ON M$_f$-EDGE COLORINGS OF GRAPHS

JAROSLAV IVANČO and ALFRÉD ONDERKO

Institute of Mathematics, P.J. Šafárik University
Jesenná 5, 040 01 Košice, Slovakia

e-mail: jaroslav.ivanco@upjs.sk
alfred.onderko@student.upjs.sk

Abstract

An edge coloring $\varphi$ of a graph $G$ is called an M$_f$-edge coloring if $|\varphi(v)| \leq f(v)$ for every vertex $v$ of $G$, where $\varphi(v)$ is the set of colors of edges incident with $v$ and $f$ is a function which assigns a positive integer $f(v)$ to each vertex $v$. Let $K_f(G)$ denote the maximum number of colors used in an M$_f$-edge coloring of $G$. In this paper we establish some bounds on $K_f(G)$, present some graphs achieving the bounds and determine exact values of $K_f(G)$ for some special classes of graphs.

Keywords: edge coloring, anti-Ramsey number, dominating set.

2010 Mathematics Subject Classification: 05C15.

1. Introduction

We consider finite undirected graphs without loops and multiple edges. If $G$ is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of $G$, respectively. The subgraph of a graph $G$ induced by $U \subseteq V(G)$ is denoted by $G[U]$. Similarly, if $A \subseteq E(G)$, then $G[A]$ denotes the subgraph of $G$ induced by $A$ (i.e., the subgraph with the edge set $A$ and the vertex set consisting of all vertices incident with an edge in $A$). The set of vertices of $G$ adjacent to a vertex $v \in V(G)$ is denoted by $N_G(v)$. The cardinality of this set, denoted $\text{deg}_G(v)$, is called the degree of $v$. As usual $\Delta(G)$ and $\delta(G)$ stand for the maximum and minimum degree among vertices of $G$. The set of vertices of degree $d$ in $G$ is denoted by $V_d(G)$.

1This work was supported by the Slovak VEGA Grant 1/0368/16 and by the Slovak Research and Development Agency under the contract No. APVV-15-0116.
An edge coloring of a graph $G$ is an assignment of colors to the edges of $G$, one color to each edge. So any mapping $\varphi$ from $E(G)$ onto a non-empty set is an edge coloring of $G$. The set of colors used in an edge coloring $\varphi$ of $G$ is denoted by $\varphi(G)$, i.e., $\varphi(G) = \{\varphi(e) : e \in E(G)\}$. For any vertex $v \in V(G)$, let $\varphi(v)$ denote the set of colors of edges incident with $v$, i.e., $\varphi(v) = \{\varphi(eu) : u \in N_G(v)\}$. Let $f$ be a function which assigns a positive integer $f(v)$ to each vertex $v \in V(G)$. An edge coloring $\varphi$ of $G$ is an $M_f$-edge coloring if at most $f(v)$ colors appear at any vertex $v$ of $G$, i.e., $|\varphi(v)| \leq f(v)$ for every vertex $v \in V(G)$. The maximum number of colors used in an $M_f$-edge coloring of $G$ is denoted by $K_f(G)$. If $f(v) = i$ for all $v \in V(G)$, then an $M_f$-edge coloring is called an $M_i$-edge coloring and the maximum number of colors used in an $M_i$-edge coloring is denoted by $K_i(G)$.

The $M_i$-edge coloring is a natural generalization of the $M_1$-edge coloring. The concept of $M_i$-edge colorings was introduced by Czap [4]. In [3] authors establish a tight bound on $K_2(G)$ depending on the size of a maximum matching in $G$. In [4] and [5], the exact values of $K_2(G)$ for subcubic graphs and complete graphs are determined. In [7] it is determined $K_2(G)$ for cacti, trees, graph joins and complete multipartite graphs. In [10] there are established some bounds on $K_2(G)$ and presented graphs achieving the bounds. Exact values of $K_2(G)$ for dense graphs are also determined. $K_3(G)$ and $K_4(G)$ for complete graphs are determined in [6]. A vertex variant of the $M_2$-edge coloring was studied in [2].

However before, Feng et al. [8] introduced a maximum edge $q$-coloring problem which arises from wireless mesh networks. It is really the problem of finding an $M_q$-edge coloring of a given graph $G$ which uses $K_q(G)$ colors (for an integer $q$, $q \geq 2$). There are studied mainly algorithmic aspects of the maximum edge $q$-coloring problem. In [8] there is provided a 2-approximation algorithm for $q = 2$ and a $\left(1 + \frac{4q-2}{3q^2-9q+2}\right)$-approximation for $q > 2$. In [1] there is proved that the maximum edge $q$-coloring problem is NP-hard. Also, for graphs with perfect matching there is presented a $\frac{5}{3}$-approximation algorithm in case $q = 2$. A related problem is studied in [12].

