ROMAN \{2\}-DOMINATION PROBLEM IN GRAPHS\(^1\)

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Abstract

For a graph \(G = (V, E)\), a Roman \{2\}-dominating function (R2DF) \(f : V \to \{0, 1, 2\}\) has the property that for every vertex \(v \in V\) with \(f(v) = 0\), either there exists a neighbor \(u \in N(v)\), with \(f(u) = 2\), or at least two neighbors \(x, y \in N(v)\) having \(f(x) = f(y) = 1\). The weight of an R2DF \(f\) is the sum \(f(V) = \sum_{v \in V} f(v)\), and the minimum weight of an R2DF on \(G\) is the Roman \{2\}-domination number \(\gamma_{\{R2\}}(G)\). An R2DF is independent if the set of vertices having positive function values is an independent set. The independent Roman \{2\}-domination number \(i_{\{R2\}}(G)\) is the minimum weight of an independent Roman \{2\}-dominating function on \(G\). In this paper, we show that the decision problem associated with \(\gamma_{\{R2\}}(G)\) is NP-complete even when restricted to split graphs. We design a linear time algorithm for computing the value of \(i_{\{R2\}}(T)\) in any tree \(T\), which answers an open problem raised by Rahmouni and Chellali [Independent Roman \{2\}-domination in graphs, Discrete Appl. Math. 236 (2018) 408–414]. Moreover, we present a linear time algorithm for computing the value of \(\gamma_{\{R2\}}(G)\) in any block graph \(G\), which is a generalization of trees.

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Let $G = (V,E)$ be a simple graph. The open neighborhood $N_G(v)$ of a vertex $v$ consists of the vertices adjacent to $v$ and its closed neighborhood is $N_G[v] = N_G(v) \cup \{v\}$. $N_G^2[v] = \{u : d_G(u,v) \leq 2\}$, where $d_G(u,v)$ is the distance between $u$ and $v$ in graph $G$. For an edge $e = uv$, it is said that $u$ (respectively, $v$) is incident to $e$, denoted by $u \in e$ (respectively, $v \in e$). A Roman dominating function (RDF) on graph $G$ is a function $f : V \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of a Roman dominating function $f$ is the value $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a Roman dominating function on a graph $G$ is called the Roman domination number $\gamma_R(G)$ of $G$. Roman domination and its variations have been studied in a number of recent papers (see, for example, [1, 6, 9]).

Chellali, Haynes, Hedetniemi and McRae [4] introduced a variant of Roman dominating functions. For a graph $G = (V,E)$, a Roman $\{2\}$-dominating function (R2DF) $f : V \rightarrow \{0, 1, 2\}$ has the slightly different property that only for every vertex $v \in V$ with $f(v) = 0$, $f(N(v)) \geq 2$, that is, either there exists a neighbor $u \in N(v)$, with $f(u) = 2$, or at least two neighbors $x, y \in N(u)$ have $f(x) = f(y) = 1$. The weight of a Roman $\{2\}$-dominating function is the sum $f(V) = \sum_{v \in V} f(v)$, and the minimum weight of a Roman $\{2\}$-dominating function $f$ is the Roman $\{2\}$-domination number, denoted $\gamma_{\{R2\}}(G)$. Roman $\{2\}$-domination is also called Italian domination by some scholars ([8]). Suppose that $f : V \rightarrow \{0, 1, 2\}$ is an R2DF on a graph $G = (V,E)$. Let $V_i = \{v : f(v) = i\}$, for $i \in \{0, 1, 2\}$. If $V_1 \cup V_2$ is an independent set, then $f$ is called an independent Roman $\{2\}$-dominating function (IR2DF), which was introduced by Rahmouni and Chellali [11] in a recent paper. The minimum weight of an independent Roman $\{2\}$-dominating function $f$ is the independent Roman $\{2\}$-domination number, denoted $i_{\{R2\}}(G)$. The authors in [4, 11] have showed that the associated decision problems for Roman $\{2\}$-domination and independent Roman $\{2\}$-domination are NP-complete for bipartite graphs. The authors in [4] have showed that $\gamma_{\{R2\}}(T)$ can be computed by a linear time algorithm for any tree $T$. In [11], the authors raised some interesting open problems, one of which is whether there is a linear time algorithm for computing $i_{\{R2\}}(T)$ for any tree $T$.

A graph $G = (V,E)$ is a split graph if $V$ can be partitioned into $C$ and $I$, where $C$ is a clique and $I$ is an independent set of $G$. Split graph is an important subclass of chordal graphs, and it turns out to be very important in the domination theory (see [2, 7]). A maximal connected induced subgraph without a cut-vertex is called a block of $G$. We use $K_n$ to denote the complete graph of order $n$. A graph $G$ is a block graph if every block in $G$ is a complete graph. If every block of $G$ is a $K_2$, then $G$ is a tree. Hence, block graphs contain trees.
as its subclass. There are widely research on variations of domination in block graphs (see, for example, [3, 5, 10, 14]).

In this paper, we first show that the decision problem associated with $\gamma_\{R2\}(G)$ is NP-complete for split graphs. Then, we give a linear time algorithm for computing $i_\{R2\}(T)$ in any tree $T$. Moreover, we present a linear time algorithm for computing $\gamma_\{R2\}(G)$ in any block graph $G$.

2. Complexity Result

In this section, we consider the decision problem associated with Roman $\{2\}$-dominating functions.

**ROMAN $\{2\}$-DOMINATING FUNCTION (R2D)**

**INSTANCE:** A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

**QUESTION:** Does $G$ have a Roman $\{2\}$-dominating function of weight at most $k$?

A vertex cover of $G$ is a subset $V' \subseteq V$ such that for each edge $uv \in E$, at least one of $u$ and $v$ belongs to $V'$. Vertex Cover (VC) problem is a well-known NP-complete problem. We show R2D problem is NP-complete by reducing the Vertex Cover (VC) to R2D.

**VERTEX COVER (VC)**

**INSTANCE:** A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

**QUESTION:** Is there a vertex cover of size $k$ or less for $G$?

**Theorem 1.** R2D is NP-complete for split graphs.

**Proof.** R2D is a member of NP, since we can check in polynomial time that a function $f : V \rightarrow \{0, 1, 2\}$ has weight at most $k$ and is a Roman $\{2\}$-dominating function. The proof is given by reducing the VC problem in general graphs to the R2D problem in split graphs.

Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. Let $V^1 = \{v'_1, v'_2, \ldots, v'_n\}$. We construct the graph $G' = (V', E')$ with $V' = V^1 \cup V \cup E$, $E' = \{v_iv_j : v_i \neq v_j, v_i \in V, v_j \in V\} \cup \{v'_iv'_j : i = 1, \ldots, n\} \cup \{ve : v \in e, e \in E\}$.

Notice that $G'$ is a split graph whose vertex set $V'$ is the disjoint union of the clique $V$ and the independent set $V^1 \cup E$. It is clear that $G'$ can be constructed in polynomial time from $G$.

If $G$ has a vertex cover $C$ of size at most $k$, let $f : V' \rightarrow \{0, 1, 2\}$ be a function
H. Chen and C. Lu defined as follows.
\[ f(v) = \begin{cases} 
2, & \text{if } v \in C; \\
1, & \text{if } v \in V^1 \text{ and let } v' \text{ be a neighbor of } v \text{ such that } v' \in V \setminus C; \\
0, & \text{otherwise.}
\end{cases} \]

It is clear that \( f \) is a Roman \( \{2\} \)-dominating function of \( G' \) with weight at most \( 2k + (n - k) \).

On the other hand, suppose that \( G' \) has a Roman \( \{2\} \)-dominating function of weight at most \( 2k + (n - k) \). Among all such functions, let \( g = (V_0, V_1, V_2) \) be one chosen so that:

(C1) \( |V^1 \cap V_2| \) is minimized;
(C2) subject to condition (C1): \( |E \cap V_0| \) is maximized;
(C3) subject to conditions (C1) and (C2): \( |V \cap V_1| \) is minimized;
(C4) subject to conditions (C1), (C2) and (C3): the weight of \( g \) is minimized.

We make the following remarks.

(i) No vertex in \( V^1 \) belongs to \( V_2 \). Indeed, suppose to the contrary that \( g(v'_i) = 2 \) for some \( i \). We reassign \( 0 \) to \( v'_i \) instead of \( 2 \) and reassign \( 2 \) to \( v_i \). Then it provides an R2DF on \( G' \) of weight at most \( 2k + (n - k) \) but with less vertices of \( V^1 \) assigned \( 2 \), contradicting the condition (C1) in the choice of \( g \).

(ii) No vertex in \( E \) belongs to \( V_2 \). Indeed, suppose that \( g(e) = 2 \) for some \( e \in E \) and \( v_j, v_k \in e \). By reassigning \( 0 \) to \( e \) instead of \( 2 \) and reassigning \( 2 \) to \( v_j \) instead of \( g(v_j) \), we obtain an R2DF on \( G' \) of weight at most \( 2k + (n - k) \) but with more vertices of \( E \) assigned \( 0 \), contradicting the condition (C2) in the choice of \( g \).

