TOTAL ROMAN \{2\}-DOMINATING FUNCTIONS IN GRAPHS

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Abstract

A Roman \{2\}-dominating function (R2F) is a function \( f : V \to \{0, 1, 2\} \)
with the property that for every vertex \( v \in V \) with \( f(v) = 0 \) there is a
neighbor \( u \) of \( v \) with \( f(u) = 2 \), or there are two neighbors \( x, y \) of \( v \) with
\( f(x) = f(y) = 1 \). A total Roman \{2\}-dominating function (TR2DF) is an
R2F \( f \) such that the set of vertices with \( f(v) > 0 \) induce a subgraph with no
isolated vertices. The weight of a TR2DF is the sum of its function values
over all vertices, and the minimum weight of a TR2DF of \( G \) is the total
Roman \{2\}-domination number \( \gamma_{tR2}(G) \). In this paper, we initiate the study
of total Roman \{2\}-dominating functions, where properties are established.
Moreover, we present various bounds on the total Roman \{2\}-domination
number. We also show that the decision problem associated with \( \gamma_{tR2}(G) \) is
NP-complete for bipartite and chordal graphs. Moreover, we show that it is
possible to compute this parameter in linear time for bounded clique-width graphs (including trees).

**Keywords:** Roman domination, Roman \( \{2\} \)-domination, total Roman \( \{2\} \)-domination.

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1. Introduction

In this paper, \( G \) is a simple graph with vertex set \( V = V(G) \) and edge set \( E = E(G) \). The order \( |V| \) of \( G \) is denoted by \( n(G) \). For every vertex \( v \in V \), the open neighborhood \( N(v) \) is the set \( \{ u \in V(G) : uv \in E(G) \} \) and the closed neighborhood of \( v \) is the set \( N[v] = N(v) \cup \{ v \} \). The degree of a vertex \( v \in V \) is \( \deg_G(v) = |N(v)| \). The minimum and maximum degree of a graph \( G \) are denoted by \( \delta = \delta(G) \) and \( \Delta = \Delta(G) \), respectively. A leaf of \( G \) is a vertex of degree one, while a support vertex of \( G \) is a vertex adjacent to a leaf. A support vertex is said to be weak (respectively, strong) if it is adjacent to exactly one leaf (respectively, at least two leaves).

Let \( P_n, C_n \) and \( K_n \) be the path, cycle and complete graph of order \( n \) and \( K_{p,q} \) the complete bipartite graph with one partite set of cardinality \( p \) and the other of cardinality \( q \). The complement of a graph \( G \) is denoted by \( \overline{G} \). The join of two graphs \( G \) and \( H \), denoted by \( G \vee H \), is a graph obtained from \( G \) and \( H \) by joining each vertex of \( G \) to all vertices of \( H \). A tree is an acyclic connected graph. A double star is a tree containing exactly two vertices that are not leaves. A double star with respectively \( p \) and \( q \) leaves attached at each support vertex is denoted by \( S_{p,q} \). The corona of a graph \( H \) is the graph obtained from \( H \) by appending a path of degree 1 to each vertex of \( H \). The distance \( d_G(u,v) \) between two vertices \( u \) and \( v \) in a connected graph \( G \) is the length of a shortest \( u-v \) path in \( G \). The diameter of \( G \), denoted by \( \text{diam}(G) \), is the maximum value among distances between all pair of vertices of \( G \).

A subset \( S \subseteq V \) is a dominating set if every vertex in \( V \setminus S \) has a neighbor in \( S \), and \( S \) is a total dominating set, abbreviated TSD, if every vertex in \( V \) has a neighbor in \( S \), that is, \( N(v) \cap S \neq \emptyset \) for all \( v \in V \). The domination number \( \gamma(G) \) of a graph \( G \) is the minimum cardinality of a dominating set of \( G \), and the total domination number \( \gamma_t(G) \) is the minimum cardinality of a TDS of \( G \).

