BOUNDS ON THE DOUBLE ITALIAN DOMINATION NUMBER OF A GRAPH

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Abstract

For a graph $G$, a Roman $\{3\}$-dominating function is a function $f : V \rightarrow \{0, 1, 2, 3\}$ having the property that for every vertex $u \in V$, if $f(u) \in \{0, 1\}$, then $f(N[u]) \geq 3$. The weight of a Roman $\{3\}$-dominating function is the sum $w(f) = f(V) = \sum_{v \in V} f(v)$, and the minimum weight of a Roman $\{3\}$-dominating function is the Roman $\{3\}$-domination number, denoted by $\gamma_{(R3)}(G)$. In this paper, we present a sharp lower bound for the double Italian domination number of a graph, and improve previous bounds given in [D.A. Mojdeh and L. Volkmann, Roman $\{3\}$-domination (double Italian domination), Discrete Appl. Math. (2020), in press]. We also present a probabilistic upper bound for a generalized version of double Italian domination number of a graph, and show that the given bound is asymptotically best possible.

Keywords: Italian domination, double Italian domination, probabilistic methods.

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1. Introduction

For a (simple) graph $G = (V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$, we denote by $|V(G)| = n(G) = n$ the order of $G$. The open neighborhood of a vertex $v$ is $N(v) = \{u \in V : uv \in E\}$ and the closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of $S$ is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of $S$ is $N[S] = N(S) \cup S$. The degree of a vertex $v$ is $\deg(v) = |N(v)|$. The maximum and minimum degree among the vertices of $G$ are denoted by $\Delta(G)$ and $\delta(G)$, respectively. For a set $S \subseteq V$ in a graph $G$ and a vertex $v \in V$, we say that $S$ dominates $v$ if $v \in S$ or
\( v \) is adjacent to some vertex of \( S \). A set \( S \) is called a dominating set in \( G \) if \( S \) dominates every vertex of \( G \). The domination number \( \gamma(G) \) of \( G \) is the minimum cardinality of a dominating set in \( G \). For other definitions and notations not given here we refer to [6].

Cockayne et al. [5] introduced the concept of Roman domination in graphs, although this notion was inspired by the work of ReVelle et al. in [11], and Stewart in [12]. Let \( f : V(G) \to \{0, 1, 2\} \) be a function having the property that for every vertex \( v \in V \) with \( f(v) = 0 \), there exists a neighbor \( u \in N(v) \) with \( f(u) = 2 \). Such a function is called a Roman dominating function or just an RDF. The weight of an RDF \( f \) is the sum \( f(V) = \sum_{v \in V} f(v) \). The minimum weight of an RDF on \( G \) is called the Roman domination number of \( G \), and is denoted by \( \gamma_R(G) \). Several varieties of Roman domination are already studied, and the reader can consult [3, 4].

A generalization of Roman domination called Italian domination (or Roman \( \{2\}\)-domination) was introduced by Chellali et al. in [2], Klostermeyer and MacGillivray [8], and Henning and Klostermeyer [7]. An Italian dominating function (IDF) on a graph \( G = (V, E) \) is a function \( f : V \to \{0, 1, 2\} \) satisfying the property that for every vertex \( v \in V \), with \( f(v) = 0 \), \( \sum_{u \in N(v)} f(u) \geq 2 \). The weight of an IDF \( f \) is the value \( w(f) = f(V) = \sum_{u \in V} f(u) \). The minimum weight of an IDF on a graph \( G \) is called the Italian domination number of \( G \), denoted by \( \gamma_I(G) \). This same concept was called Roman \( \{2\}\)-domination and what we called \( \gamma_I(G) \) is called \( \gamma_{\{2\}\}(G) \). A \( \gamma_{\{2\}\}(G) \)-function can be represented by a triple \( f = (V_0, V_1, V_2) \) (or \( f = (V_0^l, V_1^l, V_2^l) \) to refer to \( f \)), where \( V_i = \{v \in V(G) : f(v) = i\} \) for \( i = 0, 1, 2 \).