(iii) No vertex in \( E \) belongs to \( V_1 \). Suppose that \( g(e) = 1 \) for some \( e \in E \) and \( v_j, v_k \in e \). If \( g(v'_j) = 0 \), then \( g(v_j) = 2 \) (by the definition of R2DF). By reassigning \( 0 \) to \( e \) instead of \( 1 \), we obtain an R2DF on \( G' \) of weight at most \( 2k + (n - k) \) but with more vertices of \( E \) assigned \( 0 \), contradicting the condition (C2) in the choice of \( g \). Hence we may assume that \( g(v'_j) = 1 \) (by (i)). Clearly we can reassign \( 2 \) to \( v_j \) instead of \( 0 \), \( 0 \) to \( v'_j \) instead of \( 1 \) and \( 0 \) to \( e \) instead of \( 1 \). We also obtain a R2DF on \( G' \) of weight at most \( 2k + (n - k) \) but with more vertices of \( E \) assigned \( 0 \), contradicting the condition (C2) in the choice of \( g \).

(iv) No vertex in \( V \) belongs to \( V_1 \). Suppose to the contrary that \( g(v_i) = 1 \) for some \( i \), then \( g(v'_i) = 1 \) (by (i) and the definition of R2DF). We reassign \( 0 \) to \( v'_i \) instead of \( 1 \) and \( 2 \) to \( v_i \) instead of \( 1 \). It provides a R2DF on \( G' \) of weight at most
2k + (n − k) but with less vertices of V assigned 1, contradicting the condition (C3) in the choice of g.

(v) If a vertex in V is assigned 2, then its neighbor in V^1 is assigned 0 by the condition (C4) in the choice of g.

(vi) If a vertex in V is assigned 0, then its neighbor in V^1 is assigned 1 by the definition of R2DF and (i).

Therefore, according to the previous items, we conclude that V^1 ∩ V_2 = ∅, E ⊆ V_0, and V ∩ V_i = ∅. Hence V_2 ⊆ V. Let C = \{v : g(v) = 2\}. Since each vertex in \(E \cup (V \setminus C)\) belongs to V_0 in G', it is clear that C is a vertex cover of G by the definition of R2DF. Then \(g(V^1) + g(V) + g(E) = 2|C| + (n − |C|) \leq 2k + (n − k),\) implying that |C| ≤ k. Consequently, C is a vertex cover for G of size at most k.

Since the vertex cover problem is NP-complete, the Roman \{2\}-domination problem is NP-complete for split graphs.

3. Independent Roman \{2\}-Domination in Trees

In this section, a linear time dynamic programming style algorithm is given to compute the exact value of the independent Roman \{2\}-dominating number in any tree. This algorithm is constructed using the methodology of Wimer [13]. A rooted tree is a pair (T, r) with T is a tree and r is a vertex of T. We call r is the root of tree T. A rooted tree (T, r) is trivial if V(T) = r. Given two rooted trees (T_1, r_1) and (T_2, r_2) with V(T_1) ∩ V(T_2) = ∅, the composition of them is (T_1, r_1) ∘ (T_2, r_2) = (T, r_1) with V(T) = V(T_1) ∪ V(T_2) and E(T) = E(T_1) ∪ E(T_2) ∪ \{r_1r_2\}. It is clear that any rooted tree can be constructed recursively from trivial rooted trees using the defined composition.

Let f : V(T) → \{0, 1, 2\} be a function on T. Then f splits two functions f_1 and f_2 according to this decomposition. We express this as follows: (T, f, r) = (T_1, f_1, r_1) ∘ (T_2, f_2, r_2), where r = r_1, f_i = f|_{T_i} is the function f restricted to the vertices of T_i, i = 1, 2. On the other hand, let f_i : V(T_i) → \{0, 1, 2\} be a function on T_i (i = 1, 2). We can define the composition as follows: (T_1, f_1, r_1) ∘ (T_2, f_2, r_2) = (T, f, r), where V(T) = V(T_1) ∪ V(T_2), E(T) = E(T_1) ∪ E(T_2) ∪ \{r_1r_2\}, r = r_1 and f = f_1 ∘ f_2 : V(T) → \{0, 1, 2\} with f(v) = f_1(v) if v ∈ V(T_i), i = 1, 2. Before presenting the algorithm, let us give the following observation.

Observation 2. Let f be an IR2DF of T = T_1 ⊔ T_2 and f_i = f|_{T_i} (i = 1, 2). If \(f_i(r_i) \neq 0\), then \(f_i\) is an IR2DF of \(T_i\). If \(f_i(r_i) = 0\), then \(f_i\) restricted to the vertices of \(T_i − r_i\) is an IR2DF of \(T_i − r_i\).

In order to construct an algorithm for computing the independent Roman \{2\}-domination number, we must characterize the possible tree-subset tuples (T, f, r). For this purpose, we introduce some additional notations as follows:
IR2DF(T) = \{ f : f is an IR2DF of T \},
IR2DFr(T) = \{ f : f \notin IR2DF(T), but f|_{T-r} is an IR2DF of T \}.

Then we consider the following five classes:

A = \{ (T, f, r) : f \in IR2DF(T) and f(r) = 2 \},
B = \{ (T, f, r) : f \in IR2DF(T) and f(r) = 1 \},
C = \{ (T, f, r) : f \in IR2DF(T) and f(r) = 0 \},
D = \{ (T, f, r) : f \in IR2DFr(T) and f(N[r]) = 1 \},
F = \{ (T, f, r) : f \in IR2DFr(T) and f(N[r]) = 0 \}.

Let M, N \in \{ A, B, C, D, F \}. If (T_1, f_1, r_1) \in M and (T_2, f_2, r_2) \in N, we use M \circ N to denote the set of (T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2). Let (T, r) = (T_1, r_1) \circ (T_2, r_2) and r = r_1. Suppose that f_1 (respectively, f_2) is a function on T_1 (respectively, T_2). Define f as the function on T with f|_{T_1} = f_1 and f|_{T_2} = f_2.

Next, we provide some lemmas.

**Lemma 3.** A = (A \circ C) \cup (A \circ D) \cup (A \circ F).

**Proof.** It is clear that the following items are true.

(i) If (T_1, f_1, r_1) \in A and (T_2, f_2, r_2) \in C, then (T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A.
(ii) If (T_1, f_1, r_1) \in A and (T_2, f_2, r_2) \in D, then (T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A.
(iii) If (T_1, f_1, r_1) \in A and (T_2, f_2, r_2) \in F, then (T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A.

Thus, (A \circ C) \cup (A \circ D) \cup (A \circ F) \subseteq A.

Now we prove that A \subseteq (A \circ C) \cup (A \circ D) \cup (A \circ F). Let (T, f, r) \in A and (T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2), then f_1(r_1) = f(r) = 2. Since f \in IR2DF(T), then f_1 \in IR2DF(T_1). So (T_1, f_1, r_1) \in A. From the independence of V_1 \cup V_2, we have f_2(r_2) = f(r_2) = 0. If f_2 \notin IR2DF(T_2), then we obtain (T_2, f_2, r_2) \in C. If f_2 \notin IR2DF(T_2), then (T_2, f_2, r_2) \in D or F. Hence, we conclude that A \subseteq (A \circ C) \cup (A \circ D) \cup (A \circ F).

**Lemma 4.** B = (B \circ C) \cup (B \circ D).

**Proof.** It is easy to check the following items.

(i) If (T_1, f_1, r_1) \in B and (T_2, f_2, r_2) \in C, then (T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B.
(ii) If (T_1, f_1, r_1) \in B and (T_2, f_2, r_2) \in D, then (T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B.

So, (B \circ C) \cup (B \circ D) \subseteq B.

Next we need to show B \subseteq (B \circ C) \cup (B \circ D). Let (T, f, r) \in B and (T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2), then f_1(r_1) = f(r) = 1. It is clear that f_1 \in IR2DF(T_1). So we conclude that (T_1, f_1, r_1) \in B. From the definition of IR2DF, we must have f_2(r_2) = f(r_2) = 0. If f_2 \in IR2DF(T_2), then we obtain (T_2, f_2, r_2) \in C. If f_2 \notin IR2DF(T_2), then f_2(N_{T_2}[r_2]) = 1 and f_2|_{T_2-r_2} \in IR2DF(T_2 - r_2) using the fact that (T, f, r) \in B. Therefore, we have f_2 \in IR2DF_{T_2}(T_2), implying that (T_2, f_2, r_2) \in D. Hence, we deduce that B \subseteq (B \circ C) \cup (B \circ D).
Lemma 5. $C = (C \circ A) \cup (C \circ B) \cup (C \circ C) \cup (D \circ A) \cup (D \circ B) \cup (F \circ A)$.

Proof. It is easy to check the following remarks by definitions.

