]). Let $T$ be a tree with $l$ edges. If $G$ is a $2l$-regular graph, then $G$ admits a $T$-decomposition.
In the same paper, Håggkvist proved that Conjecture 1 is true when the girth of $G$ is at least the diameter of $T$. In the past several decades, this conjecture interested many researchers and many related results were presented. Fink [5] stated that if $T$ is any tree having $n$ edges ($n \geq 1$), then the $n$-cube $Q_n$ can be decomposed into $2^{n-1}$ edge-disjoint induced subgraphs, each of which is isomorphic to $T$. Erde [4] confirmed that if $n$ is odd and $k \leq n$ such that $k|n2^{n-1}$, then $Q_n$ can be decomposed into paths of length $k$. In [7], Jacobson, Truszczynski and Tuza proved that: (1) there is a wide class of $r$-regular bipartite graphs that can be decomposed into any tree of size $r$; (2) every $r$-regular bipartite graph can be decomposed into any double star of size $r$; (3) every 4-regular bipartite graph can be decomposed into paths of length 4. As one corollary of main result in [8], Jao, Kostochka and West confirmed Conjecture 1 for a $2l$-regular graph which has a 2-factorization such that every cycle consisting of edges from distinct 2-factors has length greater than the diameter of $T$. El-Zanati et al. [3] verified Conjecture 1 when $T$ is a double-star, and further they proved that the double-star $S_{k,k-1}$ can decompose every 2$k$-regular graph which contains a perfect matching.

It is natural to ask whether Conjecture 1 holds if $T$ is a path. Unfortunately, there is no definitive answer for general graphs. Kouider and Lonc [9] proposed a strengthening of Conjecture 1 in the case where $T$ is a path, and it is still open. We say a path decomposition $\mathcal{D}$ is balanced if each vertex is a terminal of exactly two paths of $\mathcal{D}$.

**Conjecture 2** (Kouider and Lonc [9]). Let $l$ be a positive integer. If $G$ is a $2l$-regular graph, then $G$ admits a balanced $P_l$-decomposition.

By Petersen’s Factorization Theorem (see Theorem 3.1), Botler et al. [1] proposed an equivalent form of Conjecture 2.

**Conjecture 3** (Botler et al. [1]). Let $m$ and $l$ be positive integers. Then every $2ml$-regular graph admits a balanced $P_l$-decomposition.

In the same paper, they proved that if $m \geq \lceil (l-2)/(g-2) \rceil$, then every $2ml$-regular graph with girth at least $g$ admits a $P_l$-decomposition. Furthermore, every $2ml$-regular graph with girth at least $l-1$ admits a $P_l$-decomposition for $m \geq 1$. By controlling the girth, Kouider and Lonc [9] confirmed Conjecture 2 for a $2l$-regular graph $G$ with girth at least $(l+3)/2$.

**Theorem 4** (Kouider and Lonc [9]). If $l \leq 2g-3$, then every $2l$-regular graph $G$ of girth $g$ has a balanced $P_l$-decomposition.

By Theorem 4, Conjecture 2 is true for $l = 1, 2$ and 3. When $l = 4$ or 5, every $2l$-regular graph $G$ without triangles has a balanced $P_l$-decomposition. For later use, we will present a short proof when $l = 3$ in Conjecture 2 in Section 3. Based on analysis of the structure of the graph, Botler and Talon [2] used a different method from that in [9] to confirm the conjecture for $l = 4$. 


Theorem 5 (Botler and Talon [2]). If $G$ is an 8-regular graph, then $G$ admits a balanced $P_4$-decomposition.

Motivated by Theorem 5, we want to solve the case $l = 5$. However, the structure of a $P_5$-decomposition in a 10-regular graph is more complex than the structure of a $P_4$-decomposition in an 8-regular graph. Thus we consider $P_5$-decompositions of 10-regular graphs which contain no 4-cycles, and get the main result of this paper.