Beeler et al. [1] introduced the concept of double Roman domination in graphs. A function \( f : V \to \{0, 1, 2, 3\} \) is a double Roman dominating function (or just DRDF) on a graph \( G \) if the following conditions hold, where \( V_i \) denote the set of vertices assigned \( i \) under \( f \), for \( i = 0, 1, 2, 3 \): (1) If \( f(v) = 0 \), then \( v \) must have at least two neighbors in \( V_2 \) or one neighbor in \( V_3 \); (2) If \( f(v) = 1 \), then \( v \) must have at least one neighbor in \( V_2 \cup V_3 \). The weight of a DRDF \( f \) is the value \( w(f) = f(V) = \sum_{v \in V} f(v) \). The double Roman domination number, \( \gamma_{dR}(G) \), is the minimum weight of a DRDF on \( G \), and a DRDF of \( G \) with weight \( \gamma_{dR}(G) \) is called a \( \gamma_{dR}\)-function of \( G \).

Recently, Mojdeh and Volkmann [9] considered an extension of Roman \( \{2\}\)-domination as follows. For a graph \( G \), a Roman \( \{3\}\)-dominating function is a function \( f : V \to \{0, 1, 2, 3\} \) having the property that for every vertex \( u \in V \), if \( f(u) \in \{0, 1\} \), then \( f(N[u]) \geq 3 \). The weight of a Roman \( \{3\}\)-dominating function is the sum \( w(f) = f(V) = \sum_{v \in V} f(v) \), and the minimum weight of a Roman \( \{3\}\)-dominating function is the Roman \( \{3\}\)-domination number, denoted by \( \gamma_{\{3\}\}(G) \). For a Roman \( \{3\}\)-dominating function \( f \), one can denote \( f = (V_0, V_1, V_2, V_3) \), where \( V_i = \{v \in V : f(v) = i\} \), for \( i = 0, 1, 2, 3 \). This concept was further studied.
in [10]. Among other results, Mojdeh et al. presented the following lower bound in [9].

**Theorem 1** (Mojdeh and Volkmann [9]). If \( G \) is a connected graph of order \( n \) and maximum degree \( \Delta \), then \( \gamma_{(R3)}(G) \geq \min \left\{ \frac{3n}{\Delta+2}, \frac{2n+\Delta}{\Delta+1} \right\} \).

In this paper we present upper and lower bounds for the Roman \( \{3\}\)-domination number of a graph. In Section 2, we present a sharp lower bound for the Roman \( \{3\}\)-domination number of a graph and improve the bound given in Theorem 1. In Section 3, we present a probabilistic upper bound for a generalized version of the Roman \( \{3\}\)-domination number, namely, the Roman \( \{k\}\)-domination number for every \( k \geq 3 \), of a graph and show that the given bound is asymptotically best possible.

In this paper, for an event \( F \) we denote by \( Pr(F) \) the probability that \( F \) occurs. We also denote by \( E(X) \) the expectation of \( X \) if \( X \) is a random variable.

## 2. Lower Bound

In this section we present a new sharp lower bound for the Roman \( \{3\}\)-domination number of a graph. We begin with the following observation.

**Observation 2.** For every connected graph \( G \) of order \( n \) and maximum degree \( \Delta \), \( \gamma_{(R3)}(G) \leq 2(n - \Delta) + 1 \).

**Proof.** Let \( v \) be a vertex of maximum degree. Let \( f \) be a function defined on \( V(G) \) by \( f(v) = 3 \), \( f(x) = 0 \) if \( x \in N(v) \) and \( f(x) = 2 \) otherwise. Then \( f \) is a R3DF for \( G \), and so \( \gamma_{(R3)}(G) \leq 2(n - \Delta) + 1 \), as desired. \( \blacksquare \)

**Lemma 3.** If \( G \) is a connected graph of maximum degree \( \Delta(G) = \Delta \geq 1 \) and \( f = (V_0, V_1, V_2, V_3) \) is a \( \gamma_{(R3)}(G) \)-function, then \( 3|V_0| \leq (\Delta - 2)|V_1| + 2\Delta|V_2| + 3\Delta|V_3| \).