For a graph \( G \) and a positive integer \( k \), let \( f : V(G) \to \{0,1,2, \ldots , k\} \) be a function, and let \( (V_0, V_1, V_2, \ldots , V_k) \) be the ordered partition of \( V = V(G) \) induced by \( f \), where \( V_i = \{ v \in V : f(v) = i \} \) for \( i \in \{0,1,2, \ldots , k\} \). There is a 1-1 correspondence between the functions \( f : V \to \{0,1,2, \ldots , k\} \) and the ordered partitions \( (V_0, V_1, V_2, \ldots , V_k) \) of \( V \), so we will write \( f = (V_0, V_1, V_2, \ldots , V_k) \). The weight of \( f \) is the value \( f(V(G)) = \sum_{u \in V(G)} f(u) \).
A function \( f : V(G) \rightarrow \{0, 1, 2\} \) is a Roman dominating function, abbreviated RDF, on \( G \) if every vertex \( u \in V \) for which \( f(u) = 0 \) is adjacent to at least one vertex \( v \) for which \( f(v) = 2 \). The Roman domination number \( \gamma_{R}(G) \) is the minimum weight of a RDF on \( G \). Roman domination was introduced by Cockayne et al. in [4] and was inspired by the work of ReVelle and Rosing [10], and Stewart [11].

The definition of Roman dominating functions was motivated by an article in Scientific American by Stewart entitled “Defend the Roman Empire” [11] and suggested even earlier by ReVelle [9]. Each vertex in our graph represents a location in the Roman Empire. A location (vertex \( v \)) is considered unsecured if no legions are stationed there (i.e., \( f(v) = 0 \)) and secured otherwise (i.e., if \( f(v) \in \{1, 2\} \)). An unsecured location (vertex \( v \)) can be secured by sending a legion to \( v \) from a neighboring location (an adjacent vertex \( u \)). But Constantine the Great (Emperor of Rome) issued a decree in the 4th century A.D. for the defense of the regions, where a legion cannot be sent from a secured location having only one legion stationed there to an unsecured location, for otherwise it leaves that location unsecured. Thus, two legions must be stationed at a location (\( f(v) = 2 \)) before one of the legions can be sent to a neighboring location. In this way, Emperor Constantine the Great can defend the Roman Empire. Since it is expensive to maintain a legion at a location, the Emperor would like to station as few legions as possible, while still defending the Roman Empire. A Roman dominating function of weight \( \gamma_{R}(G) \) corresponds to such an optimal assignment of legions to locations.

In [2], Chellali et al. introduced the Roman \( \{2\}\)-domination (called in [7] and elsewhere Italian domination) defined as follows: a Roman \( \{2\}\)-dominating function is a function \( f = (V_{0}, V_{1}, V_{2}) \) with the property that for every vertex \( v \in V_{0} \) there is a vertex \( u \in N(v) \), with \( u \in V_{2} \) or there are two vertices \( x, y \in N(v) \) with \( x, y \in V_{1} \). The Roman \( \{2\}\)-domination number \( \gamma_{\{R2\}}(G) \) is the minimum weight of a Roman \( \{2\}\)-dominating function on \( G \).