Theorem 6. Let $G$ be a 10-regular graph. If $G$ does not contain any 4-cycles, then $G$ admits a balanced $P_5$-decomposition.

2. Notations and Terminologies

A trail $T = x_0x_1\cdots x_l$ is a graph for whose $V(T) = \{x_i | 0 \leq i \leq l\}$, $E(T) = \{x_ix_{i+1} | 0 \leq i \leq l-1\}$ and $x_ix_{i+1} \neq x_jx_{j+1}$, for every $i \neq j$. Denote the vertices $x_0$ and $x_l$ as the terminal vertices of $T$, $x_1$ and $x_{l-1}$ as the preterminal vertices of $T$. If a trail has $l$ edges, then we call it an $l$-trail. If a set of edge-disjoint trails $\mathcal{B}$ of a graph $G$ is such that $\bigcup_{B \in \mathcal{B}} E(B) = E(G)$, then $\mathcal{B}$ is a decomposition of $G$ into trails. If every trail of $\mathcal{B}$ has length $l$, then $\mathcal{B}$ is a decomposition into $l$-trails (or an $l$-trail decomposition). For a trail decomposition $\mathcal{B}$ of $G$, if every vertex of $G$ is a terminal of exactly two trails of $\mathcal{B}$, then $\mathcal{B}$ is called balanced. If every trail of $\mathcal{B}$ is a path, then $\mathcal{B}$ is a decomposition into paths (or a path decomposition).

We use $\tau(\mathcal{B})$ to denote the number of elements of $\mathcal{B}$ that are cycles.

A tour of a connected graph $G$ is a closed walk that traverses each edge of $G$ at least once, and an Eulerian tour one that traverses each edge exactly once. A graph is Eulerian if it admits an Eulerian tour. Since an Eulerian tour traverses each edge exactly once, $d(v)$ is even for every vertex $v \in V(G)$. On the other hand, if $G$ is a connected graph and every vertex has even degree, then $G$ has an Eulerian tour by Fleury’s Algorithm. A graph in which each vertex has even degree is called an even graph. Therefore, a graph is Eulerian if and only if is even and connected. An orientation $O$ of a subset $E'$ of $E(G)$ is an attribution of a direction to each edge of $E'$. If an edge $xy$ is directed from $x$ to $y$ in $O$, we say that $xy$ leaves $x$ and enters $y$. For a vertex $v$ of $G$, let $d^+_G(v)$ (respectively, $d^-_G(v)$) be the number of edges leaving (respectively, entering) $v$ with respect to $O$. If $O$ is an orientation of $G$ and every vertex $v$ has $d^+_G(v) = d^-_G(v)$, then $O$ is an Eulerian orientation of $G$. It is easy to see that $G$ is even if it has an Eulerian orientation. If $G$ is even, then each of its components has an Eulerian tour. We can get an Eulerian orientation of $G$ by assigning each edge of $G$ an orientation in such a way that the Eulerian tour of each component of $G$ is a directed Eulerian tour. Thus a graph has an Eulerian orientation if and only if it is even.
3. Proof of Main Theorem

First of all, we will present Petersen’s Factorization Theorem [10].

**Theorem 7** (Petersen’s 2-Factorization Theorem [10]). Every 2k-regular graph admits a 2-factorization.

Nextly, we will introduce a approach used in [2] to get a trial decomposition. By adjusting this decomposition, we finally get the desired result.