(i) If $(T_1, f_1, r_1) \in C$ and $(T_2, f_2, r_2) \in A$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in C$.

(ii) If $(T_1, f_1, r_1) \in C$ and $(T_2, f_2, r_2) \in B$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in C$.

(iii) If $(T_1, f_1, r_1) \in C$ and $(T_2, f_2, r_2) \in C$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in C$.

(iv) If $(T_1, f_1, r_1) \in D$ and $(T_2, f_2, r_2) \in A$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in C$.

(v) If $(T_1, f_1, r_1) \in D$ and $(T_2, f_2, r_2) \in B$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in C$.

(vi) If $(T_1, f_1, r_1) \in F$ and $(T_2, f_2, r_2) \in A$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in C$.

Hence, we deduce that $(C \circ A) \cup (C \circ B) \cup (C \circ C) \cup (D \circ A) \cup (D \circ B) \cup (F \circ A) \subseteq C$.

Therefore, we need to prove $C \subseteq (C \circ A) \cup (C \circ B) \cup (C \circ C) \cup (D \circ A) \cup (D \circ B) \cup (F \circ A)$. Let $(T, f, r) \in C$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then $f \in IR2DF(T)$ and $f_1(r_1) = f(r) = 0$. Consider the following cases.

Case 1. $f(r_2) = 2$. Since $f \in IR2DF(T)$, $f_2 \in IR2DF(T_2)$. Hence, $(T_2, f_2, r_2) \in A$. If $f_1 \in IR2DF(T_1)$, then we obtain that $(T_1, f_1, r_1) \in C$. If $f_1 \notin IR2DF(T_1)$, we have $(T_1, f_1, r_1) \in D$ or $F$.

Case 2. $f(r_2) = 1$. Since $f \in IR2DF(T)$, $f_2 \in IR2DF(T_2)$. So $(T_2, f_2, r_2) \in B$. If $f_1 \in IR2DF(T_1)$, then we deduce $(T_1, f_1, r_1) \in C$. If $f_1 \notin IR2DF(T_1)$, therefore, it implies that $(T_1, f_1, r_1) \in D$.

Case 3. $f(r_2) = 0$. It is clear that $f_1$ and $f_2$ are both IR2DF. Then we obtain that $(T_1, f_1, r_1) \in C$ and $(T_2, f_2, r_2) \in C$.

Hence, $C \subseteq (C \circ A) \cup (C \circ B) \cup (C \circ C) \cup (D \circ A) \cup (D \circ B) \cup (F \circ A)$. ■

Lemma 6. $D = (D \circ C) \cup (F \circ B)$.

Proof. It is easy to check the following remarks by definitions.

(i) If $(T_1, f_1, r_1) \in D$ and $(T_2, f_2, r_2) \in C$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in D$.

(ii) If $(T_1, f_1, r_1) \in F$ and $(T_2, f_2, r_2) \in B$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in D$.

Thus, $(D \circ C) \cup (F \circ B) \subseteq D$.

On the other hand, we show $D \subseteq (D \circ C) \cup (F \circ B)$. Let $(T, f, r) \in D$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$. Then $f_1(r_1) = f(r) = 0$. By the definition of $D$, $f_2 \in IR2DF(T_2)$. Using the fact that $f(N_T[r_1]) = 1$, we deduce that $f(r_2) < 2$. Consider the following cases.

Case 1. $f(r_2) = 1$. It is clear that $(T_2, f_2, r_2) \in B$ because $f_2$ is an IR2DF of $T_2$. Since $f_1(N_{T_1}[r_1]) = 0$, we obtain $f_1[r_1] \in IR2DF(T_1 - r_1)$. Hence, we have $f_1 \in IR2DF_{r_1}(T_1)$, implying that $(T_1, f_1, r_1) \in F$.

Case 2. $f(r_2) = 0$. Then $f_2$ is an IR2DF of $T_2$, implying that $(T_2, f_2, r_2) \in C$. Using the fact that $f(N_T[r_1]) = 1$ and $f(r_2) = 0$, we know $f_1(N_{T_1}[r_1]) = 1$. So $f_1 \in IR2DF_{r_1}(T_1)$. It implies that $(T_1, f_1, r_1) \in D$.
Lemma 7. $F = F \circ C$.

Proof. If $(T_1, f_1, r_1) \in F$ and $(T_2, f_2, r_2) \in C$, then it is clear that $(T, f, r) \in F$. Hence, $(F \circ C) \subseteq F$.

On the other hand, let $(T, f, r) \in F$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$. Then $f_1(r_1) = f(r) = 0$. By the definition of $F$, we deduce that $f(r_2) = 0$. Using the fact that $(T, f, r) \in F$, we have that $f_2 \in \text{IR2DF}(T_2)$. So $(T_2, f_2, r_2) \in C$. Notice that $(T, f, r) \in F$, we have $f_1(N_{T_1}[r_1]) = 0$, implying that $(T_1, f_1, r_1) \notin D$. We can easily check that $f_1 \in \text{IR2DF}_{T_1}(T)$. Hence, we have $(T_1, f_1, r_1) \in F$, implying that $F \subseteq (F \circ C)$. \hfill \blacksquare

Let $T = (V, E)$ be a tree with $n$ vertices. It is well known that the vertices of $T$ have an ordering $v_1, v_2, \ldots, v_n$ such that for each $1 \leq i \leq n-1$, $v_i$ is adjacent to exactly one vertex $v_j$ with $j > i$ (see [12]). The ordering is called a tree ordering where the only neighbor $v_j$ with $j > i$ is called the father of $v_i$ and $v_i$ is a child of $v_j$. For each $1 \leq i \leq n-1$, the father of $v_i$ is denoted by $F(v_i) = v_j$.

For each vertex $v_i$ $(1 \leq i \leq n)$, define a vector $l[i, 1..5]$. Let $T_{v_i}$ be a tree such that $v_i$ is the root of $T_{v_i}$. For each rooted tree $(T_{v_i}, v_i)$, let $f_{v_i} : V(T_{v_i}) \to \{0, 1, 2\}$ be a function on $T_{v_i}$ and define $w(f_{v_i}) = f_{v_i}(V(T_{v_i}))$. In this case, for a tree, the only basis graph is a single vertex. Then, the vector $l[i, 1..5]$ is initialized by

$$
\begin{align*}
\min_{(T_{v_i}, f_{v_i}, v_i) \in A} w(f_{v_i}), & \quad \min_{(T_{v_i}, f_{v_i}, v_i) \in B} w(f_{v_i}), & \quad \min_{(T_{v_i}, f_{v_i}, v_i) \in C} w(f_{v_i}), & \quad \min_{(T_{v_i}, f_{v_i}, v_i) \in D} w(f_{v_i}), \\
\min_{(T_{v_i}, f_{v_i}, v_i) \in F} w(f_{v_i}). & \\
\end{align*}
$$

It means $l[i, 1..5] = [2, 1, \infty, \infty, 0]$, where '$\infty$' means undefined. Now, we are ready to present the algorithm.

**Algorithm 1: INDEPENDENT-ROMAN {2}-DOM-IN-TREE**

**Input:** A tree $T = (V, E)$ with a tree ordering $v_1, v_2, \ldots, v_n$.

**Output:** The independent Roman {2}-domination number $i_{\{R2\}}(T)$.

1. if $T = K_1$ then
   2. return $i_{\{R2\}}(T) = 1$;
3. for $i := 1$ to $n$ do
   4. initialize $l[i, 1..5]$ to $[2, 1, \infty, \infty, 0]$;
5. for $j := 1$ to $n - 1$ do
   6. $v_k = l[v_j]$;
   7. $l[k, 1] = \min\{l[k, 1] + l[j, 3], l[k, 1] + l[j, 4], l[k, 1] + l[j, 5]\}$;
   8. $l[k, 2] = \min\{l[k, 2] + l[j, 3], l[k, 2] + l[j, 4]\}$;
   9. $l[k, 3] = \min\{l[k, 3] + l[j, 1], l[k, 3] + l[j, 2], l[k, 3] + l[j, 3], l[k, 4] + l[j, 1]\}$;
   10. $l[k, 4] = \min\{l[k, 4] + l[j, 2], l[k, 5] + l[j, 1]\}$;
   11. $l[k, 5] = \min\{l[k, 5] + l[j, 3]\}$;
12. return $i_{\{R2\}}(T) = \min\{l[n, 1], l[n, 2], l[n, 3]\}$;
From the above argument, we can obtain the following theorem.

**Theorem 8.** Algorithm **INDEPENDENT-ROMAN** \{2\}-**DOM-IN-TREE** can output the independent Roman \{2\}-domination number of any tree \(T = (V, E)\) in linear time \(O(n)\), where \(n = |V|\).