The anti-Ramsey number has been extensively studied in the area of extremal graph theory (see [9] for a survey). For given graphs $G$ and $H$ the anti-Ramsey number $\text{ar}(G, H)$ is defined to be the maximum number $k$ such that there exists an assignment of $k$ colors to the edges of $G$ in which every copy of $H$ in $G$ has at least two edges with the same color. A coloring of $G$ is an $M_q$-edge coloring if and only if each subgraph $K_{1,q+1}$ (a star with $q + 1$ edges) of $G$ has two edges with the same color. Therefore $K_q(G)$ is equal to $\text{ar}(G, K_{1,q+1})$. Thereby, in [11] there is determined $K_q(K_{n,n})$ exactly and $K_q(K_n)$ within 1, for all positive integers $n$ and $q$. Similarly, an upper bound on the value of $K_q(G)$ if $\delta(G) \geq q + 5$, or if $G$ is $K_3$-free and $\delta(G) \geq q + 2$, is presented in [13]. Some applications of this bound (e.g., exact values of $K_q(G)$ for hypercubes) are also produced.
In this paper we establish some bounds of $K_f(G)$ depending on dominating sets of $G$. We also determine exact values of $K_f(G)$ for some particular classes of graphs, especially for trees, forests, some cactuses, and dense graphs with a dominating vertex. Accordingly, we extend some known results, proved in [11] and [13], on $K_q(G)$ (as anti-Ramsey number) for complete graphs and complete multipartite graphs.

2. Auxiliary Results

It is easy to see that $|\varphi(v)| \leq \deg_G(v)$ for any edge coloring $\varphi$ of a graph $G$ and each vertex $v \in V(G)$. Therefore, throughout the paper we suppose that the function $f$ satisfies

\[
1 \leq f(v) \leq \deg_G(v) \quad \text{for every } v \in V(G).
\]

The following two claims are evident.

**Observation 1.** Let $f$ be a function from the vertex set of a graph $G$ to positive integers. Assume that $G$ has $k$ connected components. Let $G_j$, $j \in \{1, \ldots, k\}$, be a component of the graph $G$ and let $f_j$ be a restriction of $f$ to $V(G_j)$. Then

\[
K_f(G) = \sum_{j=1}^{k} K_{f_j}(G_j).
\]

Given a graph $G$, let $e = uv$ be an edge of $G$ such that $\deg_G(v) \geq 2$. By $S(G; e, v)$ we denote the graph with the vertex set $V(G) \cup \{v'\}$ and the edge set $(E(G) \setminus \{e\}) \cup \{uv'\}$.

**Observation 2.** Let $f$ be a function from the vertex set of a graph $G$ to integers satisfying (1). Let $v$ be a vertex of $G$ such that $f(v) = \deg_G(v) \geq 2$. For an edge $e$ incident with $v$ let $h$ be a function from the vertex set of $S(G; e, v)$ to integers given by

\[
h(u) = \begin{cases} f(u) & \text{if } u \notin \{v, v'\}, \\ \deg_{S(G; e, v)}(u) & \text{if } u \in \{v, v'\}. \end{cases}
\]

Then

\[
K_f(G) = K_h(S(G; e, v)).
\]

Let $\varphi$ be an $M_f$-edge coloring of $G$. For a set $U \subseteq V(G)$, let $\varphi(U)$ denote the set of colors of edges incident with vertices of $U$ in $G$. Thus, $\varphi(U) = \bigcup_{v \in U} \varphi(v)$.

**Lemma 1.** Let $\varphi$ be an $M_f$-edge coloring of a graph $G$ and let $U$ be a non-empty subset of $V(G)$. Then the following statements hold.
(i) \( |\varphi(U)| \leq c + \sum_{u \in U} (f(u) - 1) \), where \( c \) denotes the number of connected components of \( G[U] \).

(ii) If \( G[U] \) is a 2-connected graph and \( |\varphi(U)| = 1 + \sum_{u \in U} (f(u) - 1) \), then 
\[ |\{ \varphi(e) : e \in E(G[U]) \}| = 1. \]

**Proof.** (i) First suppose that \( G[U] \) is a connected graph. Denote the vertices of \( U \) by \( u_1, u_2, \ldots, u_k \) in such a way that the set \( X_i = \{ u_1, u_2, \ldots, u_i \} \) induces a connected subgraph of \( G \) for every \( i \in \{1, 2, \ldots, k\} \). As \( G[X_i] \) is connected for \( i \geq 2 \), there is \( j \) (\( 1 \leq j < i \)) such that \( u_i u_j \) is an edge of \( G \). Therefore, \( \varphi(u_i u_j) \in \varphi(X_{i-1}) \cap \varphi(u_i) \) and
\[
|\varphi(X_i)| = |\varphi(X_{i-1}) \cup \varphi(u_i)| = |\varphi(X_{i-1})| + |\varphi(u_i)| - |\varphi(X_{i-1}) \cap \varphi(u_i)|
\leq |\varphi(X_{i-1})| + f(u_i) - 1.
\
Clearly, \( |\varphi(X_1)| = |\varphi(u_1)| \leq f(u_1) = 1 + \sum_{u \in X_1} (f(u) - 1) \). Thus, by induction we get
\[
|\varphi(X_i)| \leq |\varphi(X_{i-1})| + (f(u_i) - 1) \leq 1 + \sum_{u \in X_i} (f(u) - 1)
\]
and consequently \( |\varphi(U)| = |\varphi(X_k)| \leq 1 + \sum_{u \in U} (f(u) - 1) \).