Let $G$ be an $r$-regular graph ($r \geq 6$ and is even), $F$ be a 2-factorization of $G$ given by Theorem 7. By combining the elements of $F$, we obtain a decomposition of $G$ into an $(r - 4)$-factor and a 4-factor, say $F_1$ and $F_2$, respectively. Let $O$ be an Eulerian orientation of $F_2$. Suppose $F_1$ has a balanced $P_{(r-4)/2}$-decomposition $D$. So every vertex $v$ of $G$ is a terminal of exactly two paths in $D$. Note that $d_O^+(v) = 2$ for every vertex $v$ of $F_2$. Thus, we can extend every path $P = x_1x_2\cdots x_{(r-4)/2+1}x_{(r-4)/2+2}$ in $D$ to a $(D, O)$-extension $Q_P = x_0x_1\cdots x_{(r-4)/2+2}$ such that $x_0x_1$ and $x_{(r-4)/2+1}x_{(r-4)/2+2}$ are two edges in $F_2$ leaving $x_1$ and $x_{(r-4)/2+1}$, respectively, and further every edge of $F_2$ is used exactly once. Therefore, $\{Q_P \mid P \in D\}$ is a decomposition into $(D, O)$-extensions of $G$, which may not be a decomposition into paths, just into trails. Obviously, each decomposition into $(D, O)$-extensions is balanced.

In this paper, we focus on the path decompositions of a 10-regular graph which does not contain any 4-cycles. Let $F_1$ be a 6-factor of $G$, $F_2$ be a 4-factor such that $F_1 \cup F_2 = G$. $O$ be an Eulerian orientation of $F_2$. By Theorem 4, $F_1$ has a balanced $P_3$-decomposition $D$. Following the method above, we first obtain a decomposition into $(D, O)$-extensions of $G$ from $D$, and then adjust this trail decomposition to a path decomposition of $G$.

Let $G$ be a 6-regular graph. We present a brief proof that $G$ has a balanced $P_3$-decomposition.

**Lemma 8.** If $G$ is a 6-regular graph, then $G$ admits a balanced $P_3$-decomposition $D$ and every vertex of $G$ is a preterminal of exactly two paths in $D$.

**Proof.** Let $F$ be a 2-factorization of $G$ given by Theorem 7. By combining the elements of $F$, we obtain a decomposition of $G$ into a 2-factor and a 4-factor, say $F_3$ and $F_4$, respectively. Obviously, $F_3$ has a balanced $P_1$-decomposition, denoted by $D_1$. Because every vertex of $F_4$ has even degree, there is an Eulerian orientation $O$ on $F_4$. Let $D$ be a decomposition of $G$ into $(D_1, O)$-extensions which minimizes $\tau(D)$. If every element in $D$ is a $P_3$, then we are done. Suppose there is a triangle $C = x_0x_1x_2x_3$ in $D$, $x_0 = x_3$, $x_1x_2 \in D_1$. There is an element $T = y_0y_1y_2y_3$ of $D$ such that $y_1 = x_1$, $y_1y_2 \in D_1$, $y_1y_2 \neq x_1x_2$ and $T \neq C$. Let $C' = y_0x_1x_2x_3$ and $T' = x_0y_1y_2y_3$. Obviously, $C'$ is a path of length 3. Because, $G$ is simple, $y_1$, $y_2$ and $y_3$ are distinct vertices, $x_0 \neq y_1$ and $x_0 \neq y_2$. If $T'$
is a triangle, then \( y_3 = x_0 = x_3 \) and \( d_O(x_0) = 3 \), which is a contradiction to the assumption before that \( O \) is an Eulerian orientation on \( F_1 \). Hence, \( T' \) is a path of length 3. Let \( D' = (D - \{T, C\}) \cup \{T', C'\} \). \( D' \) is a decomposition of \( G \) into \((D_1, O)\)-extensions and \( \tau(D') \leq \tau(D) - 1 \), which is a contradiction to the minimality of \( \tau(D) \). Therefore, \( D \) is a balanced \( P_3 \)-decomposition of \( G \). By the construction of \( D \), we can find that every vertex of \( G \) is a preterminal of exactly two paths in \( D \).