**Proof.** It is clear that the running time of Algorithm 1 is linear. We only need to show \(i_{\{R2\}}(T) = \min\{l[n, 1], l[n, 2], l[n, 3]\}\). Suppose that \(f \in IR2DF(T)\). Then, \((T, f, r) \in A \cup B \cup C\). By the Algorithm 1 and Lemmas 3–7, we have \(l[n, 1] = \min_{(T, f, r) \in A} f(V), l[n, 2] = \min_{(T, f, r) \in B} f(V),\) and \(l[n, 3] = \min_{(T, f, r) \in C} f(V)\). By the definition of \(i_{\{R2\}}(T)\), we deduce that

\[
i_{\{R2\}}(T) = \min_{(T, f, r) \in A \cup B \cup C} f(V) = \min\{l[n, 1], l[n, 2], l[n, 3]\}.
\]

\[\blacksquare\]

### 4. Roman \{2\}-Domination in Block Graph

Let \(G(\not\cong K_n)\) be a connected block graph. The **block-cutpoint graph** of \(G\) is a bipartite graph \(T_G = (C \cup B, E)\) in which one partite set \(C\) consists of the cutvertices of \(G\), and the other \(B\) has a vertex \(h\) for each block \(H\) of \(G\). Let \(v \in C\) and \(h \in B\). We include \(vh\) as an edge of \(T_G\) if and only if \(v\) is in \(H\), where \(H\) is the block of \(G\) represented by \(h\). Obviously, \(T_G\) is a tree and can be constructed from \(G\) in linear time (see [12]). In this section, we call each vertex in \(C\) a **C-vertex** and each vertex in \(B\) a **B-vertex**.

Let \(H\) be a block of \(G\). Suppose that \(S = \{v : v \in H\) and \(v\) is a cutvertex of \(G\}\). We say \(H\) is a block of **type 0** if \(|H| = |S|\) and \(H\) is a block of **type 1** if \(|H| = |S| + 1\). If \(|H| \geq |S| + 2\), we say \(H\) is a block of **type 2**. Let \(f : V(G) \to \{0, 1, 2\}\) be a function of a block graph \(G(\not\cong K_n)\). \(f_s : V(T_G) \to \mathbb{Z}\) is defined as follows:

\[
f_s(v) = \begin{cases} f(v), & \text{if } v \text{ is a C-vertex}, \\ f(H) - f(S), & \text{if } v \text{ is a B-vertex representing the block } H. \end{cases}
\]

We say that \(f_s\) is the function induced by \(f\). Now we present a key result on the relationship between \(f\) and \(f_s\).

**Theorem 9.** Let \(f : V(G) \to \{0, 1, 2\}\) be a function of a connected block graph \(G(\not\cong K_n)\) and \(f_s\) be the function induced by \(f\). Then, \(f\) satisfies the following properties:

1. \(f(v) = 0\) or \(1\) if \(v \in H\) is not a cut-vertex of \(G\), where \(H\) is a block of type 1 of \(G\).
(2) $f(v) = 0$ if $v \in H$ is not a cut-vertex of $G$, where $H$ is a block of type 2 of $G$.

(3) $f$ is an R2DF of $G$, if and only if $f_*$ satisfies the following properties:

(a) $f_*(v) = 0$ or 1 if $v$ is a B-vertex and the block $H$ represented by $v$ is type 1.
(b) $f_*(v) = 0$ if $v$ is a B-vertex and the block $H$ represented by $v$ is not type 1.
(c) If $v$ is a C-vertex with $f_*(v) = 0$, then there exists either $u \in N^2_{G_1}(v)$ with $f_*(u) = 2$ or $u_1, u_2 \in N^2_{G_1}(v)$ with $f_*(u_1) = f_*(u_2) = 1$.
(d) If $v$ is a B-vertex with $f_*(v) = 0$ and the block $H$ represented by $v$ is not type 0, then there exists either $u \in N_{G_1}(v)$ with $f_*(u) = 2$ or $u_1, u_2 \in N_{G_1}(v)$ with $f_*(u_1) = f_*(u_2) = 1$.

**Proof.** If $f$ satisfies the above properties, it is clear that $f_*$ satisfies the above items (a), (b). Suppose that $v$ is a C-vertex with $f_*(v) = 0$. By the definition of $f_*$, $f(v) = 0$. If there exists a vertex $u \in N_{G}(v)$ with $f(u) = 2$, then $u$ is a cut-vertex of $G$, and hence $u \in N^2_{G_1}(v)$ with $f_*(u) = 2$. Otherwise, there exists at least two vertices $x, y \in N_{G_1}(v)$ having $f(x) = f(y) = 1$. If $x$ and $y$ are both cut-vertices of $G$, then we obtain $x, y \in N^2_{G_1}(v)$ having $f_*(x) = f_*(y) = 1$. If $x$ is not a cut-vertex of $G$ and $H$ is the block containing $x$, we deduce that $H_1$ is type 1 by the second property of $f$. It implies that $f_*(h_1) = 1$ and $vh \in E(T_{G_1})$, where $h$ is the B-vertex representing the block $H$. In this case, $f_*$ also satisfies item (c). Suppose that $v$ is a B-vertex with $f_*(v) = 0$ and the block $H$ represented by $v$ is not type 0. Let $S = \{u : u \in H$ and $u$ is a cut-vertex of $G\}$. By the definition of $f_*$, we know that $f(x) = 0$ for each $x \in H \setminus S$. Since $f$ is an R2DF of $G$, then there exists either $u \in N_{G}(v)$ with $f(u) = 2$ or $u_1, u_2 \in N_{G}(v)$ such that $f(u_1) = f(u_2) = 1$. It is clear that $u, u_1, u_2$ are cut-vertices. It means that $f_*(u) = 2$ and $f_*(u_1) = f_*(u_2) = 1$. So $f_*$ satisfies item (d).

On the other hand, if $f_*$ satisfies the above properties, by the definition of $f_*$, it is easy to know that $f$ satisfies items (1) and (2).

We now need to show that $f$ is an R2DF of $G$. Suppose that $v$ is a cut-vertex with $f(v) = 0$. Hence, $f_*(v) = f(v) = 0$. If there exists $u \in N^2_{T_{G_1}}(v)$ such that $f_*(u) = 2$, we deduce that $u$ is a cut-vertex of $G$, $f_*(u) = 2$ and $u \in N_{G}(v)$. Otherwise, there exists $h_1, h_2 \in N^2_{T_{G_1}}(v)$ such that $f_*(h_1) = f_*(h_2) = 1$. If $h_1$ and $h_2$ are both C-vertex, then we have $h_1, h_2 \in N_{G}(v)$ and $f_*(h_1) = f_*(h_2) = 1$. If $h_1$ is a B-vertex and $h_1$ represent block $H_1$ in $T_{G_1}$. We deduce that $H_1$ is a block of type 1. Hence, there exists $v_1 \in H_1$ and $v_1$ is not a cut-vertex of $G$ such that $f_*(v_1) = f_*(h_1) = 1$. Therefore, we obtain $f(N(v)) \geq 2$. Suppose that $H$ is a block containing $v$ and $v$ is not a cut-vertex with $f(v) = 0$. Then $f_*(h) = f(v) = 0$, where $h$ is the B-vertex representing the block $H$. As $H$ is not type 0, there either exists $u \in N_{T_{G_1}}(h)$ such that $f_*(u) = 2$ or exists $u_1, u_2 \in N_{T_{G_1}}(h)$
such that \( f_*(u_1) = f_*(u_2) = 1 \). It is clear that \( u, u_1, u_2 \) are cut-vertices and \( u, u_1, u_2 \in N_G(v) \). We also obtain \( f(u) = f_*(u) = 2 \) and \( f(u_1) = f(u_2) = 1 \). Therefore, we deduce \( f(N(v)) \geq 2 \).

**Lemma 10.** There exists an \( R2DF \) \( f \) of \( G \) with weight \( \gamma_{\{R2\}}(G) \), which satisfies the following properties:

1. \( f(v) = 0 \) or \( 1 \) if \( v \in H \) is not a cut-vertex of \( G \), where \( H \) is a block of type 1 of \( G \).

2. \( f(v) = 0 \) if \( v \in H \) is not a cut-vertex of \( G \), where \( H \) is a block of type 2 of \( G \).

**Proof.** Let \( f \) be an \( R2DF \) of weight \( \gamma_{\{R2\}}(G) \) and \( u \in H \) be a cut-vertex of \( G \), where \( H \) is not a block of type 0, \( S = \{ v : v \in H \text{ and } f(v) \text{ is a cut-vertex of } G \} \) and \( f(u) = \max_{v \in S} f(v_0) \). Suppose \( v \in H \) is not a cut-vertex of \( G \). If \( f(v) = 2 \), we can reassign 0 to \( v \) and 2 to \( u \). Hence, \( f(v) = 0 \) or 1. Furthermore, if \( H \) is a block of type 2, we suppose that there exists a vertex \( v \in H \) such that \( f(v) = 1 \). If \( f(u) \geq 1 \), then we can reassign 2 to \( u \) and 0 to \( v \), a contradiction. Suppose that \( f(u) = 0 \), then there exists a vertex \( w \in H \), such that \( w \) is not a cut-vertex and \( f(w) \geq 1 \). We reassign 2 to \( u \) and 0 to \( v, w \), a contradiction.