If \( G[U] \) is a disconnected graph, then the set \( U \) can be partitioned into disjoint subsets \( U_1, U_2, \ldots, U_c \) in such a way that \( G[U_i] \) is a connected component of \( G[U] \) for every \( i \in \{1, 2, \ldots, c\} \). Therefore,
\[
|\varphi(U)| = |\varphi \left( \bigcup_{i=1}^c U_i \right)| \leq \sum_{i=1}^c |\varphi(U_i)| \leq \sum_{i=1}^c \left( 1 + \sum_{u \in U_i} (f(u) - 1) \right)
\leq c + \sum_{u \in U} (f(u) - 1).
\]

(ii) Now suppose that \( G[U] \) is 2-connected and \( |\{ \varphi(e) : e \in E(G[U]) \}| > 1 \). Then there are edges \( uw \) and \( vw \) in \( E(G[U]) \) such that \( \varphi(uw) \neq \varphi(vw) \). Therefore, \( |\varphi(w) \cap (\varphi(u) \cup \varphi(v))| \geq 2 \) and consequently \( |\varphi(w) \cap \varphi(U \setminus \{w\})| \geq 2 \). As \( G[U] \) is 2-connected, \( G[U \setminus \{w\}] \) is connected and by (i)
\[
|\varphi(U \setminus \{w\})| \leq 1 + \sum_{u \in U \setminus \{w\}} (f(u) - 1).
\]

Hence \( |\varphi(U)| \leq |\varphi(U \setminus \{w\})| + f(w) - 2 \leq \sum_{u \in U} (f(u) - 1) \), which completes the proof. \( \blacksquare \)

A subgraph \( H \) of a graph \( G \) is called an \textit{f-subgraph} of \( G \) if \( \deg_H(v) < f(v) \) for every \( v \in V(H) \). The maximum number of edges in an \textit{f}-subgraph of \( G \) is
denoted by \( \alpha_f(G) \) and the maximum number of edges in an \( f \)-subgraph of \( G[U] \) (\( U \subset V(G) \)) is denoted by \( \alpha_f(U) \) (i.e., \( \alpha_f(G) = \alpha_f(V(G)) \)). If \( f(v) = i \) for all \( v \in V(G) \), then \( \alpha_f(G) \) and \( \alpha_f(U) \) is denoted by \( \alpha_i(G) \) and \( \alpha_i(U) \), respectively.

**Lemma 2.** Let \( H \) be an \( f \)-subgraph of a graph \( G \). Then there is an \( M_f \)-edge coloring of \( G \) such that \( |\varphi(G)| = c + |E(H)| \), where \( c \) denotes the number of connected components of \( G[E(G) \setminus E(H)] \).

**Proof.** Denote by \( e_1, e_2, \ldots, e_k \) edges of \( H \) and by \( C_1, C_2, \ldots, C_c \) components of \( G[E(G) \setminus E(H)] \) (\( c = 0 \) when \( E(H) = E(G) \)). Consider a mapping \( \varphi \) from \( E(G) \) onto \( \{1, 2, \ldots, h + c\} \) given by

\[
\varphi(e) = \begin{cases} 
j & \text{if } e \in E(H) \text{ and } e = e_j, \\
h + j & \text{if } e \notin E(H) \text{ and } e \in C_j.
\end{cases}
\]

Clearly, \( |\varphi(v)| \leq \deg_H(v) + 1 \leq f(v) \), for any vertex \( v \in V(G) \). Therefore, \( \varphi \) is a desired \( M_f \)-edge coloring of \( G \).

**Lemma 3.** Let \( G \) be a connected graph of order at least 2. Let \( c(v) \) denote the number of components of \( G - v \) and \( d(v) = \min\{c(v), f(v)\} \) for every \( v \in V(G) \). Then there is an \( M_f \)-edge coloring \( \varphi \) of \( G \) such that

\[
|\varphi(G)| = 1 + \sum_{v \in V(G)} (d(v) - 1) \quad \text{and} \quad |\varphi(v)| = d(v) \quad \text{for every } v \in V(G).
\]

**Proof.** Denote vertices of \( U = \{u \in V(G) : d(u) > 1\} \) by \( u_1, u_2, \ldots, u_k \). Put \( U_0 = \emptyset, s_0 = 0 \) and \( U_i = U_{i-1} \cup \{u_i\}, s_i = s_{i-1} + d(u_i) - 1 \), for \( i \in \{1, 2, \ldots, k\} \). Evidently, \( s_i = \sum_{v \in U_i} (d(v) - 1) \). For all \( i \in \{0, 1, \ldots, k\} \), define the \( M_f \)-edge coloring \( \varphi_i \) of \( G \) recursively in the following way.

Let \( \varphi_0 \) be a mapping from \( E(G) \) to \( \{0\} \). As \( \varphi_0(e) = 0 \), for every edge \( e \in E(G) \), \( |\varphi_0(G)| = 1 = 1 + s_0 \) and \( |\varphi_0(v)| = 1 \) for each \( v \in V(G) \).