![Figure 1. Extensions.](image)

Now, let \( G \) be a 10-regular graph without a \( C_4 \), \( F_1 \) be a 6-factor of \( G \), \( F_2 \) be a 4-factor such that \( F_1 \cup F_2 = G \). Let \( O \) be an Eulerian orientation of \( F_2 \) and \( D_1 \) be a balanced \( P_3 \)-decomposition of \( F_1 \), and further, \( T = \{Q_P \mid P \in D_1\} \) be a decomposition into \((D_1, O)\)-extensions of \( G \). Let \( T = x_0x_1x_2x_3x_4x_5 \in T \). Because \( D_1 \) is a balanced \( P_3 \)-decomposition of \( F_1 \) and \( G \) does not contain any \( C_4 \), we have \( x_1, x_2, x_3 \) and \( x_4 \) are distinct vertices, \( x_0 \neq x_4 \), \( x_5 \neq x_1 \) and it is impossible that both \( x_0 = x_3 \) and \( x_5 = x_2 \) hold. Hence, if \( T \) is a trail of \( T \), then exactly one of the following holds: (a) \( T \) is a path of length 5; (b) \( T \) is a trail of length 5 which contains a triangle; (c) \( T \) is a cycle of length 5 (see Figure 1).

In the figures throughout this section, we illustrate the edges of \( F_1 \) as straight edges, and the edges of \( F_2 \) as dashed edges. The next result shows that every 10-regular graph admits a decomposition into \((D_1, O)\)-extensions which are not cycles.

**Lemma 9.** Let \( G \) be a 2l-regular graph, \( F_1 \) be a \( 2(l - 2) \)-factor of \( G \), \( F_2 = G \setminus E(F_1) \) and \( O \) be an Eulerian orientation of \( F_2 \). If there is a balanced \( P_{(l-2)} \)-decomposition \( D_1 \) of \( F_1 \), then \( G \) admits a decomposition into \((D_1, O)\)-extensions which are not cycles.

**Proof.** Let \( G, F_1, F_2, D_1, \) and \( O \) be as in the statement above. Now, let \( D \) be a decomposition of \( G \) into \((D_1, O)\)-extensions which minimizes \( \tau(D) \).

Suppose, for contradiction, that \( \tau(D) > 0 \). Let \( T = x_0x_1x_2 \cdots x_{l-1}x_l \) be a cycle of length \( l \) in \( D \), where \( L_1 = x_1x_2 \cdots x_{l-1} \in D_1 \) and \( x_0 = x_l \). Note that \( D_1 \) is balanced. Let \( L_2 = y_1y_2 \cdots y_{l-1} \) be the element of \( D_1 \) such that \( L_2 \neq L_1 \) and \( y_1 = x_1 \). Suppose \( Q = y_0y_1y_2 \cdots y_{l-1}y_l \) is the \((D_1, O)\)-extension of \( L_2 \) in \( D \). Let \( T' = y_0x_1x_2 \cdots x_{l-1}x_i \) and \( Q' = x_0y_1y_2 \cdots y_{l-1}y_l \). Clearly, \( T' \) and \( Q' \) are \((D_1, O)\)-extensions. Because \( G \) is simple, \( y_0 \neq x_1 \). Hence, \( T' \) is not a cycle. Moreover, if \( Q' \) is a cycle, then the edges \( x_0x_1, x_{l-1}x_l, \) and \( y_{l-1}y_l \) are directed.
towards \(x_0\), which implies \(d_G(x_0) \geq 3\), contrary to the fact that \(O\) is an Eulerian orientation of \(F_2\). Therefore, \(\mathcal{D}' = (\mathcal{D} - \{T, Q\}) \cup \{T', Q'\}\) is a decomposition into \((\mathcal{D}_1, O)\)-extensions of \(G\) such that \(\tau(\mathcal{D}') \leq \tau(\mathcal{D}) - 1\), which is a contradiction to the minimality of \(\tau(\mathcal{D})\). This completes the proof of Lemma 9.

In the following, we will define a special Eulerian orientation, which is important for the proof of Theorem 6.