Let \( f \) be an \( R2DF \) of block graph \( G(\neq K_n) \) and \( f_* \) be the function induced by \( f \). We say \( f_* \) is an induced Roman \( \{2\} \)-domination function (\( R2DF_* \)) of \( T_G \) if it satisfies the four properties in Theorem 9. By Theorem 9 and Lemma 10, we can transform the Roman \( \{2\} \)-domination problem on block graph \( G \) into the induced Roman \( \{2\} \)-domination problem on tree \( T_G \). Then, we can also use the method of tree composition and decomposition in Section 3. For convenience, \( T_G = (C \cup B, E) \) is denoted by \( T \) and \( v \in C \) (respectively, \( v \in B \)) is used to represent that \( v \) is a \( C \)-vertex (respectively, \( B \)-vertex) of \( T_G \) if there is no ambiguity.

Suppose that \( T \) is a tree rooted at \( r \) and \( f : V(T) \to \{0, 1, 2\} \) is a function on \( T \). \( T' \) is defined as a new tree rooted at \( r' \) and \( f' : V(T') \to \{0, 1, 2\} \) is a function on \( T' \), where \( V(T') = V(T) \cup \{r'\} \) and \( E(T') = E(T) \cup \{rr'\} \), \( f' |_{T'} = f \).

In order to construct an algorithm for computing the Roman \( \{2\} \)-domination number, we must characterize the possible tree-subset tuples \((T, f, r)\). For this purpose, we introduce some additional notations as follows:

\[
\begin{align*}
R2DF_*(T) & = \{ f : f \text{ is an } R2DF_* \text{ of } T \}, \\
F_1(T) & = \{ f : f \in R2DF_* (T) \text{ with } f(r) = 1 \}, \\
F_2(T) & = \{ f : f \in R2DF_* (T) \text{ with } f(r) = 2 \}, \\
R2DF_*(T+1) & = \{ f : f \notin R2DF_* (T), f' \in F_1(T') \text{ and } f'|_T = f \}, \\
R2DF_*(T+2) & = \{ f : f \notin R2DF_* (T), f' \in F_2(T') \text{ and } f'|_T = f \} - R2DF_*(T+1).
\end{align*}
\]
Then we consider the following eleven classes:

\[ A_1 = \{ (T, f, r) : f \in R^{2DF}(T), r \in C \text{ and } f(r) = 2 \}, \]

\[ A_2 = \{ (T, f, r) : f \in R^{2DF}(T), r \in C \text{ and } f(r) = 1 \}, \]

\[ A_3 = \{ (T, f, r) : f \in R^{2DF}(T), r \in C \text{ and } f(r) = 0 \}, \]

\[ A_4 = \{ (T, f, r) : f \in R^{2DF}(T+1), r \in C \}, \]

\[ A_5 = \{ (T, f, r) : f \in R^{2DF}(T+2), r \in C \}, \]

\[ B_1 = \{ (T, f, r) : f \in R^{2DF}(T), r \in B \text{ and } f(N[r]) \geq 2 \}, \]

\[ B_2 = \{ (T, f, r) : f \in R^{2DF}(T), r \in B \text{ and } f(N[r]) = 1 \}, \]

\[ B_3 = \{ (T, f, r) : f \in R^{2DF}(T), r \in B \text{ and } f(N[r]) = 0 \}, \]

\[ B_4 = \{ (T, f, r) : f \in R^{2DF}(T+1), r \in B \text{ and } f(N[r]) = 1 \}, \]

\[ B_5 = \{ (T, f, r) : f \in R^{2DF}(T+1), r \in B \text{ and } f(N[r]) = 0 \}, \]

\[ B_6 = \{ (T, f, r) : f \in R^{2DF}(T+2), r \in B \}. \]

Let \( (T, r) = (T_1, r_1) \circ (T_2, r_2) \) and \( r = r_1 \). Suppose that \( f_1 \) (respectively, \( f_2 \)) is a function on \( T_1 \) (respectively, \( T_2 \)). Define \( f \) as the function on \( T \) with \( f|_{T_1} = f_1 \) and \( f|_{T_2} = f_2 \). In order to give the algorithm, we present the following lemmas.

\textbf{Lemma 11.} \( A_1 = (A_1 \circ B_1) \cup (A_1 \circ B_2) \cup (A_1 \circ B_3) \cup (A_1 \circ B_4) \cup (A_1 \circ B_5) \cup (A_1 \circ B_6) \).

\textbf{Proof.} For each \( 1 \leq i \leq 6 \), if \( (T_1, f_1, r_1) \in A_1 \) and \( (T_2, f_2, r_2) \in B_i \), it is clear that \( f \) is an R\( 2DF \) of \( T \), \( r \in C \) and \( f(r) = 2 \). We deduce that \( (T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_1 \). So \( (A_1 \circ B_1) \cup (A_1 \circ B_2) \cup (A_1 \circ B_3) \cup (A_1 \circ B_4) \cup (A_1 \circ B_5) \cup (A_1 \circ B_6) \subseteq A_1 \).

Now we prove that \( A_1 \subseteq (A_1 \circ B_1) \cup (A_1 \circ B_2) \cup (A_1 \circ B_3) \cup (A_1 \circ B_4) \cup (A_1 \circ B_5) \cup (A_1 \circ B_6) \). Let \( (T, f, r) \in A_1 \) and \( (T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2) \), then \( f_1(r_1) = f(r) = 2 \). Since \( f \in R^{2DF}(T) \), \( f_1 \in R^{2DF}(T_1) \) and \( r_1 \in C \). So \( (T_1, f_1, r_1) \in A_1 \) and \( r_2 \in B \). If \( f_2 \in R^{2DF}(T_2) \), then we obtain \( (T_2, f_2, r_2) \in B_1 \), \( B_2 \) or \( B_3 \). If \( f_2 \notin R^{2DF}(T_2) \), then \( (T_2, f_2, r_2) \in B_4, B_5 \) or \( B_6 \). Hence, we conclude that \( A_1 \subseteq (A_1 \circ B_1) \cup (A_1 \circ B_2) \cup (A_1 \circ B_3) \cup (A_1 \circ B_4) \cup (A_1 \circ B_5) \cup (A_1 \circ B_6) \). \( \blacksquare \)

\textbf{Lemma 12.} \( A_2 = (A_2 \circ B_1) \cup (A_2 \circ B_2) \cup (A_2 \circ B_3) \cup (A_2 \circ B_4) \subseteq (A_2 \circ B_5) \).

\textbf{Proof.} For each \( 1 \leq i \leq 5 \), if \( (T_1, f_1, r_1) \in A_2 \) and \( (T_2, f_2, r_2) \in B_i \), it is clear that \( f \) is an R\( 2DF \) of \( T \), \( r \in C \) and \( f(r) = 2 \). We deduce that \( (T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_2 \), implying that \( (A_2 \circ B_1) \cup (A_2 \circ B_2) \cup (A_2 \circ B_3) \cup (A_2 \circ B_4) \cup (A_2 \circ B_5) \subseteq A_2 \).

Then we need to show \( A_2 \subseteq (A_2 \circ B_1) \cup (A_2 \circ B_2) \cup (A_2 \circ B_3) \cup (A_2 \circ B_4) \cup (A_2 \circ B_5) \). Let \( (T, f, r) \in A_2 \) and \( (T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2) \), then \( f_1(r_1) = f(r) = 1 \). It is clear that \( (T_1, f_1, r_1) \in A_2 \) and \( r_2 \in B \). If \( f_2 \) is an R\( 2DF \) of \( T_2 \), then we obtain \( (T_2, f_2, r_2) \in B_1, B_2 \) or \( B_3 \). If \( f_2 \) is not an R\( 2DF \) of \( T_2 \), then \( f_2(N_{T_2}[r_2]) \leq 1 \) and \( f_2 \in R^{2DF}(T_2+1) \) by using the fact
that $(T, f, r) \in A_2$. Therefore, we have $(T_2, f_2, r_2) \in B_4$ or $B_5$. Hence, $A_2 \subseteq (A_2 \circ B_1) \cup (A_2 \circ B_2) \cup (A_2 \circ B_3) \cup (A_2 \circ B_4) \cup (A_2 \circ B_5)$. ■

**Lemma 13.** $A_3 = (A_3 \circ B_1) \cup (A_3 \circ B_2) \cup (A_3 \circ B_3) \cup (A_4 \circ B_1) \cup (A_4 \circ B_2) \cup (A_5 \circ B_1)$.