There are many papers in the literature devoted to the study of Roman domination type problems and its variations. One of the questions that arise naturally when a Roman domination type problem is studied is to focus on its complexity and algorithmic aspects. In 2008, Liedloff et al. [8] showed, among other results, that it is possible to compute the Roman domination number of a graph with bounded cliquewidth in linear time. Clearly, this implies that there exists algorithms for computing the Roman domination number of trees in linear time. Chellali et al. [2] proved that Roman \( \{2\}\)-domination problem is NP-complete for bipartite graphs while Chen and Lu [3] recently showed it is NP-complete even when restricted to split graphs. Moreover, the authors [3] presented a linear time algorithm to obtain the Roman \( \{2\}\)-domination number of a block graph.
In this paper, we initiate the study of the total version of Roman \(2\)-dominating function. A total Roman \(2\)-dominating function, abbreviated TR2DF, is a Roman \(2\)-dominating function \(f = (V_0, V_1, V_2)\) such that the subgraph induced by \(V_1 \cup V_2\) has no isolated vertices. The total Roman \(2\)-domination number \(\gamma_{tR2}(G)\) is the minimum weight of a TR2DF on \(G\). A TR2DF on \(G\) with weight \(\gamma_{tR2}(G)\) is called a \(\gamma_{tR2}(G)\)-function. Total Roman \(2\)-domination number is well-defined for all graphs \(G\) with no isolated vertices since assigning a 1 to every vertex of \(G\) provides a TR2DF of \(G\). Hence for all graphs \(G\) of order \(n\) with \(\delta(G) \geq 1\), \(2 \leq \gamma_{tR2}(G) \leq n\). We present various bounds on the total Roman \(2\)-domination number and several properties are established. We show that the decision problem associated with \(\gamma_{tR2}(G)\) is NP-complete for bipartite and chordal graphs. Moreover, we show that it is possible to compute this parameter in linear time for bounded clique-width graphs (including trees).

We note that throughout this paper, we only consider nontrivial connected graphs that we will call ntc graphs.

2. Preliminary Results

We begin by giving some properties of total Roman \(2\)-dominating functions. The following two facts lead to our first observation. Clearly assigning a 2 to every vertex in a minimum total dominating set of a ntc graph \(G\) and a 0 to the remaining vertices of \(G\) provides a TR2DF. Also, for every TR2DF \(f = (V_0, V_1, V_2)\) the set \(V_1 \cup V_2\) total dominates \(V(G)\).

**Observation 1.** For every ntc graph \(G\),

\[
\gamma_t(G) \leq \gamma_{tR2}(G) \leq 2\gamma_t(G).
\]

Note that the lower bound of Observation 1 is attained for \(K_2 \vee K_{n-2}\) while the upper bound is attained for the double star \(S_{3,3}\).

It is well-known that \(\gamma_t(G) \leq 2\gamma_t(G)\) for every ntc graph \(G\). So by Observation 1, we will have \(\gamma_{tR2}(G) \leq 4\gamma_t(G)\). Our next result improves this upper bound to \(\gamma_{tR2}(G) \leq 3\gamma_t(G)\). We need the following result due to Bollobás and Cockayne [1]. If \(S\) is a set of vertices, then we say that a vertex \(v\) is a private neighbor of a vertex \(u \in S\) (with respect to \(S\)) if \(N[v] \cap S = \{u\}\). The external private neighborhood \(epn(u, S)\) of \(u\) with respect to \(S\) consists of those private neighbors of \(u\) in \(V \setminus S\). For a TR2DF \(f = (V_0, V_1, V_2)\) of an ntc graph, let \(V_{02} = \{w \in V_0 : N(w) \cap V_2 \neq \emptyset\}\) and \(V_{01} = V_0 \setminus V_{02}\).

**Theorem 2** (Bollobás and Cockayne [1]). If \(G\) is a graph without isolated vertices, then \(G\) has a minimum dominating set \(D\) such that for all \(d \in D\), there exists a neighbor \(f(d) \in V \setminus D\) of \(d\) such that \(f(d)\) is not a neighbor of any vertex \(x \in D \setminus \{d\}\).
Proposition 3. For every ntc graph $G$, $\gamma_{tR2}(G) \leq 3\gamma(G)$.

Proof. Let $D$ be a minimum dominating set of $G$ satisfying the property of Theorem 2. Since each vertex of $D$ has an external private neighbor in $V \setminus D$, let $W$ be a subset of $V \setminus D$ formed by the private neighbors chosen so that each vertex of $D$ has exactly one external private neighbor in $D$. Clearly $|W| = |D|$. Now define the function $f$ as follows: $f(x) = 2$ for all $x \in D$, $f(x) = 1$ for all $x \in W$, and $f(x) = 0$ otherwise. Clearly $f$ is a TR2DF of $G$ of weight $2|D| + |W| = 3\gamma(G)$, and thus $\gamma_{tR2}(G) \leq 3\gamma(G)$.