**Proof.** Let \( G \) be a connected graph of maximum degree \( \Delta(G) = \Delta \geq 1 \) and \( f = (V_0, V_1, V_2, V_3) \) be a \( \gamma_{(R3)}(G) \)-function. If \( \Delta = 1 \), then \( G = K_2 \), and since \( \gamma_{(R3)}(K_2) = 3 \), we obtain that either \( |V_0| = |V_3| = 1 \) and \( |V_1| = |V_2| = 0 \) or \( |V_0| = |V_3| = 0 \) and \( |V_1| = |V_2| = 1 \). Thus the inequality holds.

Hence we assume that \( \Delta \geq 2 \). We partition \( V_0 \) into four sets \( V_0^3, V_0^{12}, V_0^1 \) and \( V_0^2 \), and \( V_1 \) into three sets \( V_1^1, V_1^2 \) and \( V_1^3 \) as follows. Let

\[
\begin{align*}
V_0^3 & = \{ v \in V_0 : N(v) \cap V_3 \neq \emptyset \}, \\
V_0^{12} & = \{ v \in V_0 \setminus V_0^3 : N(v) \cap V_1 \neq \emptyset, N(v) \cap V_2 \neq \emptyset \}, \\
V_0^1 & = \{ v \in V_0 \setminus (V_0^3 \cup V_0^{12}) : N(v) \subseteq V_0 \cup V_1 \}, \\
V_0^2 & = \{ v \in V_0 \setminus (V_0^3 \cup V_0^{12}) : N(v) \subseteq V_0 \cup V_2 \}.
\end{align*}
\]
\[ V_1^1 = \{ x \in V_1 : N(x) \cap (V_2 \cup V_3) = \emptyset \}, \]
\[ V_1^2 = \{ x \in V_1 : N(x) \cap V_2 = \emptyset \}, \]
\[ V_1^3 = V_1 \setminus (V_1^1 \cup V_1^2). \]

For \( i = 1, 2, 3 \), let \( |V_i^j| = m_i \). We first present an upper bound for \( |V_0^3| \) in terms of \( |V_3| \) and \( m_3 \). Each vertex in \( V_3 \) with no neighbor in \( V_1 \) dominates at most \( \Delta \) vertices of \( V_0^3 \), and every vertex in \( V_3 \) with at least one neighbor in \( V_1 \) dominates at most \( \Delta - 1 \) vertices of \( V_0^3 \). Thus,

\[
|V_0^3| \leq \Delta(|V_3| - m_3) + (\Delta - 1)m_3 = \Delta|V_3| - m_3. \tag{1}
\]

Let \( |V_0^{12}| = x \). We next present an upper bound for \( |V_0^2| \) in terms of \( |V_2| \), \( x \) and \( m_2 \). Clearly, every vertex of \( V_1^2 \cup V_0^{12} \) has a neighbor in \( V_2 \). Since \( |V_1^2| = m_2 \) and \( |V_0^{12}| = x \), there are at most \( \Delta|V_2| - x - m_2 \) edges which have an end-point in \( V_2 \). Since any vertex of \( V_0^2 \) is adjacent to at least two vertices of \( V_2 \), we obtain that

\[
|V_0^2| \leq \frac{\Delta|V_2| - x - m_2}{2}. \tag{2}
\]

We next present an upper bound for \( |V_0^1| \) in terms of \( |V_1| \), \( x \), \( m_1 \), \( m_2 \) and \( m_3 \). Note that every vertex of \( V_1^1 \) is adjacent to at least two vertices of \( V_1 \), every vertex of \( V_1^2 \) is adjacent to at least one vertex of \( V_2 \) and every vertex of \( V_1^3 \) is adjacent to at least one vertex of \( V_3 \). Also every vertex of \( V_0^{12} \) is adjacent to a vertex in \( V_1 \). Thus, there are at most \( \Delta|V_1| - 2m_1 - m_2 - m_3 - x \) edges which have an end-point in \( V_1 \). Since any vertex of \( V_0^1 \) is adjacent to at least three vertices of \( V_1 \), we obtain that

\[
|V_0^1| \leq \frac{\Delta|V_1| - 2m_1 - m_2 - m_3 - x}{3}. \tag{3}
\]