**Proof.** We make some remarks.

(i) For each $1 \leq i \leq 3$, if $(T_1, f_1, r_1) \in A_3$ and $(T_2, f_2, r_2) \in B_1$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_3$. Indeed, if $(T_1, f_1, r_1) \in A_3$ and $(T_2, f_2, r_2) \in B_1$, then $f_1$ is an R2DF$_s$ of $T_1$ and $f_2$ is an R2DF$_s$ of $T_2$. Hence, $f$ is an R2DF$_s$ of $T$, $r \in C$ and $f(r) = 0$. So $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_3$.

(ii) For each $1 \leq i \leq 2$, if $(T_1, f_1, r_1) \in A_4$ and $(T_2, f_2, r_2) \in B_1$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_3$. Indeed, if $(T_1, f_1, r_1) \in A_4$, then we have that $f_1 \in$ R2DF$_s(T^+_1)$, $r \in C$, $f(r) = 0$ and $f(N^2_{T_1}[r]) = 1$. By the definition of $B_1$, we obtain $f(N^2_{T_1}[r]) \geq 2$ and $f \in$ R2DF$_s(T)$. Hence, $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_3$.

(iii) If $(T_1, f_1, r_1) \in A_5$ and $(T_2, f_2, r_2) \in B_1$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_3$. Indeed, if $(T_1, f_1, r_1) \in A_5$, then we have that $f_1 \in$ R2DF$_s(T^+_1)$, $r \in C$, $f(r) = 0$ and $f(N^2_{T_1}[r]) = 0$. By the definition of $B_1$, we obtain $f(N^2_{T_1}[r]) \geq 2$ and $f \in$ R2DF$_s(T)$. It means that $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_3$. Hence, $(A_3 \circ B_1) \cup (A_3 \circ B_2) \cup (A_3 \circ B_3) \cup (A_4 \circ B_1) \cup (A_4 \circ B_2) \cup (A_5 \circ B_1) \subseteq A_3$.

Therefore, we need to prove $A_3 \subseteq (A_3 \circ B_1) \cup (A_3 \circ B_2) \cup (A_3 \circ B_3) \cup (A_4 \circ B_1) \cup (A_4 \circ B_2) \cup (A_5 \circ B_1)$.

Let $(T, f, r) \in A_3$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then we have that $f_1(r_1) = f(r) = 0$, $r_1 \in C$ and $f_2 \in$ R2DF$_s(T_2)$. So $r_2 \in B$. If $f_1 \in$ R2DF$_s(T_1)$, then we obtain $(T_1, f_1, r_1) \in A_3$, implying that $(T_2, f_2, r_2) \in B_1$, $B_2$ or $B_3$. Suppose that $f_1 \notin$ R2DF$_s(T_1)$. Consider the following cases.

**Case 1.** $f_1(N^2_{T_1}[r]) = 1$. Then we obtain $f_1 \in$ R2DF$_s(T^+_1)$, implying that $(T_1, f_1, r_1) \in A_4$. Since $(T, f, r) \in A_3$, we have $f_2(N^2_{T_2}[r_2]) \geq 1$. So $(T_2, f_2, r_2) \in B_1$ or $B_2$.

**Case 2.** $f_1(N^2_{T_1}[r]) = 0$. So we have $f_1 \in$ R2DF$_s(T^{+2}_1)$. Then $(T_1, f_1, r_1) \in A_5$. Since $(T, f, r) \in A_3$, we obtain $f_2(N^2_{T_2}[r_2]) \geq 2$. Hence, $(T_2, f_2, r_2) \in B_1$. ■

**Lemma 14.** $A_4 = (A_4 \circ B_3) \cup (A_5 \circ B_2)$.

**Proof.** It is easy to check the following remarks by definitions.

(i) If $(T_1, f_1, r_1) \in A_4$ and $(T_2, f_2, r_2) \in B_3$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_4$.

(ii) If $(T_1, f_1, r_1) \in A_5$ and $(T_2, f_2, r_2) \in B_2$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_4$. Therefore, $(A_4 \circ B_3) \cup (A_5 \circ B_2) \subseteq A_4$.

On the other hand, we show $A_4 \subseteq (A_4 \circ B_3) \cup (A_5 \circ B_2)$. Let $(T, f, r) \in A_4$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$. Then we have that $f \in$ R2DF$_s(T^+_1)$ and $r_1 \in C$, implying that $f(N^2_{T_1}[r_1]) = 1$. It means that $r_2 \in B$. By the definition
of $A_4$, $f_2 \in \text{R}2\text{DF}_*(T_2)$. Using the fact that $f(N_2^3[r_1]) = 1$, we deduce that $f_2(N[r_2]) < 2$. Consider the following cases.

Case 1. $f_2(N[r_2]) = 1$. It is clear that $(T_2, f_2, r_2) \in B_2$. Since $f_1(N_2^3[r_1]) = f(N_2^3[r_1]) - f_2(N[r_2]) = 0$, we obtain $(T_1, f_1, r_1) \in A_5$.

Case 2. $f_2(N[r_2]) = 0$. Then $(T_2, f_2, r_2) \in B_3$. Since $f_1(N_2^3[r_1]) = f(N_2^3[r_1]) - f_2(N[r_2]) = 1$, we have $(T_1, f_1, r_1) \in A_4$.

Consequently, we deduce that $A_4 \subseteq (A_4 \circ B_3) \cup (A_5 \circ B_2)$.

**Lemma 15.** $A_5 = A_5 \circ B_3$.

**Proof.** It is easy to check that $(A_3 \circ B_3) \subseteq A_5$ by the definitions. On the other hand, let $(T, f, r) \in A_5$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$. Then we obtain $f \in \text{R}2\text{DF}_*(T_{i+2})$, $r_1 \in C$ and $f_1(N_2^3[r_1]) = f(N_2^3[r]) = 0$. It implies that $(T_1, f_1, r_1) \in A_5$ and $r_2 \in B$. Using the fact that $(T, f, r) \in A_5$, we deduce $f_2(N[r_2]) = 0$ and $f_2 \in \text{R}2\text{DF}_*(T_2)$. Therefore, $(T_2, f_2, r_2) \in B_3$. Then we obtain $A_5 \subseteq (A_5 \circ B_3)$.

**Lemma 16.** $B_1 = (B_1 \circ A_1) \cup (B_1 \circ A_2) \cup (B_1 \circ A_3) \cup (B_1 \circ A_4) \cup (B_1 \circ A_5) \cup (B_2 \circ A_1) \cup (B_2 \circ A_2) \cup (B_3 \circ A_1) \cup (B_4 \circ A_1) \cup (B_4 \circ A_2) \cup (B_5 \circ A_1) \cup (B_6 \circ A_1)$.

**Proof.** We make some remarks.

(i) For each $1 \leq i \leq 5$, if $(T_1, f_1, r_1) \in B_1$ and $(T_2, f_2, r_2) \in A_4$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_1$. It is easy to check it by the definitions of $B_1$ and $A_4$.

(ii) For each $2 \leq i \leq 6$, if $(T_1, f_1, r_1) \in B_i$ and $(T_2, f_2, r_2) \in A_1$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_1$. We can easily check it by definitions too.

(iii) For each $i \in \{2, 4\}$, if $(T_1, f_1, r_1) \in B_i$ and $(T_2, f_2, r_2) \in A_2$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_1$. Indeed, it is clear that $f \in \text{R}2\text{DF}_*(T)$, $r \in B$ and $f(N[r]) = f_1(N[r_1]) + f_2(r_2) = 2$. Hence, $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_1$.

Therefore, we need to prove $B_1 \subseteq (B_1 \circ A_1) \cup (B_1 \circ A_2) \cup (B_1 \circ A_3) \cup (B_1 \circ A_4) \cup (B_1 \circ A_5) \cup (B_2 \circ A_1) \cup (B_2 \circ A_2) \cup (B_3 \circ A_1) \cup (B_4 \circ A_1) \cup (B_4 \circ A_2) \cup (B_5 \circ A_1) \cup (B_6 \circ A_1)$.

Let $(T, f, r) \in B_1$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then we have $f \in \text{R}2\text{DF}_*(T)$, $r_1 \in B$ and $f(N[r]) \geq 2$. It means that $r_2 \in C$. Consider the following cases.

Case 1. $f(r_2) = 2$. Then we have $f_2 \in \text{R}2\text{DF}_*(T_2)$, implying that $(T_2, f_2, r_2) \in A_1$. If $f_1 \in \text{R}2\text{DF}_*(T_1)$, we obtain $(T_1, f_1, r_1) \in B_1 \cup B_2$ or $B_3$. Suppose that $f_1 \notin \text{R}2\text{DF}_*(T_1)$, then $f_1 \in \text{R}2\text{DF}_*(T_{i+1})$ or $f_1 \in \text{R}2\text{DF}_*(T_{i+2})$. Hence, $(T_1, f_1, r_1) \in B_4$, $B_5$ or $B_6$.

Case 2. $f(r_2) = 1$. It is clear that $(T_2, f_2, r_2) \in A_2$. We also have $f_1(N[r_1]) = f(N[r_1]) - f_2(r_2) \geq 2 - 1 \geq 1$. If $f_1 \in \text{R}2\text{DF}_*(T_1)$, we obtain $(T_1, f_1, r_1) \in B_1$ or $B_2$. 