**Definition 10.** Let \(G\) be a 10-regular graph, \(F\) be a 6-factor of \(G\), \(\mathcal{D}\) be a balanced \(P_3\)-decomposition of \(F\), \(H = G \setminus E(F)\). We say that an Eulerian orientation \(O\) on \(H\) is good if the following holds. For each path \(U = v_1v_2v_3\) of \(H\) and distinct vertices \(x_2, x_3, y_2, y_4, z_2, z_4, v_1, v_2, v_3\), if there exists three elements \(T_1 = x_1x_2x_3x_4, T_2 = y_1y_2y_3y_4, T_3 = z_1z_2z_3z_4 \in \mathcal{D}\) and \(x_1 = v_1 = y_1, y_3 = v_2 = z_3, x_4 = v_3 = z_1\), then \(U\) is a directed path under orientation \(O\), no matter which direction it goes (see Figure 2).

![Figure 2. A good orientation on \(U = v_1v_2v_3\).](image)

**Lemma 11.** Let \(G\) be a 10-regular graph without \(C_4\), \(F\) be a 6-factor of \(G\), and \(H = G \setminus E(F)\). Then, there is a good Eulerian orientation on the edges of \(H\).

**Proof.** By Lemma 8, we assume that \(\mathcal{D}\) is a balanced \(P_3\)-decomposition of \(F\) such that every vertex of \(G\) is a preterminal of exactly two paths in \(\mathcal{D}\). Let \(U = v_1v_2v_3\) be a path of length 2 in \(H\). Because \(G\) is simple and does not contain any 4-cycles, if there exists three elements \(T_1 = x_1x_2x_3x_4, T_2 = y_1y_2y_3y_4, T_3 = z_1z_2z_3z_4 \in \mathcal{D}\) and \(x_1 = v_1 = y_1, y_3 = v_2 = z_3, x_4 = v_3 = z_1\), then \(x_2, x_3, y_2, y_4, z_2, z_4, v_1, v_2\) and \(v_3\) are distinct vertices. Hence, \(U\) and \(T_1, T_2, T_3\) form a structure defined in Definition 10. In order to obtain a good Eulerian orientation on \(H\), we need to construct a new even graph \(H'\) from \(H\).

Let \(\mathcal{U} = \{v_i^1v_i^2v_i^3 \mid 1 \leq i \leq k\}\) be the set of all the paths of length 2 in \(H\) which are contained in the structure defined in Definition 10. Note that these paths in \(\mathcal{U}\) are not necessarily edge-disjoint. We claim that \(v_i^1 \neq v_j^1\) for \(U_i = v_i^1v_i^2v_i^3\), \(U_j = v_j^1v_j^2v_j^3 \in \mathcal{U}\) and \(i \neq j\). If not, suppose that \(v_i^1 = v_j^1\). Without loss of generality, let \(U_i\) and three elements \(T_1, T_2, T_3\) of \(\mathcal{D}\) be contained in the structure depicted in Definition 10 such that \(v_i^1, v_2^1 \in V(T_1), v_1^3, v_3^1 \in V(T_2), v_i^1, v_3^1 \in V(T_3)\). If \(|E(U_i) \cap E(U_j)| = 1\), then without loss of generality let \(v_i^1 = v_j^1\). This implies that there
are two paths $T_1$ and $T_3$ of $D$ (because $G$ is simple, $E(U_i), E(U_j) \subseteq E(H)$ and $E(T_m) \subseteq E(F)$ $(1 \leq m \leq 5$), $T_k \neq T_q, k \in \{4, 5\}, q \in \{1, 2, 3\}$) together with $U_j$ and $T_1$ form another structure defined in Definition 10, such that $v_1^j, v_2^j \in V(T_1), v_3^j, v_4^j \in V(T_3), v_5^j \in V(T_5)$. If $|E(U_i) \cap E(U_j)| = 0$, then this implies that there are three paths $T_1, T_3$ and $T_5$ of $D$ (because $G$ is simple, $E(U_i), E(U_j) \subseteq E(H)$ and $E(T_m) \subseteq E(F)$ $(1 \leq m \leq 6$), $T_k \neq T_q, k \in \{4, 5, 6\}, q \in \{1, 2, 3\}$) together with $U_j$ form another structure defined in Definition 10, such that $v_1^j, v_2^j \in V(T_4), v_3^j, v_2^j \in V(T_5), v_1^j, v_3^j \in V(T_6)$. In the two cases, $v_2^j$ appears in at least three paths in $D$ as their preterminal vertex, contrary to that $v_2^j$ is the preterminal vertex of exactly two paths in $D$. Thus, $v_2^j \neq v_2^i$ when $i \neq j$, as claimed. This means that for every vertex $v$ of $G$, there is at most one $U_i \in \mathcal{U}$ such that edges incident with $U_i$ is contained in the subgraph induced by $E_H(v)$ which is the set of $v$ in $H$.