Since \( |V_0| = |V_0^1| + |V_0^2| + |V_0^{12}| + |V_0^3| \), from (1), (2) and (3) we obtain that

\[
3|V_0| \leq \Delta|V_1| - 2m_1 - m_2 - m_3 - x + \frac{3\Delta|V_2|}{2} - \frac{3x}{2} - \frac{3m_2}{2} + 3x + 3\Delta|V_3| - 3m_3.
\]

Since \( |V_1| = m_1 + m_2 + m_3 \), we obtain that

\[
3|V_0| \leq (\Delta - 2)|V_1| + 2\Delta|V_2| + 3\Delta|V_3| - \frac{m_2}{2} - 2m_3 + \frac{x}{2} - \frac{\Delta|V_2|}{2}.
\]

It is evident that \( x \leq \Delta|V_2| \). Thus \( 3|V_0| \leq (\Delta - 2)|V_1| + 2\Delta|V_2| + 3\Delta|V_3| \), as desired. \( \blacksquare \)

**Corollary 4.** If \( G \) is a connected graph of order \( n \) with maximum degree \( \Delta(G) = \Delta \geq 1 \) and \( f = (V_0, V_1, V_2, V_3) \) is a \( \gamma_{(R3)}(G) \)-function, then \( \gamma_{(R3)}(G) \geq \frac{3n - |V_2|}{\Delta + 1} \).
Proof. Let \( f = (V_0, V_1, V_2, V_3) \) be a \( \gamma_{(R3)}(G) \)-function for \( G \). Then
\[
(\Delta + 1)\gamma_{(R3)}(G) = (\Delta + 1)|V_1| + 2(\Delta + 1)|V_2| + 3(\Delta + 1)|V_3|
\]
\[
= (\Delta - 2)|V_0| + 2\Delta|V_2| + 3\Delta|V_3| + 3|V_1| + 2|V_2| + 3|V_3|
\]
\[
\geq 3|V_0| + 3|V_1| + 2|V_2| + 3|V_3| \quad \text{(by Lemma 3)}
\]
\[
= 3n - |V_2|.
\]
Thus the result follows.

Now we present the main result of this section.

Theorem 5. If \( G \) is a connected graph of order \( n > 1 \) and maximum degree \( \Delta \geq 1 \), then
\[
\gamma_{(R3)}(G) \geq \left\lceil \max \left\{ \frac{3n}{\Delta + 2}, \frac{2n + \Delta}{\Delta + 1} \right\} \right\rceil + 1.
\]
This bound is sharp.

Proof. Let \( G \) be a connected graph of order \( n \) and maximum degree \( \Delta \) and \( f = (V_0, V_1, V_2, V_3) \) be a \( \gamma_{(R3)}(G) \)-function for \( G \). If \( |V_2| = 0 \), then from Corollary 4, we obtain that \( \gamma_{(R3)}(G) \geq \frac{3n}{\Delta + 2} > \max \left\{ \frac{3n}{\Delta + 2}, \frac{2n + \Delta}{\Delta + 1} \right\} \), and so \( \gamma_{(R3)}(G) \geq \left\lceil \max \left\{ \frac{3n}{\Delta + 2}, \frac{2n + \Delta}{\Delta + 1} \right\} \right\rceil + 1 \).

Thus assume that \( |V_2| \neq 0 \). By Corollary 4, \( \gamma_{(R3)}(G) \geq \frac{3n - |V_2|}{\Delta + 1} \). Using Observation 2, we obtain \( |V_2| \leq \frac{\gamma_{(R3)}(G)}{2} \leq n - \Delta + \frac{1}{2} \) which implies that \( |V_2| \leq n - \Delta \).