Now suppose that a mapping \( \varphi_i \) from \( E(G) \) onto \( \{0, 1, \ldots, s_i\} \) is an \( M_f \)-edge coloring of \( G \) such that \( |\varphi_i(v)| = d(v) \) for \( v \in U_i \) and \( |\varphi_i(v)| = 1 \) for \( v \in V(G) \setminus U_i \). As \( u_{i+1} \notin U_i \), \( |\varphi_i(u_{i+1})| = 1 \). Since \( d(u_{i+1}) > 1 \), the graph \( G - u_{i+1} \) is disconnected with \( c(u_{i+1}) \) components. As \( c(u_{i+1}) \geq d(u_{i+1}) \), we can choose components \( C_1, C_2, \ldots, C_t \) (where \( t = d(u_{i+1}) - 1 \)) of \( G - u_{i+1} \). For each \( j \in \{1, 2, \ldots, t\} \), let \( H_j \) be a subgraph of \( G \) induced by \( V(C_j) \cup \{u_{i+1}\} \). Consider a mapping \( \varphi_{i+1} \) from \( E(G) \) onto \( \{0, 1, \ldots, s_{i+1}\} \) given by

\[
\varphi_{i+1}(e) = \begin{cases} 
s_i + j & \text{if } \varphi_i(e) \in \varphi_i(u_{i+1}) \text{ and } e \in E(H_j), \\
\varphi_i(e) & \text{otherwise.}
\end{cases}
\]

Evidently, \( |\varphi_{i+1}(v)| = |\varphi_i(v)| \) for \( v \in V(G) \setminus \{u_{i+1}\} \), and \( |\varphi_{i+1}(u_{i+1})| = 1 + t = d(u_{i+1}) \). Therefore, \( \varphi_{i+1} \) is an \( M_f \)-edge coloring of \( G \) such that \( |\varphi_{i+1}(G)| = \)
1 + s_i + t = 1 + s_{i+1}. Moreover, |ϕ_{i+1}(v)| = d(v) for v ∈ U_{i+1} and |ϕ_{i+1}(v)| = 1 for v ∈ V(G) \ U_{i+1}.

Thus, there is an M_f-edge coloring ϕ (ϕ = ϕ_k) of G such that |ϕ(G)| = 1 + s_k, |ϕ(v)| = d(v) for v ∈ U_k = U, and |ϕ(v)| = 1 for v ∈ V(G) \ U. As d(v) = 1 for each v ∈ V(G) \ U, ϕ is a desired coloring.

3. Main Results

A set D ⊆ V(G) is called dominating in G, if for each v ∈ V(G) \ D there exists a vertex u ∈ D adjacent to v.

Theorem 1. Let D be a dominating set of a graph G. If c denotes the number of connected components of G[D], then

\[ K_f(G) ≤ c + \sum_{u \in D} (f(u) - 1) + \alpha_f(V(G) \setminus D). \]

Proof. Let ϕ be an M_f-edge coloring of G which uses K_f(G) colors, i.e., |ϕ(G)| = K_f(G). Suppose that A is a subset of E(G) containing exactly one edge of each color belonging to ϕ(G) \ ϕ(D). Let H be a subgraph of G induced by A. Evidently, the graph H is an f-subgraph of G and V(H) ⊆ V(G) \ D. Therefore, |A| = |E(H)| ≤ \alpha_f(V(G) \setminus D). Thus,

\[ K_f(G) = |ϕ(D)| + |A| ≤ |ϕ(D)| + \alpha_f(V(G) \setminus D). \]

According to Lemma 1, |ϕ(D)| ≤ c + \sum_{u \in D} (f(u) - 1) and the desired inequality follows.

The following result presents some graphs achieving the bound established in Theorem 1.

Theorem 2. Let D be a dominating set of a connected graph G satisfying

(i) |D| ≥ 2;
(ii) G[D] is a connected subgraph of G;
(iii) if u ∈ D and c(u) is the number of connected components of G[D] − u, then there is at least f(u) − c(u) vertices in V(G) \ D adjacent to u;
(iv) f(v) = \deg_G(v) for all v ∈ V(G) \ D.

Then

\[ K_f(G) = 1 + |E(G[V(G) \setminus D])| + \sum_{u \in D} (f(u) - 1). \]
Proof. For each vertex $v \in V(G) \setminus D$ there is a vertex in $D$ adjacent to $v$. Thus, $\deg_{G[V(G) \setminus D]}(v) < \deg_G(v) = f(v)$. Therefore, $G[V(G) \setminus D]$ is an $f$-subgraph of $G$ and $\alpha_f(V(G) \setminus D) = |E(G[V(G) \setminus D])|$. According to (ii), $G[D]$ is a connected subgraph of $G$, and by Theorem 1 we have

$$K_f(G) \leq 1 + \sum_{u \in D} (f(u) - 1) + \alpha_f(V(G) \setminus D) = 1 + \sum_{u \in D} (f(u) - 1) + |E(G[V(G) \setminus D])|.$$  