Suppose that $f_1 \notin R_{2DF^*}(T_1)$, then $f_1 \in R_{2DF^*}(T_1^{+1})$. Therefore, $(T_1, f_1, r_1) \in B_4$.

Case 3. $f(r_2) = 0$. Then we obtain $f_1(N[r_1]) = f(N[r]) - f_2(r_2) \geq 2$ and $f_1 \in R_{2DF^*}(T_1)$, implying that $(T_1, f_1, r_1) \in B_1$. If $f_2 \in R_{2DF^*}(T_2)$, we deduce $(T_1, f_1, r_1) \in A_3$. Suppose that $f_2 \notin R_{2DF^*}(T_2)$, then $f_2 \in R_{2DF^*}(T_2^{+1})$ or $f_2 \in R_{2DF^*}(T_2^{+2})$. Therefore, $(T_2, f_2, r_2) \in A_4$ or $A_5$.

Hence, $B_1 \subseteq (B_1 \circ A_1) \cup (B_1 \circ A_2) \cup (B_1 \circ A_3) \cup (B_1 \circ A_4) \cup (B_1 \circ A_5) \cup (B_2 \circ A_1) \cup (B_2 \circ A_2) \cup (B_3 \circ A_1) \cup (B_4 \circ A_1) \cup (B_4 \circ A_2) \cup (B_5 \circ A_1) \cup (B_6 \circ A_1)$.

Lemma 17. $B_2 = (B_2 \circ A_3) \cup (B_2 \circ A_4) \cup (B_3 \circ A_2) \cup (B_5 \circ A_2)$.

Proof. We make some remarks.

(i) For each $3 \leq i \leq 4$, if $(T_1, f_1, r_1) \in B_2$ and $(T_2, f_2, r_2) \in A_i$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_2$. It is easy to check it by the definitions.

(ii) For each $i \in \{3, 5\}$, if $(T_1, f_1, r_1) \in B_i$ and $(T_2, f_2, r_2) \in A_2$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_2$. Indeed, if $(T_1, f_1, r_1) \in B_i$ and $(T_2, f_2, r_2) \in A_2$, we obtain that $f \in R_{2DF^*}(T)$, $r \in B$ and $f(N[r]) = f_1(N[r_1]) + f_2(r_2) = 1$. Hence, we deduce $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_2$. Thus, $(B_2 \circ A_3) \cup (B_2 \circ A_4) \cup (B_3 \circ A_2) \cup (B_5 \circ A_2) \subseteq B_2$.

Now we need to prove $B_2 \subseteq (B_2 \circ A_3) \cup (B_2 \circ A_4) \cup (B_3 \circ A_2) \cup (B_5 \circ A_2)$. Let $(T, f, r) \in B_2$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then we have that $f \in R_{2DF^*}(T)$, $r_1 \in B$ and $f(N[r]) = 1$. It implies $r_2 \in C$. Consider the following cases.

Case 1. $f(r_2) = 1$. Then we have $f_1(N[r_1]) = f(N[r]) - f_2 = 0$ and $f_2(r_2) = 1$, implying that $f_2 \in R_{2DF^*}(T_2)$. So $(T_2, f_2, r_2) \in A_2$. If $f_1 \in R_{2DF^*}(T_1)$, we obtain $(T_1, f_1, r_1) \in B_3$. Suppose that $f_1 \notin R_{2DF^*}(T_1)$, then $f_1(r_1) = 0$ because $f \in R_{2DF^*}(T)$. Since $f_1(N[r_1]) = 0$, we have that $(T_1, f_1, r_1) \in B_5$.

Case 2. $f(r_2) = 0$. It is clear that $f_1(N[r_1]) = f(N[r]) - f_2 = 1$. Since $f_1 = f_{T_1}$ and $f \in R_{2DF^*}(T)$, we have $f_1 \in R_{2DF^*}(T_1)$. Hence, $(T_1, f_1, r_1) \in B_2$. If $f_2 \in R_{2DF^*}(T_2)$, we deduce that $(T_2, f_2, r_2) \in A_3$. Suppose that $f_2 \notin R_{2DF^*}(T_2)$, then $f_2(N^{+2}[r_2]) = 1$. It implies $f_2 \in R_{2DF^*}(T_2^{+1})$. Therefore, $(T_2, f_2, r_2) \in A_4$.

Hence, $B_2 \subseteq (B_2 \circ A_3) \cup (B_2 \circ A_4) \cup (B_3 \circ A_2) \cup (B_5 \circ A_2)$.

Lemma 18. $B_3 = B_3 \circ A_3$.

Proof. It is easy to check that $(B_3 \circ A_3) \subseteq B_3$ by the definitions. On the other hand, let $(T, f, r) \in B_3$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$. Then we obtain $f_1(N[r_1]) = f(N[r]) - r_1 \in B$ and $f_2(r_2) = 0$. It means that $r_2 \in C$. Since $f \in R_{2DF^*}(T)$ and $f(r_2) = 0$, we obtain that $f_1 \in R_{2DF^*}(T_1)$, implying that $(T_1, f_1, r_1) \in B_3$. Using the fact that $f_1(N[r_1]) = 0$ and $f_2(r_2) = 0$, we deduce that $f_2 \in R_{2DF^*}(T_2)$. Therefore, $(T_2, f_2, r_2) \in A_3$. Then $B_3 \subseteq (B_3 \circ A_3)$. 


Lemma 19. $B_4 = (B_2 \circ A_5) \cup (B_4 \circ A_3) \cup (B_4 \circ A_4) \cup (B_4 \circ A_5) \cup (B_6 \circ A_2)$.

Proof. It is easy to check the following remarks by definitions.

(i) If $(T_1, f_1, r_1) \in B_2$ and $(T_2, f_2, r_2) \in A_5$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_4$.

(ii) For each $3 \leq i \leq 5$, if $(T_1, f_1, r_1) \in B_4$ and $(T_2, f_2, r_2) \in A_i$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_4$.

(iii) If $(T_1, f_1, r_1) \in B_6$ and $(T_2, f_2, r_2) \in A_2$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_4$.

Therefore, we need to prove $B_4 \subseteq (B_2 \circ A_5) \cup (B_4 \circ A_3) \cup (B_4 \circ A_4) \cup (B_4 \circ A_5) \cup (B_6 \circ A_2)$. Let $(T, f, r) \in B_4$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then we have $f \in \:\text{R2DF}_s(T^{+1}), r_1 \in B$ and $f(N[r]) = 1$. It implies $r_2 \in C$. Consider the following cases.

Case 1. $f(r_2) = 1$. Then we have $f_1(N[r_1]) = f(N[r]) - f(r_2) = 0$ and $f_2(r_2) = 1$, implying that $f_2 \in \:\text{R2DF}_s(T_2)$. So $(T_2, f_2, r_2) \in A_2$ and $f_1 \notin \:\text{R2DF}_s(T_1)$. Since $f_1(N[r_1]) = 0$ and $(T, f, r) \in B_4$, we obtain $(T_1, f_1, r_1) \in B_6$.

Case 2. $f(r_2) = 0$. It is clear that $f_1(N[r_1]) = f(N[r]) - f(r_2) = 1$. If $f_2 \in \:\text{R2DF}_s(T_2)$, we deduce that $(T_2, f_2, r_2) \in A_3$, implying $(T_1, f_1, r_1) \in B_4$. Suppose that $f_2 \notin \:\text{R2DF}_s(T_2)$, then $f_2(N^2[r_2]) = 0$ or 1. If $f_2(N^2[r_2]) = 0$, we obtain $(T_2, f_2, r_2) \in A_5$. Then, we have $(T_1, f_1, r_1) \in B_2$ or $B_4$. If $f_2(N^2[r_2]) = 1$, we obtain $(T_2, f_2, r_2) \in A_4$. Then, we have $(T_1, f_1, r_1) \in B_4$.

Hence, $B_4 \subseteq (B_2 \circ A_5) \cup (B_4 \circ A_3) \cup (B_4 \circ A_4) \cup (B_4 \circ A_5) \cup (B_6 \circ A_2)$.

Lemma 20. $B_5 = (B_3 \circ A_4) \cup (B_5 \circ A_3) \cup (B_5 \circ A_4)$.

Proof. It is easy to check the following remarks by definitions.

(i) If $(T_1, f_1, r_1) \in B_3$ and $(T_2, f_2, r_2) \in A_4$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_5$.

(ii) For each $3 \leq i \leq 4$, if $(T_1, f_1, r_1) \in B_5$ and $(T_2, f_2, r_2) \in A_i$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_5$. Thus, $(B_3 \circ A_4) \cup (B_5 \circ A_3) \cup (B_5 \circ A_4) \subseteq B_5$.