For the sharpness of the bound in Proposition 3, consider the tree $T$ obtained from two stars $K_{1,4}$ by adding an edge between a leaf of one star to a leaf of the other star. The next observation is straightforward and is tight for double stars.

Observation 4. For every ntc graph $G$, $\gamma_{R2}(G) \leq \gamma_{tR2}(G)$.

Proposition 5. Let $G$ be an ntc graph. Then for every $\gamma_{tR2}(G)$-function $f = (V_0, V_1, V_2)$ such that $V_2 = \emptyset$, we have the following.

(i) Each vertex in $V_2$ has at least two private neighbors in $V_0$ with respect to $V_2$.
(ii) $2|V_2| \leq |V_0|$.

Proof. (i) Suppose there exists a vertex $v \in V_2$ with at most one private neighbor in $V_0$ with respect to $V_2$. Then reassigning $v$ and its private neighbor (if any) the value 1 instead of 2 and 0, respectively provides a $\gamma_{tR2}(G)$-function with less vertices assigned a 2, which contradicts the choice of $f$.

(ii) Follows from (i).

Proposition 6. Let $G$ be an ntc graph with maximum degree $\Delta \leq 2$. Then there exists a $\gamma_{tR2}(G)$-function $f = (V_0, V_1, V_2)$ such that $V_2 = \emptyset$.

Proof. Among all $\gamma_{tR2}(G)$-functions, let $f = (V_0, V_1, V_2)$ be a one such that $|V_2|$ is as small as possible. If $V_2 \neq \emptyset$, then by Proposition 5, every vertex of $V_2$ has at least two private neighbors in $V_0$ with respect to $V_2$. But then since $\Delta \leq 2$, each vertex in $V_2$ would be isolated in $G[V_1 \cup V_2]$, a contradiction. Hence $V_2 = \emptyset$.

Recall that a subset $S$ of $V$ is a double dominating set of $G$ if for every vertex $v \in V$, $|N[v] \cap S| \geq 2$, that is, $v$ is in $S$ and has at least one neighbor in $S$ or $v$ is in $V \setminus S$ and has at least two neighbors in $S$. The double domination number $\gamma_{x2}(G)$ is the minimum cardinality of a double dominating set of $G$. Double domination was introduced by Harary and Haynes [6] who proved that $\gamma_{x2}(P_n) = \left\lceil \frac{2n+2}{3} \right\rceil$ and $\gamma_{x2}(C_n) = \left\lceil \frac{2n}{3} \right\rceil$. The following result shows that the equality between $\gamma_{tR2}(G)$ and $\gamma_{x2}(G)$ occurs under certain conditions.

Proposition 7. Let $G$ be an ntc graph. If $G$ has a $\gamma_{tR2}(G)$-function $f = (V_0, V_1, V_2)$ such that $V_2 = \emptyset$, then $\gamma_{tR2}(G) = \gamma_{x2}(G)$. 
Proof. If $S$ is a double dominating set of $G$, then $(V \setminus S, S, \emptyset)$ is clearly a TR2DF on $G$, and thus $\gamma_{tR2}(G) \leq \gamma_{\times2}(G)$. Now if $f = (V_0, V_1, V_2)$ is a $\gamma_{tR2}(G)$-function such that $V_2 = \emptyset$, then $V_1$ double dominates $V(G)$, and thus $\gamma_{\times2}(G) \leq \gamma_{tR2}(G)$. Therefore $\gamma_{tR2}(G) = \gamma_{\times2}(G)$.

The following results are immediate consequences of Propositions 6, 7 and the exact values of the double domination number of paths and cycles given above.