On the other hand, according to (i) and (ii), $G[D]$ is a connected graph of order at least 2. For every vertex $u \in D$, set $A(u) = \{uv \in E(G) : v \in V(G) \setminus D\}$, $d(u) = \min\{c(u), f(u)\}$, and $t(u) = f(u) - d(u)$. By (iii), $|A(u)| \geq t(u)$. Thus, there is a set $A^*(u)$ such that $A^*(u) \subseteq A(u)$ and $|A^*(u)| = t(u)$. Clearly,

$$\left|E(G[V(G) \setminus D]) \cup \bigcup_{u \in D} A^*(u)\right| = |E(G[V(G) \setminus D])| + \sum_{u \in D} t(u).$$

Therefore, there is a bijection $\zeta$ from $E(G[V(G) \setminus D]) \cup \bigcup_{u \in D} A^*(u)$ onto a set $B$, where $|B| = |E(G[V(G) \setminus D])| + \sum_{u \in D} t(u)$.

According to Lemma 3, there is an $M_f$-edge coloring $\varphi$ of $G[D]$ such that $|\varphi(G[D])| = 1 + \sum_{u \in D} (d(u) - 1) + |\varphi(u)| = d(u)$ for each $u \in D$. Moreover, we can assume that $\varphi(G[D])$ and $B$ are disjoint sets. Now suppose that $\xi$ is any mapping from $D$ to $\varphi(G[D])$ satisfying $\xi(u) \in \varphi(u)$ for each $u \in D$. Consider the edge coloring $\psi$ of $G$ defined in the following way

$$\psi(e) = \begin{cases} \varphi(e) & \text{if } e \in E(G[D]), \\ \zeta(e) & \text{if } e \in A^*(u), \\ \xi(u) & \text{if } e \in A(u) \setminus A^*(u), \\ \zeta(e) & \text{if } e \in E(G[V(G) \setminus D]). \end{cases}$$

We have $|\psi(u)| = |\varphi(u)| + |A^*(u)| = d(u) + t(u) = f(u)$, for any vertex $u \in D$, and $|\psi(v)| \leq \deg_G(v) = f(v)$, for any vertex $v \in V(G) \setminus D$. So, $\psi$ is an $M_f$-edge coloring of $G$ which uses $|\varphi(G[D])| + |B|$ colors. Hence

$$|\psi(G)| = 1 + \sum_{u \in D} (d(u) - 1) + |E(G[V(G) \setminus D])| + \sum_{u \in D} t(u)$$

$$= 1 + \sum_{u \in D} (d(u) - 1 + t(u)) + |E(G[V(G) \setminus D])|$$

$$= 1 + \sum_{u \in D} (f(u) - 1) + |E(G[V(G) \setminus D])|,$$

i.e., $K_f(G) \geq 1 + \sum_{u \in D} (f(u) - 1) + |E(G[V(G) \setminus D])|$.  

Recall that a connected graph in which every edge belongs to at most one cycle is called a cactus.

**Corollary 3.** Let \( G \) be a cactus of order at least \( 2 \). For every vertex \( u \in V(G) \), let \( \nu(u) \) denote the number of cycles of \( G \) containing \( u \). If \( f(u) + \nu(u) \leq \deg_G(u) \), for all \( u \in V(G) \), then

\[
K_f(G) = 1 + \sum_{u \in V(G)} (f(u) - 1).
\]

**Proof.** Evidently, \( D = V(G) \) is a dominating set of \( G \). As \( G \) is a cactus, \( c(u) \), the number of connected components of \( G - u \), is equal to \( \deg_G(u) - \nu(u) \) for every vertex \( u \in V(G) \). Then, \( f(u) - c(u) = f(u) + \nu(u) - \deg_G(u) \leq 0 \). Therefore, the conditions of Theorem 2 are satisfied. Moreover, \( |E(G[V(G) \setminus D])| = 0 \). According to Theorem 2, the result follows.

**Corollary 4.** Let \( T \) be a tree of order at least \( 2 \). Let \( f \) be a function from \( V(T) \) to positive integers satisfying (1). Then

\[
K_f(T) = 1 + \sum_{u \in V(T)} (f(u) - 1) = |E(T)| - \sum_{u \in V(T)} (\deg_T(u) - f(u)).
\]

Especially, if \( q \) is a positive integer, then

\[
K_q(T) = 1 + (q - 1)|V(T)| - \sum_{j=1}^{q-1} (q - j)|V_j(T)|.
\]

**Proof.** Each tree is a cactus without cycles. Therefore, by Corollary 3,

\[
K_f(T) = 1 + \sum_{u \in V(T)} (f(u) - 1) = 1 + \sum_{u \in V(T)} f(u) - |V(T)|
\]

\[
= \sum_{u \in V(T)} f(u) - |E(T)| = |E(T)| + \sum_{u \in V(T)} f(u) - 2|E(T)|
\]

\[
= |E(T)| + \sum_{u \in V(T)} f(u) - \sum_{u \in V(T)} \deg_T(u)
\]

\[
= |E(T)| - \sum_{u \in V(T)} (\deg_T(u) - f(u)).
\]