We first show that \( \gamma_{(R3)}(G) \geq \frac{2n + \Delta}{\Delta + 1} \). If \( |V_2| < n - \Delta \), then we obtain that \( \gamma_{(R3)}(G) \geq \frac{3n - |V_2|}{\Delta + 1} > \frac{2n + \Delta}{\Delta + 1} \), as desired. Thus assume that \( |V_2| = n - \Delta \). Suppose that \( \gamma_{(R3)}(G) = \frac{3n - |V_2|}{\Delta + 1} \). Let \( |V_i| = v_i \) for \( i = 0, 1, 2, 3 \). Then from \( \gamma_{(R3)}(G) = \frac{3n - |V_2|}{\Delta + 1} \) we find that \( (\Delta + 1)(v_1 + 2v_2 + 3v_3) = 3v_0 + 3v_1 + 2v_2 + 3v_3 \), since \( n = v_0 + v_1 + v_2 + v_3 \). Since \( \Delta = n - v_2 = v_0 + v_1 + v_3 \), we obtain by a simple calculation that \( v_1 + 2v_2 + 3v_3 = \frac{3v_0 + 2v_1}{v_0 + v_1 + v_3} \). If \( v_3 = 3 \) and \( v_1 = 0 \), then \( 2v_2 = 3 \), a contradiction. If \( v_3 = 0 \) and \( v_1 = 0 \), then \( 2v_2 = 3 - v_1 \) and \( \frac{2v_2}{v_0 + v_1 + v_3} \) is a contradiction. Thus \( v_3 \neq 3 \) and \( v_1 \neq 0 \). Then \( 2v_2 = 3 - v_1 - v_3 + \frac{v_1 + 3v_3}{v_0 + v_1 + v_3} \). If \( v_3 = 0 \) and \( v_1 = 0 \), then \( 2v_2 = 3 - v_1 \) and \( \frac{2v_2}{v_0 + v_1 + v_3} \) is a contradiction. Thus \( v_3 \neq 0 \) and \( v_1 \neq 0 \). Then \( 2v_2 = 3 - v_1 + v_3 \) and \( \frac{v_1 + 3v_3}{v_0 + v_1 + v_3} < 0 \), a contradiction. We conclude that \( \gamma_{(R3)}(G) \neq \frac{3n - |V_2|}{\Delta + 1} \), and so \( \gamma_{(R3)}(G) > \frac{3n - |V_2|}{\Delta + 1} > \frac{2n + \Delta}{\Delta + 1} \), as desired.

We next show that \( \gamma_{(R3)}(G) \geq \frac{3n}{\Delta + 2} \). If \( |V_2| \geq \frac{3n}{\Delta + 2} \), then \( \gamma_{(R3)}(G) \geq 2|V_2| > \frac{3n - |V_2|}{\Delta + 1} \). Thus assume that \( |V_2| < \frac{3n}{\Delta + 2} \). Then a simple calculation shows that \( \gamma_{(R3)}(G) > \frac{3n}{\Delta + 2} \), as desired.
Hence, $\gamma_{\{R_3\}}(G) > \max\left\{\frac{2n+\Delta}{\Delta+1}, \frac{3n}{\Delta+2}\right\}$. This completes the proof of lower bound.

To see the sharpness, consider a complete graph of order at least two. ■

We end this section by remarking that Lemma 3 holds for each R3DF. Moreover, it holds if $\Delta = 0$.

3. Upper Bound

In this section we present an upper bound for a generalization of the Roman $\{3\}$-domination number namely $\text{Roman } \{k\}$-domination number for every integer $k \geq 3$ that is defined as follows. For a graph $G$ and an integer $k \geq 3$, a $\text{Roman } \{k\}$-dominating function is a function $f : V \rightarrow \{0, 1, \ldots, k\}$ having the property that for every vertex $u \in V$, if $f(u) < \left\lceil \frac{k}{2} \right\rceil$, then $f(N[u]) \geq k$. The weight of a $\text{Roman } \{k\}$-dominating function is the sum $w(f) = f(V) = \sum_{v \in V} f(v)$, and the minimum weight of a $\text{Roman } \{k\}$-dominating function is the $\text{Roman } \{k\}$-domination number, denoted by $\gamma_{\{R_k\}}(G)$. For a $\text{Roman } \{k\}$-dominating function $f$, we denote $f = (V_0, V_1, \ldots, V_k)$, where $V_i = \{v \in V : f(v) = i\}$, for $i = 0, 1, \ldots, k$.