**Proposition 8.** For $n \geq 2$, $\gamma_{tR2}(P_n) = \lceil \frac{2n+2}{3} \rceil$.

**Proposition 9.** For $n \geq 3$, $\gamma_{tR2}(C_n) = \lceil \frac{2n}{3} \rceil$.

3. Complexity

Our aim in this section is to study the complexity of the following decision problem, to which we shall refer as TOTAL ROMAN $\{2\}$-DOMINATION.

TOTAL ROMAN $\{2\}$-DOMINATION

**Instance:** Graph $G = (V, E)$, positive integer $k \leq |V|$.

**Question:** Does $G$ have a total Roman $\{2\}$-dominating function of weight at most $k$?

We show that this problem is NP-complete by reducing the well-known NP-complete problem, Exact-3-Cover (X3C), to TOTAL ROMAN $\{2\}$-DOMINATION.

EXACT 3-COVER (X3C)

**Instance:** A finite set $X$ with $|X| = 3q$ and a collection $C$ of 3-element subsets of $X$.

**Question:** Is there a subcollection $C'$ of $C$ such that every element of $X$ appears in exactly one element of $C'$?

**Theorem 10.** TOTAL ROMAN $\{2\}$-DOMINATION is NP-Complete for bipartite graphs.

**Proof.** TOTAL ROMAN $\{2\}$-DOMINATION is a member of NP, since we can check in polynomial time that a function $f : V \to \{0, 1, 2\}$ has weight at most $k$ and is a total Roman $\{2\}$-dominating function. Now let us show how to transform any instance of X3C into an instance $G$ of TOTAL ROMAN $\{2\}$-DOMINATION so that one of them has a solution if and only if the other one has a solution. Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$ be an arbitrary instance of X3C.

For each $x_i \in X$, we build a graph $H_i$ obtained from a path $P_2 : x_i-y_i$ and two stars $K_{1,3}$ centered at $a_i$ and $b_i$, by adding edges $y_ia_i$ and $y_ib_i$. Hence, each $H_i$
has order 10. For each $C_j \in C$, we build a double star $S_{3,3}$ with support vertices $u_j$ and $v_j$. Let $c_j$ be a leaf of the double star $S_{3,3}$. Let $Y = \{c_1, c_2, \ldots, c_t\}$. Now to obtain a graph $G$, we add edges $c_jx_i$ if $x_i \in C_j$. Clearly $G$ is a bipartite graph.

Set $k = 4t + 16q$. Observe that for every total Roman $\{2\}$-dominating function $f$ on $G$, each $H_i$ has weight at least 5 and each double star $S_{3,3}$ has weight at least 4.

Suppose that the instance $X, C$ of $X3C$ has a solution $C'$. We construct a total Roman $\{2\}$-dominating function $f$ on $G$ of weight $k$. For each $i$, assign the value 2 to $a_i, b_i$; the value 1 to $y_i$ and 0 to the remaining vertices of $H_i$. In addition, for every $c_j$, assign the value 1 if $C_j \in C'$ and the value 0 if $C_j \notin C'$. Note that since $C'$ exists, its cardinality is precisely $q$, and so the number of $c_j$’s with value 1 is $q$, having disjoint neighborhoods in $\{x_1, x_2, \ldots, x_{3q}\}$. Since $C'$ is a solution for $X3C$, every vertex in $X$ is adjacent to two vertices assigned a 1. Hence, it is straightforward to see that $f$ is a total Roman $\{2\}$-dominating function with weight $f(V) = 4t + q + 15q = k$.