Now consider a function \( t \) from \( V(T) \) to positive integers given by

\[
t(u) = \min\{\deg_T(u), q\}.
\]
Then
\[
\sum_{u \in V(T)} t(u) = \left( \sum_{j=1}^{\Delta(T)} j \sum_{u \in V(T) \atop \deg_T(u) = j} t(u) \right) = \sum_{j=1}^{q-1} j|V_j(T)| + \sum_{j=q}^{\Delta(T)} q|V_j(T)|
\]
\[
= \sum_{j=1}^{\Delta(T)} q|V_j(T)| - \sum_{j=1}^{q-1} (q-j)|V_j(T)| = q|V(T)| - \sum_{j=1}^{q-1} (q-j)|V_j(T)|.
\]
Evidently, \( K_q(T) = K_t(T) \). Thus
\[
K_q(T) = 1 + \sum_{u \in V(T)} (t(u) - 1) = 1 + \sum_{u \in V(T)} t(u) - |V(T)|
\]
\[
= 1 + q|V(T)| - \sum_{j=1}^{q-1} (q-j)|V_j(T)| - |V(T)|
\]
\[
= 1 + (q - 1)|V(T)| - \sum_{j=1}^{q-1} (q-j)|V_j(T)|,
\]
which completes the proof.

**Corollary 5.** Let \( F \) be a forest whose every component is of order at least 2. Let \( f \) be a function from \( V(F) \) to positive integers satisfying (1). Then
\[
K_f(F) = |E(F)| - \sum_{u \in V(F)} (\deg_F(u) - f(u)).
\]

**Proof.** Let \( T_j, j \in \{1, \ldots, k\} \), be a component of \( F \) and let \( f_j \) be a restriction of \( f \) to \( V(T_j) \). Every component of \( F \) is a tree, thus, by Observation 1 and Corollary 4, we have
\[
K_f(F) = \sum_{j=1}^{k} K_{f_j}(T_j) = \sum_{j=1}^{k} \left( |E(T_j)| - \sum_{u \in V(T_j)} (\deg_{T_j}(u) - f(u)) \right)
\]
\[
= |E(F)| - \sum_{u \in V(F)} (\deg_F(u) - f(u)),
\]
which completes the proof.

**Corollary 6.** Let \( f \) be a function from the vertex set of a graph \( G \) to positive integers satisfying (1). If every cycle of \( G \) contains a vertex \( v \) such that \( f(v) = \deg_G(v) \), then
\[
K_f(G) = |E(G)| - \sum_{u \in V(G)} (\deg_G(u) - f(u)).
\]
Proof. Suppose that $G$ is a counterexample with the minimum number of cycles. According to Corollary 5, $G$ contains a cycle $C$. Then there is a vertex $v$ of $C$ such that $f(v) = \deg_G(v)$. Let $e$ be an edge of the cycle $C$ incident with $v$. Consider a graph $H = S(G; e, v)$ and a function $h$ from $V(H)$ to positive integers defined by

$$h(u) = \begin{cases} f(u) & \text{if } u \in V(H) \setminus \{v, v'\}, \\ \deg_H(u) & \text{if } u \in \{v, v'\}. \end{cases}$$

Clearly, every cycle of $H$ is also a cycle in $G$ and it contains a vertex $w$ such that $h(w) = \deg_H(w)$. Moreover, $H$ has less cycles than $G$ and so it is not a counterexample. Then, $K_h(H) = |E(H)| - \sum_{u \in V(H)} (\deg_H(u) - h(u))$. By Observation 2, $K_f(G) = K_h(H)$. Therefore,

$$K_f(G) = K_h(H) = |E(H)| - \sum_{u \in V(H)} (\deg_H(u) - h(u))$$

$$= |E(G)| - \sum_{u \in V(G)} (\deg_G(u) - f(u)),$$

a contradiction to the choice of $G$.

The following result present other graphs achieving the bound established in Theorem 1.

**Theorem 3.** Let $D$ be a dominating set of a graph $G$ such that $|D| \geq 2$ and $G[D]$ is a connected subgraph of $G$. Let $I$ be a set of isolated vertices in $G[V(G) \setminus D]$. If there is a spanning subgraph $B$ of $G$ satisfying

(i) every edge of $B$ is incident with a vertex in $I$,

(ii) $\deg_B(u) = f(u) - 1$ if $u \in D$,

(iii) $\deg_B(u) < f(u)$ if $u \in I$ and $\deg_G(u) > f(u),

then

$$K_f(G) = 1 + \sum_{u \in D} (f(u) - 1) + \alpha_f(V(G) \setminus D).$$

**Proof.** Set $k = \sum_{u \in D} (f(u) - 1)$ and $\alpha = \alpha_f(V(G) \setminus D)$. According to (i), every edge of $B$ connects a vertex from $I$ with one from $D$. Moreover, by (ii), $|E(B)| = k$. Let $H$ be an $f$-subgraph of $G[V(G) \setminus D]$ having $\alpha$ edges. Clearly, no edge of $H$ is incident with a vertex in $I$.