Now we can split edges of $U_i$ in the following way: delete edges $v_1^i, v_2^i$ and $v_3^i v_4^i$, add a new vertex $z$ and two edges $v_1^i z, z v_4^i$. By operating on all elements in $\mathcal{U}$ as described above, we can get a new graph $H'$ from $H$. Let $O'$ be an Eulerian orientation on $H'$. By identifying $z$ and $v_2^i (1 \leq i \leq k)$ in $H'$ and preserving the orientation of $O'$ on all edges after identifying, we get an Eulerian orientation $O$ on $H$. It is obviously that $O$ is good.

Now we are able to prove Theorem 6. For a 5-trail decomposition $B$ of a 10-regular graph $G$, we use $\tau'(B)$ to denote the number of elements of $B$ that are paths.

**Proof of Theorem 6.** Let $G$ be a 10-regular graph without $C_4$, $F$ be a 6-factor of $G$, $D$ be a balanced $P_3$-decomposition of $F$, $H = G \setminus E(F)$, and $O$ be a good Eulerian orientation of $H$. By Lemma 9, $G$ has a decomposition $B$ into $(D, O)$-extensions which are not cycles. Further, we may assume that $\tau'(B)$ is maximum. If $\tau'(B) = |B|$, then we are done. Suppose that $\tau'(B) < |B|$. Let $T \in B$ be a trail containing a triangle.

Let $T = x_0 x_1 x_2 x_3 x_4 x_5$, where $x_1 x_2 x_3 x_4 \in D$, $x_0 = x_3$. There is a trail $Q = y_0 y_1 y_2 y_3 y_4 y_5 \in B$ with $Q \neq T$ such that $y_1 y_2 y_3 y_4 \in D$ and $y_1 = x_1$. We put $T' = y_0 x_1 x_2 x_3 x_4 x_5, Q' = x_0 y_1 y_2 y_3 y_4 y_5$. Because $G$ is simple and does not contain $C_4$, $y_0 \notin V(T)$, which implies that $T'$ is a path. Moreover, $x_0 \neq y_3$ which follows from the fact that $G$ does not contain $C_4$. Hence, if $Q'$ contains a triangle only if $Q$ contains the triangle $y_2 y_3 y_4 y_5$. If $Q'$ is not a cycle, then $B' = (B \setminus \{T, Q\}) \cup \{T', Q'\}$ is a decomposition of $G$ into $(D, O)$-extensions with $\tau'(B') \geq \tau'(B) + 1$, which is a contradiction to the maximality of $\tau'(B)$. In the following, we assume $Q'$ is a cycle.