Conversely, suppose that $G$ has a total Roman $\{2\}$-dominating function with weight at most $k$. Among all such functions, let $g = (V_0, V_1, V_2)$ be one such that $\{y_1, y_2, \ldots, y_{3q}\} \cap V_2$ is as small as possible. As observed above, since each $H_i$ has weight at least 5, we may assume that $g(a_i) = g(b_i) = 2$ and $g(y_i) > 0$ so that vertices $a_i, b_i$ are not isolated in the subgraph induced by $V_1 \cup V_2$. Hence each leaf neighbor of $a_i$ or $b_i$ is assigned a 0 under $g$. If $g(y_i) = 2$ for some $i$, then we must have $g(x_i) = 0$. In this case, reassigning a 1 to each of $y_i$ and $x_i$ instead of 2 and 0, respectively, provides a total Roman $\{2\}$-dominating function $g'$ with less vertices $y_i$ assigned a 2 than under $g$, contradicting our choice of $g$. Hence $g(y_i) = 1$ for every $i \in \{1, 2, \ldots, 3q\}$. On the other hand, the total weight
of all double stars corresponding to elements of $C$ is at least $4t$. In this case, we can assume that $g(u_j) = g(v_j) = 2$ and so each leaf neighbor of $u_j$ or $b_j$ is assigned a 0 under $g$. Note that each $c_j$ can be assigned a 0 since $g(u_j) = 2$.

Since $w(g) \leq 4t + 16q$ and the total weight assigned to vertices of $V(G) \setminus (X \cup Y)$ is $4t + 15q$, we have to assign to vertices of $(X \cup Y)$ weights whose total not exceeding $q$ in order that each vertex $x_i \in X$ has either $g(x_i) > 0$ or has two neighbors in $V_1$. Since $|X| = 3q$, it is clear that this is only possible if there are $q$ vertices of $\{c_1, c_2, \ldots, c_t\}$ that are assigned a 1. Since each $c_j$ has exactly three neighbors in $\{x_1, x_2, \ldots, x_{3q}\}$, we deduce that $C' = \{C_j : g(c_j) = 1\}$ is an exact cover for $C$.

The next result is obtained by using the same proof of Theorem 10 on the (same) graph $G$ built for the transformation by adding all edges between the vertices labelled $c_j$ in order that the resulting graph is chordal.

**Theorem 11.** TOTAL ROMAN $\{2\}$-DOMINATION is NP-Complete for chordal graphs.

In the rest of this section, we prove that the decision problem associated to $\gamma_{R_2}(G)$ can be solved in linear time for the class of graphs with bounded clique-width, which implies that it also can be computed in linear time for trees.

We make use of several useful objects and results, which are formally described in [5, 8], related to logic structures. Namely, in what follows, a $k$-expression on the vertices of a graph $G$ with labels $\{1, 2, \ldots, k\}$ is an expression using the following operations.

- $i(x)$ create a new vertex $x$ with label $i$,
- $G_1 \oplus G_2$ create a new graph which is the disjoint union of the graphs $G_i$,
- $\eta_{ij}(G)$ add all edges in $G$ joining vertices with label $i$ with vertices with label $j$,
- $\rho_{i \rightarrow j}(G)$ change the label of all vertices with label $i$ into label $j$.

We call the **clique-width** of a graph $G$ the minimum positive integer $k$ that is needed to describe $G$ by means of a $k$-expression. For example, the complete graph $K_3$ with set of vertices $\{a, b, c\}$ can be described by the following 2-expression.

$$\rho_{2 \rightarrow 1} (\eta_{12} (\rho_{2 \rightarrow 1} (\eta_{12} (\bullet 1(a) \oplus \bullet 2(b)) \oplus \bullet 2(c)))).$$

In what follows, MSOL$(\tau_1)$ stands for the monadic second order logic with quantification over subsets of vertices. We denote by $G(\tau_1)$ the logic structure $< V(G), R >$ where $R$ is a binary relation such that $R(u, v)$ holds if and only if $uv$ is an edge in $G$. An optimization problem is a LinEMSOL$(\tau)$ optimization problem if it can be described in the following way (see [8] for more details, since this is a version of the definition given by [5] restricted to finite simple graphs),
Let us show that the total Roman \{k\} optimization problem can be expressed as a LinEMSOL(\tau_1) optimization problem. Let \( f = (V_0, V_1, V_2) \) be a total Roman \{2\}-domination function in \( G = (V, E) \) and let us define the free set-variables \( X_i \) such that \( X_i(v) = 1 \) whenever \( v \in V_i \) and \( X_i(v) = 0 \), otherwise. For the sake of congruence with the logical system notation, we denote by \( |X_i| = \sum_{v \in V} X_i(v) \), even when, clearly, is \( |X_i| = |V_i| \). Observe that the total Roman \{2\}-domination decision problem corresponds to achieve the optimum for the following expression.