Denote by $e_1, e_2, \ldots, e_k$ edges of $B$ and by $a_1, a_2, \ldots, a_\alpha$ edges of $H$. Consider the mapping $\psi$ from $E(G)$ onto $\{1, 2, \ldots, 1 + k + \alpha\}$ given by

$$\psi(e) = \begin{cases} j & \text{if } e \in E(B) \text{ and } e = e_j, \\ k + j & \text{if } e \in E(H) \text{ and } e = a_j, \\ 1 + k + \alpha & \text{if } e \notin E(B) \cup E(H). \end{cases}$$
According to (ii), $|\psi(u)| = f(u)$, for any vertex $u \in D$. By (iii), $|\psi(u)| \leq f(u)$, for any vertex $u \in I$. Similarly, $|\psi(u)| \leq f(u)$, for any vertex $u \in V(G) \setminus (D \cup I)$, because $H$ is an $f$-subgraph. Therefore, $\psi$ is an $M_f$-edge coloring of $G$. Consequently, $K_f(G) \geq |\psi(G)| = 1 + k + \alpha$. The opposite inequality follows from Theorem 1.

Recall that the join of two graphs $G$ and $H$ is obtained from vertex-disjoint copies of $G$ and $H$ by adding all edges between $V(G)$ and $V(H)$.

**Corollary 7.** Let $q$, $n$, and $m$ be integers such that $q \geq 2$, $n \geq 2$, $m \geq q - 1$ when $n \leq q$, and $m \geq n$ when $n > q$. Let $G_1$ and $G_2$ be disjoint graphs such that $|V(G_1)| = n$, $G_2$ contains $m$ isolated vertices, and let $G$ be the join of $G_1$ and $G_2$. Then

$$K_q(G) = 1 + n(q - 1) + \alpha_q(V(G_2)).$$

**Proof.** Clearly, $V(G_1)$ is a dominating set of $G$. Let $I$ be the set of isolated vertices in $G_2$. Then $G$ contains the complete bipartite subgraph with parts $V(G_1)$ and $I$ (i.e., the subgraph isomorphic to $K_{n,m}$). The graph $K_{n,m}$ contains either a subgraph isomorphic to $K_{n,q-1}$ (if $n \leq q$), or a $(q-1)$-regular subgraph of order $2n$ (if $n > q$). Thus, there is a spanning subgraph $B$ of $G$ satisfying conditions (i)–(iii) from Theorem 3, and the assertion follows.

A complete $k$-partite graph is a graph whose vertices can be partitioned into $k \geq 2$ disjoint classes $V_1, \ldots, V_k$ such that two vertices are adjacent whenever they belong to distinct classes. If $|V_i| = n_i$, $i = 1, \ldots, k$, then the complete $k$-partite graph is denoted by $K_{n_1, \ldots, n_k}$.

In [13] there are stated some results on $K_q(G)$ for complete multipartite graphs with parts of size at least $q - 1$. In the following assertion we consider complete multipartite graphs that can contain parts of size less than $q - 1$, so we extend the result from [13]. The complete $k$-partite graph $K_{n_1, \ldots, n_k}$ is the join of $K_{n_1, \ldots, n_{k-1}}$ and the totally disconnected graph of order $n_k$. Thus, according to Corollary 7, we immediately have the following statement.

**Corollary 8.** Let $q$, $k$, $n_1, \ldots, n_k$ and $p$ be integers such that $q \geq 2$, $k \geq 3$, $1 \leq n_1 \leq \cdots \leq n_k$, $p = \sum_{j=1}^{k-1} n_j$, $n_k \geq q - 1$ when $p \leq q$, and $n_k \geq p$ when $p > q$. Then

$$K_q(K_{n_1, \ldots, n_k}) = 1 + p(q - 1).$$
Corollary 9. Let $q$ be a positive integer. Let $G$ be a connected graph of order at least two and let $H$ be a graph containing at least $q - 1$ isolated vertices. Then

$$K_q(G \circ H) = 1 + |V(G)|(q - 1 + \alpha_q(H)).$$

A vertex of a graph $G$ is called a dominating vertex if it is adjacent to every other vertex of $G$.

Theorem 4. Let $w$ be a dominating vertex of a graph $G$. Let $f$ be a function from $V(G)$ to positive integers such that $\deg_G(u) \geq f(u) + \left\lceil \frac{|V(G)| + f(w) - 3}{2} \right\rceil$, for every vertex $u$ of $G$. Then

$$K_f(G) = 1 + \alpha_f(G).$$

Proof. Suppose that $\varphi$ is an $M_f$-edge coloring of $G$ which uses $K_f(G)$ colors (i.e., $|\varphi(G)| = K_f(G)$). Denote colors of $\varphi(w)$ by $c_1, \ldots, c_k$ ($k = |\varphi(w)|$) and set $U_j = \{u \in V(G) \setminus \{w\} : \varphi(wu) = c_j\}$ for each $j \in \{1, \ldots, k\}$.