Therefore, we need to prove $B_5 \subseteq (B_3 \circ A_4) \cup (B_5 \circ A_3) \cup (B_5 \circ A_4)$. Let $(T, f, r) \in B_5$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then we have $f \in \:\text{R2DF}_s(T^{+1}), r_1 \in B$ and $f(N[r]) = 0$. It implies $r_2 \in C$ and $f_2(r_2) = f(r_2) = 0$. Consider the following cases.

Case 1. If $f_2 \in \:\text{R2DF}_s(T_2)$, then we have $(T_2, f_2, r_2) \in A_3$ and $f_1 \notin \:\text{R2DF}_s(T_1)$. Since $f_1(N[r_1]) = 0$ and $(T, f, r) \in B_5$, we obtain $(T_1, f_1, r_1) \in B_5$.

Case 2. If $f_2 \notin \:\text{R2DF}_s(T_2)$, we deduce that $(T_2, f_2, r_2) \in A_4$. It is clear that $(T_1, f_1, r_1) \in B_3$ or $B_5$.

Hence, $B_5 \subseteq (B_3 \circ A_4) \cup (B_5 \circ A_3) \cup (B_5 \circ A_4)$.

Lemma 21. $B_6 = (B_3 \circ A_5) \cup (B_5 \circ A_3) \cup (B_6 \circ A_3) \cup (B_6 \circ A_4) \cup (B_6 \circ A_5)$. 
Proof. It is easy to check the following remarks by definitions.

(i) For each \( i \in \{3, 5\} \), if \((T_1, f_1, r_1) \in B_i \) and \((T_2, f_2, r_2) \in A_5\), then \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_6\).

(ii) For each \( 3 \leq i \leq 5 \), if \((T_1, f_1, r_1) \in B_6 \) and \((T_2, f_2, r_2) \in A_i\), then \((T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_6\).

Therefore, we need to prove \( B_6 \subseteq (B_3 \circ A_5) \cup (B_5 \circ A_5) \cup (B_6 \circ A_3) \cup (B_6 \circ A_4) \cup (B_6 \circ A_5) \). Let \((T, f, r) \in B_6 \) and \((T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)\), then we have \( f \in R2DF_s(T^{+2}) \), \( r_1 \in B \) and \( f(N[r]) = 0 \). It implies \( r_2 \in C \). Consider the following cases.

Case 1. \( f_1 \in R2DF_s(T_1) \). Since \( f_1(N[r_1]) = f(N[r]) = 0 \), we have \((T_1, f_1, r_1) \in B_3\). It implies \((T_2, f_2, r_2) \in A_5\).

Case 2. \( f_1 \not\in R2DF_s(T_1) \). Since \( f_1(N[r_1]) = f(N[r]) = 0 \), then we obtain \((T_1, f_1, r_1) \in B_5 \) or \( B_6 \). If \((T_1, f_1, r_1) \in B_5\), we have \( f_1 \in R2DF_s(T_1^{+1}) \). Since \( f \in R2DF_s(T^{+2}) \), it means that \( f_2 \in R2DF_s(T_2^{+2}) \). Then we deduce \((T_2, f_2, r_2) \in A_5\). If \((T_1, f_1, r_1) \in B_6\), we have \( f_1 \in R2DF_s(T_1^{+2}) \). Since \((T, f, r) \in B_6\), we deduce that \( f_2(r_2) = 0 \). So we obtain \((T_2, f_2, r_2) \in A_3, A_4 \) or \( A_5\).

Hence, \( B_6 \subseteq (B_3 \circ A_5) \cup (B_5 \circ A_5) \cup (B_6 \circ A_3) \cup (B_6 \circ A_4) \cup (B_6 \circ A_5)\).

The final step is to define the initial vector. In this case, for block-cutpoint graphs, the only basis graph is a single vertex. We can use the similar method in Section 3 to initialize the vector. It is clear that if \( v \) is a \( C \)-vertex, then the initial vector is \([2, 1, \infty, \infty, 0, \infty]\); if \( v \) is a \( B \)-vertex and \( v \) represents a block of type 0, then the initial vector is \([\infty, \infty, 0, \infty, \infty, \infty]\); if \( v \) is a \( B \)-vertex and \( v \) represents a block of type 1, then the initial vector is \([\infty, 1, \infty, \infty, \infty, 0]\); if \( v \) is a \( B \)-vertex and \( v \) represents a block of type 2, then the initial vector is \([\infty, \infty, \infty, \infty, \infty, 0]\). Among them, ‘\( \infty \)’ means undefined. From the above argument, we can obtain the following theorem.

Theorem 22. Algorithm ROMAN \{2\}-DOM-IN-BLOCK can output the Roman \{2\}-domination number of any block graphs \( G = (V, E) \) in linear time \( O(n) \), where \( n = |V|\).

Proof. One can prove Theorem 22 by the similar argument as in the proof of Theorem 8.

Now, we are ready to present the algorithm.
Algorithm 2: ROMAN \{2\}-DOM-IN-BLOCK

Input: A connected block graph $G$ ($G \not\cong K_n$) and its corresponding block-cutpoint graph $T = (V, E)$ with a tree ordering $v_1, v_2, \ldots, v_n$.

Output: The Roman \{2\}-domination number $\gamma_{\{R2\}}(G)$.

for $i := 1$ to $n$ do
  if $v_i$ is a C-vertex then
    initialize $h[i, 1.6]$ to $[2, 1, \infty, \infty, 0, \infty]$;
  else if $v_i$ is a B-vertex representing a block of type 0 then
    initialize $h[i, 1.6]$ to $[\infty, \infty, 0, \infty, \infty, \infty]$;
  else if $v_i$ is a B-vertex representing a block of type 1 then
    initialize $h[i, 1.6]$ to $[\infty, 1, \infty, \infty, 0, \infty]$;
  else
    initialize $h[i, 1.6]$ to $[\infty, \infty, \infty, \infty, 0, \infty]$;
for $j := 1$ to $n - 1$ do
  $v_k = F(v_j)$;
  if $v_k$ is a C-vertex then
    $h[k, 1] = \min\{h[k, 1] + h[j, 1], h[k, 1] + h[j, 2], h[k, 1] + h[j, 3], h[k, 1] + h[j, 4], h[k, 1] + h[j, 5], h[k, 1] + h[j, 6]\};$
    $h[k, 2] = \min\{h[k, 2] + h[j, 1], h[k, 2] + h[j, 2], h[k, 2] + h[j, 3], h[k, 2] + h[j, 4], h[k, 2] + h[j, 5], h[k, 2] + h[j, 6]\};$
    $h[k, 3] = \min\{h[k, 3] + h[j, 1], h[k, 3] + h[j, 2], h[k, 3] + h[j, 3], h[k, 3] + h[j, 4], h[k, 3] + h[j, 5], h[k, 3] + h[j, 6]\};$
    $h[k, 4] = \min\{h[k, 4] + h[j, 1], h[k, 4] + h[j, 2], h[k, 4] + h[j, 3], h[k, 4] + h[j, 4], h[k, 4] + h[j, 5], h[k, 4] + h[j, 6]\};$
    $h[k, 5] = \min\{h[k, 5] + h[j, 1], h[k, 5] + h[j, 2], h[k, 5] + h[j, 3], h[k, 5] + h[j, 4], h[k, 5] + h[j, 5], h[k, 5] + h[j, 6]\};$
  else
    $S_1 = h[k, 2];$
    $S_2 = h[k, 3];$
    $S_3 = h[k, 5];$
    $h[k, 1] = \min\{h[k, 1] + h[j, 1], h[k, 1] + h[j, 2], h[k, 1] + h[j, 3], h[k, 1] + h[j, 4], h[k, 1] + h[j, 5], h[k, 1] + h[j, 6]\};$
    $h[k, 2] = \min\{h[k, 2] + h[j, 1], h[k, 2] + h[j, 2], h[k, 2] + h[j, 3], h[k, 2] + h[j, 4], h[k, 2] + h[j, 5], h[k, 2] + h[j, 6]\};$
    $h[k, 3] = \min\{h[k, 3] + h[j, 1], h[k, 3] + h[j, 2], h[k, 3] + h[j, 3], h[k, 3] + h[j, 4], h[k, 3] + h[j, 5], h[k, 3] + h[j, 6]\};$
    $h[k, 4] = \min\{h[k, 4] + h[j, 1], h[k, 4] + h[j, 2], h[k, 4] + h[j, 3], h[k, 4] + h[j, 4], h[k, 4] + h[j, 5], h[k, 4] + h[j, 6]\};$
    $h[k, 5] = \min\{h[k, 5] + h[j, 1], h[k, 5] + h[j, 2], h[k, 5] + h[j, 3], h[k, 5] + h[j, 4], h[k, 5] + h[j, 5], h[k, 5] + h[j, 6]\};$
  \end{algorithm}

Return $\gamma_{\{R2\}}(G) = \min\{h[n, 1], h[n, 2], h[n, 3]\}$;
Outliers
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References


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