\[
\min_{X_i} \{ |X_1| + 2|X_2| : < G(\tau_1), X_0, X_1, X_2 > \models \theta(X_0, X_1, X_2) \},
\]

where \( \theta \) is the formula given by

\[
\theta(X_0, X_1, X_2) = (\forall v ((X_1(v) \lor X_2(v)) \rightarrow \exists u ((X_1(u) \lor X_2(u)) \land R(u, v)))) \\
\land (\forall v (X_1(v) \lor X_2(v)) \lor \exists u (R(u, v) \land X_2(u)) \\
\lor \exists u, w (R(u, v) \land R(w, v) \land X_1(u) \land X_1(w))).
\]

Clearly, \( \theta \) is an MSOL(\tau_1) formula that describes the total Roman \{2\}-domination problem. Namely, the formula has two main clauses. The first one requires that every vertex \( v \) with a positive label 1 or 2 must have a neighbor \( u \) with a positive label. The latter implies that the induced subgraph by the set of vertices \( V_1 \cup V_2 \) has no isolated vertices. The second clause of the formula assures that for any vertex \( v \) of the graph either the vertex itself has a positive label, or either it has a neighbor with a label 2, or either it has two different neighbors having label 1 each. Hence, when the formula \( \theta \) is satisfied, the requirements of a total Roman \{2\}-domination assignment in \( G \) holds.
As a consequence, we may derive the following corollary.

**Corollary 14.** The decision problem associated to the total Roman \{2\}-domination problem can be solved in linear time on any graph \(G\) with clique-width bounded by a constant \(k\), provided that either there exists a linear-time algorithm to construct a \(k\)-expression of \(G\), or a \(k\)-expression of \(G\) is part of the input.

Since any graph with bounded treewidth is also a bounded clique-width graph, and it is well-known that any tree graph has treewidth equal to 1, then we can deduce that the total Roman \{2\}-domination problem can be solved in linear time for the class of trees. Besides, there are several classes of graphs \(G\) with bounded clique-width \(cw(G)\) like, for example, the cographs \((cw(G) \leq 2)\) and the distance hereditary graphs \((cw(G) \leq 3)\), for which it is also possible to solve the total Roman \{2\}-domination problem in linear time.

4. **Graphs with Small or Large Total Roman \{2\}-domination**

As mentioned in Section 1, for all ntc graphs \(G\) of order \(n\), \(2 \leq \gamma_{tR2}(G) \leq n\). In this section, we characterize all ntc graphs \(G\) such that \(\gamma_{tR2}(G) \in \{2, 3, n\}\).

**Proposition 15.** Let \(G\) be an ntc graph. For any graph \(H\), we have \(\gamma_{tR2}(K_2 \lor H) = 2\). Conversely, if \(\gamma_{tR2}(G) = 2\), there is some graph \(H\) such that \(G = K_2 \lor H\).

**Proof.** If \(G = K_2 \lor H\), then clearly \(\gamma_{tR2}(G) = 2\). Conversely, assume that \(\gamma_{tR2}(G) = 2\) and let \(f = (V_0, V_1, V_2)\) be a \(\gamma_{tR2}(G)\)-function. By definition of TR2DF of \(G\), we have \(\gamma_{tR2}(G) = |V_1| + 2|V_2| = 2\). Since \(G[V_1 \cup V_2]\) has no isolated vertex, we deduce that \(|V_2| = 0\) and \(|V_1| = 2\). Now let \(V_1 = \{x, y\}\). Clearly, \(xy \in E(G)\), because \(G[V_1 \cup V_2]\) has no isolated vertex, and every vertex in \(V(G) \setminus \{x, y\}\) is adjacent to both \(x\) and \(y\). Thus \(G \cong K_2 \lor H\), and \(H\) is any graph of order \(n - 2\).