Let $A$ be a subset of $E(G)$ containing exactly one edge of each color belonging to $\varphi(G) \setminus \varphi(w)$. Let $H$ be a subgraph of $G$ such that $V(H) = V(G) \setminus \{w\}$ and $E(H) = A$. Evidently, the graph $H$ is an $f$-subgraph of $G$. Set

$$X = \{v \in V(H) : \deg_H(v) = f(v) - 1\}$$
and
$$Y = \{v \in V(H) : \deg_H(v) < f(v) - 1\}.$$

First suppose that $|Y| \leq k - 2$. As $U_1, U_2, \ldots, U_k$ are pairwise disjoint, at most $|Y|$ sets of them contain a vertex of $Y$. Then there are at least two sets, without loss of generality $U_1$ and $U_2$, such that $U_1 \cap Y = \emptyset = U_2 \cap Y$. Moreover, we can assume that $|U_1| \leq |U_2|$. Thus, $|U_1| \leq \frac{|X|}{2} = \left\lfloor \frac{|V(G)| - 1 - |Y|}{2} \right\rfloor$. Let $u^*$ be a vertex of $U_1$. As

$$|\{w\}| + |U_1 \setminus \{u^*\}| + |Y| \leq 1 + \left\lceil \frac{|V(G)| - 1 - |Y|}{2} \right\rceil - 1 + |Y|$$

$$= \left\lfloor \frac{|V(G)| + |Y| - 1}{2} \right\rfloor \leq \left\lceil \frac{|V(G)| + f(w) - 3}{2} \right\rceil,$$

there are at least $f(u^*)$ vertices of $X \setminus U_1$ that are adjacent to $u^*$ in $G$. Since $\deg_H(u^*) = f(u^*) - 1$, there is a vertex $v^* \in X \setminus U_1$ such that $u^*v^* \in E(G)$ and $u^*v^* \notin E(H)$. As $v^* \in X \setminus U_1$, there is $i, 2 \leq i \leq k$, such that $v^* \in U_i$. Since $\deg_H(v^*) = f(v^*) - 1$, for each color $c \in \varphi(v^*) \setminus \{c_i\}$, there is a vertex $x \in N_H(v^*)$ such that $\varphi(v^*x) = c$. Similarly, for each color $c \in \varphi(u^*) \setminus \{c_1\}$, there is a vertex $x \in N_H(u^*)$ such that $\varphi(u^*x) = c$. Therefore, $\varphi(u^*) \setminus \{c_1\} \cap (\varphi(v^*) \setminus \{c_i\}) = \emptyset$, because the vertices $u^*$ and $v^*$ are not adjacent in $H$. As the colors $c_1$ and $c_i$ are distinct, $\varphi(u^*) \cap \varphi(v^*) = \emptyset$. Consequently, $\varphi(u^*) \cap \varphi(v^*) = \emptyset$, a contradiction. So, this case is impossible.
Then \(|Y| \geq k - 1\) and there are vertices \(y_1, \ldots, y_{k-1}\) belonging to \(Y\). Set \(A^* = A \cup \{w y_j : 1 \leq j \leq k - 1\}\) and consider a subgraph \(F\) of \(G\) induced by \(A^*\).

Clearly, \(F\) is an \(f\)-subgraph of \(G\) and so \(|A^*| \leq \alpha_f(G)\). Hence

\[
K_f(G) = |\varphi(G)| = |\varphi(w)| + |A| = 1 + (k - 1) + |A| = 1 + |A^*| \leq 1 + \alpha_f(G).
\]

The opposite inequality follows from Lemma 2.

\[\text{Corollary 10. Let } q \text{ be a positive integer. Let } G \text{ be a graph such that } \Delta(G) = |V(G)| - 1 \text{ and } \delta(G) \geq \lceil (|V(G)| + 3q - 3)/2 \rceil.\]

Then

\[
K_q(G) = 1 + \left\lfloor \frac{(q - 1)|V(G)|}{2} \right\rfloor.
\]

**Proof.** The case when \(q = 1\) is evident, so next we consider \(q \geq 2\).

As \(\delta(G) \geq \lceil (|V(G)| + 3q - 3)/2 \rceil \geq (3q - 4)/2 + |V(G)|/2\), there are pairwise edge-disjoint Hamilton cycles \(C_1, C_2, \ldots, C_k\), where \(k = \lceil (q - 1)/2 \rceil\), in \(G\) (because of Dirac’s theorem). Suppose that \(A\) is a subset of \(E(C_1)\) such that it consists of either \(|V(G)|/2\) independent edges, when \(q\) is even, or all edges of \(C_1\), when \(q\) is odd. Set \(A^* = A \cup \bigcup_{j=2}^{k} E(C_j)\). It is easy to see that the subgraph of \(G\) induced by \(A^*\) is a \(q\)-subgraph with the maximum number of edges, i.e., \(\alpha_q(G) = |A^*| = \lfloor (q - 1)|V(G)|/2 \rfloor\). Therefore, according to Theorem 4, we have the assertion.

In [11] there is determined \(K_q(K_n)\) within 1, for \(n \geq q + 2\). Note that, by Corollary 10, \(K_q(K_n) = 1 + \lfloor (q - 1)n/2 \rfloor\), for \(n \geq 3q - 1\), which is an extension of the result from [11].

**References**


Received 3 December 2019
Revised 30 April 2020
Accepted 30 April 2020