Now, $y_5 = x_0 = x_3$. Note that $G$ is simple and does not contain any 4-cycles. We have that $y_1 \neq y_4, y_2$ and $y_3$ are not equal to any one of $\{x_1, x_2, x_3, x_4, x_5\}$, $y_4$ is not equal to any one of $\{x_1, x_2, x_3, x_4\}$. In this case, $y_4$ and $x_5$ may be the same one. Let $R = z_0 z_1 z_2 z_3 z_4 z_5$ be an element in $B$ different from $T$ and
$Q$, where $z_1z_2z_3z_4 \in \mathcal{D}$ and $z_4 = y_4$ (see Figure 3). Let $Q'' = x_0y_1y_2y_3y_4z_5$, $R' = z_0z_1z_2z_3z_4y_5$. Because $G$ is simple and does not contain any 4-cycles. We have that $z_5 \notin V(Q')$. Hence $Q''$ is a path. If $R'$ is a cycle, we have $x_0 = x_3 = z_0$, $d_G(x_0) \geq 3$, contrary to the fact that $O$ is an Eulerian orientation of $H$. Hence, $R'$ is not a cycle. If $R$ contains a triangle, then $\mathcal{B}' = (\mathcal{B} - \{T, Q, R\}) \cup \{T', Q'', R'\}$ is a decomposition of $G$ into $(\mathcal{D}, O)$-extensions with $\tau'(\mathcal{B}') \geq \tau'(\mathcal{B}) + 1$, which is a contradiction to the maximality of $\tau'(\mathcal{B})$. In the following, we assume $R$ is a path. Because $G$ is simple and does not contain $C_4$, $y_5 \neq z_1, z_3$. If $y_5 = z_2$, then let $U = x_1x_0y_4$, $T_1 = y_1y_2y_3y_4$, $T_2 = x_1x_2x_3x_4$ and $T_3 = z_4z_3z_2z_1$. Now, we want to prove that $U, T_1, T_2$ and $T_3$ form the structure defined in the Definition 10. Note that $x_0 = x_3 = y_5 = z_2$, $x_1 = y_1$ and $y_4 = z_4$. Therefore, we should check $x_0, x_1, x_2, x_4, y_2, y_3, y_4, z_1$ and $z_3$ are distinct vertices of $G$. Because $G$ is simple, $x_0, x_1, x_2, x_4, y_4, z_1$ and $z_3$ are distinct vertices, $x_0, x_1, y_2, y_3$ and $y_4$ are distinct vertices, $x_2 \neq y_2$ and $y_3 \neq z_3$. What remains is the following cases. If $z_1 = y_2$ (respectively, $y_3$), then $z_1x_1x_2x_3z_1$ (respectively, $z_1z_4z_3z_2z_1$) is a cycle of length 4, a contradiction. If $x_4 = y_2$ (respectively, $y_3$), then $y_2x_1x_2x_3y_2$ (respectively, $y_3y_2y_1x_3y_3$) is a cycle of length 4, a contradiction. If $y_2 = z_3$, then $y_2x_0x_2x_1y_2$ is a cycle of length 4, a contradiction. If $y_3 = x_2$ (respectively, $z_1$), then $y_3x_4z_3z_2x_2$ (respectively, $z_1z_4z_3z_2z_1$) is a cycle of length 4, also a contradiction. Hence, $x_0, x_1, x_2, x_4, y_2, y_3, y_4, z_1$ and $z_3$ are distinct vertices of $G$, and $U, T_1, T_2$ and $T_3$ form the structure defined in the Definition 10. But the orientation of $E(U)$ implies that $O$ is not a good Eulerian orientation of $H$, a contradiction to our assumption. Hence, $R'$ is a path and $\mathcal{B}' = (\mathcal{B} - \{T, Q, R\}) \cup \{T', Q'', R'\}$ is a decomposition of $G$ into $(\mathcal{D}, O)$-extensions with $\tau'(\mathcal{B}') \geq \tau'(\mathcal{B}) + 1$, which is a contradiction to the maximality of $\tau'(\mathcal{B})$. This completes the proof of Theorem 6.

Acknowledgments

The authors would like to thank the anonymous reviewers for their careful reading and useful suggestions. This work is supported by the National Natural Science Foundation of China (No. 11971110).
References


Received 29 October 2019
Revised 2 May 2020
Accepted 4 May 2020