**Theorem 6.** If $G$ is a graph of order $n$ with minimum degree $\delta(G) = \delta \geq 1$, then

$$\gamma_{\{R_k\}}(G) \leq k \left( \ln \left\lceil \frac{k}{2} \right\rceil + \ln(1+\delta) - \ln k + 1 \right) \frac{n}{1+\delta}.$$  

**Proof.** Let $G$ be a graph of order $n$ with minimum degree $\delta(G) = \delta \geq 1$. Let $S \subseteq V(G)$ be a set obtained by choosing each vertex $v \in V(G)$, independently, with probability $p \in [0, 1]$, and let $T = V(G) \setminus N[S]$. We form sets $V_i$, $i = 0, 1, \ldots, k$ as follows. Let $V_0 = V(G) \setminus (S \cup T)$, $V_{\left\lceil \frac{k}{2} \right\rceil} = T$, $V_k = S$ and $V_i = \emptyset$ for $i = 1, 2, \ldots, k-1, i \neq \left\lceil \frac{k}{2} \right\rceil$. Then the function $f = (V_0, V_1, \ldots, V_k)$ is a $\text{Roman } \{k\}$-dominating function for $G$. We compute the expected value of $w(f)$. Note that

$$\mathbb{E}(w(f)) = \mathbb{E}(k|S| + \left\lceil \frac{k}{2} \right\rceil |T|) = k\mathbb{E}(|S|) + \left\lceil \frac{k}{2} \right\rceil \mathbb{E}(|T|).$$

Clearly, $\mathbb{E}(|S|) = np$. If $v \in T$, then $v \notin S$ and $v \notin N(S)$. Thus, $Pr(v \notin N[S]) = (1-p)^{1+\deg(v)} \leq (1-p)^{1+\delta}$. Using the fact that $1-p \leq e^{-p}$ for $p \geq 0$, we find that $Pr(v \in T) \leq e^{-p(1+\delta)}$, and so $\mathbb{E}(|T|) \leq ne^{-p(1+\delta)}$. Therefore,

$$\mathbb{E}(w(f)) = \mathbb{E}\left( k|S| + \left\lceil \frac{k}{2} \right\rceil |T| \right) \leq knp + \left\lceil \frac{k}{2} \right\rceil ne^{-p(1+\delta)}.$$
Taking derivative of the function \( g(p) = kp + \left\lceil \frac{k}{2} \right\rceil e^{-p(1+\delta)} \) and solving the equation \( g'(p) = 0 \), we obtain that \( g(p) \) is minimized at \( p = \frac{\ln \left\lceil \frac{k}{2} \right\rceil (1+\delta)}{1+\delta} \). Then by putting these values in (4) we obtain

\[
E(w(f)) \leq nk \left( \frac{\ln \left\lceil \frac{k}{2} \right\rceil (1+\delta) + 1}{1+\delta} \right)^{1+\delta} =: \alpha.
\]

Since the average of \( w(f) \) is not more than \( \alpha \), there is a Roman \( \{k\} \)-dominating function with weight at most \( \alpha \), i.e., \( \gamma_{\{Rk\}}(G) \leq \alpha \), as desired. ■

We now prove that the upper bound of Theorem 6 is asymptotically best possible.

**Theorem 7.** When \( n \) is large, there exists a graph \( G \) of order \( n \) and minimum degree \( \delta \) such that

\[
\gamma_{\{Rk\}}(G) \geq k \left( \ln \left\lceil \frac{k}{2} \right\rceil + \ln(1+\delta) - \ln k + 1 \right)(1+o(1)).
\]

**Proof.** Let \( H \) a complete graph with \( \lfloor \delta \ln \delta \rfloor \) vertices and let \( V(H) = V \). We add a set of new vertices \( V' = \{ v_1, v_2, \ldots, v_\delta \} \) and join each of them to \( \delta \) vertices of \( V(H) \) which are chosen randomly. Let \( G \) be the resulted graph. Therefore \( G \) has \( n = \lfloor \delta \ln \delta \rfloor + \delta \) vertices. We show that

\[
\gamma_{\{Rk\}}(G) \geq \frac{k \ln \delta}{\delta} n(1+o(1)) = \frac{k \ln \delta}{\delta} (\delta \ln \delta + \delta)(1+o(1)) = k \ln^2 \delta(1+o(1)).
\]

Let \( f = (V_0, V_1, \ldots, V_k) \) be a \( \gamma_{\{Rk\}} \)-function for \( G \). If \( |V_k| \geq \ln^2 \delta - \ln \delta \ln \ln^4 \delta \), then

\[
\gamma_{\{Rk\}}(G) \geq k|V_k| \geq k \ln^2 \delta - k \ln \delta \ln \ln^4 \delta = k \ln^2 \delta(1+o(1)),
\]

as desired. Thus assume for the next that \( |V_k| < \ln^2 \delta - \ln \delta \ln \ln^4 \delta \).

We compute the probability that \( V_k \) dominates an element of \( V' \). Note that we can assume \( V_k \subseteq V \). For a vertex \( v_i \in V' \), we have

\[
Pr[V_k \text{ does not dominate } v_i] = \left( \frac{|V| - |V_k|}{|V|} \right)^{\delta} \geq \left( \frac{|V| - |V_k| - \delta}{|V| - \delta} \right)^{\delta} = \left( 1 - \frac{|V_k|}{|V| - \delta} \right)^{\delta}.
\]
Using the fact that $1 - x \geq e^{-x}(1 - x^2)$ for $x \leq 1$, we find that

$$Pr[\text{\textit{V}_k does not dominate \textit{v}_i}] \geq e^{-\frac{|\text{\textit{V}_k}|}{|\text{\textit{V}}| - \delta}} \left(1 - \left(\frac{|\text{\textit{V}_k}|}{|\text{\textit{V}}| - \delta}\right)^2\right) \delta$$

$$\geq e^{-\frac{\ln^2 \delta - \ln \delta \ln \ln^4 \delta}{\delta \ln \delta - \delta}} \delta \left(1 + o_\delta(1)\right)$$

$$\geq e^{-\frac{\ln \delta \ln \ln^4 \delta}{1 - \frac{\ln \delta}{\delta}}} \left(1 + o_\delta(1)\right)$$

$$\geq e^{\ln \left(\frac{\ln^4 \delta}{\delta}\right)(1 + o_\delta(1))} \left(1 + o_\delta(1)\right)$$

$$\geq \left(\frac{\ln^4 \delta}{\delta}\right)^{1 + o_\delta(1)} \left(1 + o_\delta(1)\right) \geq \frac{\ln^3 \delta}{\delta}.$$ 

Thus $Pr[\text{\textit{V}_k dominates \textit{v}_i}] \leq 1 - \frac{\ln^3 \delta}{\delta}$. Now the expected value of the random variable $|N(\text{\textit{V}_k}) \cap \text{\textit{V}}'|$ is bounded above as follows

$$\mathbb{E}(|N(\text{\textit{V}_k}) \cap \text{\textit{V}}'|) = \sum_{i=1}^{\delta} Pr[\text{\textit{V}_k dominates \textit{v}_i}] \leq \delta \left(1 - \frac{\ln^3 \delta}{\delta}\right) = \delta - \ln^3 \delta.$$ 

Consequently, $|\text{\textit{V}}' \setminus N(\text{\textit{V}_k})| \geq \ln^3 \delta$. Since $\text{\textit{V}_k} \subseteq \text{\textit{V}}$, we conclude that there exists a graph $G$ for which

$$\gamma_{(R_k)}(G) \geq |\text{\textit{V}}' \setminus N(\text{\textit{V}_k})| \geq \ln^3 \delta > k \ln^2 \delta(1 + o_\delta(1)),$$

as desired. 

**References**


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