**Proposition 16.** Let \(G\) be an ntc graph of order \(n \geq 5\). Then \(\gamma_{tR2}(G) = 3\) if and only if either \(G\) has exactly one vertex of degree \(n - 1\) or \(\Delta(G) \leq n - 2\) and \(G\) is obtained from two disjoint graphs \(G_1\) and \(G_2\) such that \(G_1 \in \{P_3, C_3\}\) and \(G_2\) is any graph of order \(n - 3\) by adding edges between vertices of \(G_1\) and \(G_2\) in order that every vertex of \(G_2\) has at least two neighbors in \(G_1\).

**Proof.** If \(\Delta(G) = n - 1\) and \(G\) has exactly one vertex \(u\) of degree \(n - 1\), then the function \(f\) defined on \(V(G)\) by \(f(u) = 2, f(v) = 1\) for some \(v \in V(G) \setminus \{u\}\) and \(f(w) = 0\) for all \(w \in V \setminus \{u, v\}\) is a TR2DF and so \(\gamma_{tR2}(G) \leq 3\). Since \(G\) has exactly one vertex of degree \(n - 1\), we deduce from Proposition 15 that \(\gamma_{tR2}(G) \geq 3\) and the equality follows.
Now assume that $\Delta(G) \leq n - 2$ and $G$ is obtained from two disjoint graphs $G_1$ and $G_2$ such that $G_1 \in \{P_3, C_3\}$ and $G_2$ is any graph of order $n - 3$ by adding edges between vertices of $G_1$ and $G_2$ in order that every vertex of $G_2$ has at least two neighbors in $G_1$. Then the function $f$ defined on $V(G)$ by $f(u) = 1$ for every vertex $u \in V(G_1)$ and $f(v) = 0$ for all $v \in V(G_2)$ is a TR2DF of $G$. Hence $\gamma_{tR2}(G) \leq 3$, and the equality follows as above from Proposition 15.

Conversely, assume that $\gamma_{tR2}(G) = 3$. Then $G$ has at most one vertex of degree $n - 1$, for otherwise $\gamma_{tR2}(G) = 2$ (by Proposition 15). Let $f = (V_0, V_1, V_2)$ be a $\gamma_{tR2}(G)$-function. Since $\gamma_{tR2}(G) = |V_1| + 2|V_2| = 3$, then it must be either $|V_1| = |V_2| = 1$ or either $|V_1| = 3$ and $|V_2| = 0$. If $|V_1| = |V_2| = 1$, with $V_2 = \{u\}$ and $V_1 = \{v\}$, then $uw \in E(G)$ and every vertex in $V \{x, y\}$ must be adjacent to $u$, because $f$ is a TR2DF. So $u$ is the unique vertex of degree $n - 1$. Now assume that $V_2 = \emptyset$ and $|V_1| = 3$, where $V_1 = \{u, v, w\}$. Since $G[V_1]$ has no isolated vertices, $G[V_1] \in \{P_3, C_3\}$. Moreover, every vertex in $V_0 = V \{u, v, w\}$ must be adjacent to at least two vertices of $V_1$. Clearly, if $G_1 = G[V_1]$ and $G_2 = G[V_0]$, then $G$ is an ntc graph as described in the statement.

**Theorem 17.** Let $G$ be an ntc graph of order $n$. Then $\gamma_{tR2}(G) = n$ if and only if $G \in \{K_2, K_{1,2}\}$ or every vertex of $G$ is either a leaf or a weak support vertex.

**Proof.** Assume that $\gamma_{tR2}(G) = n$. Clearly, if $n \in \{2, 3\}$, then $G \in \{K_2, K_{1,2}\}$. Hence assume that $n \geq 4$. Suppose first that $G$ has a vertex $w$ which is neither a leaf nor a support vertex. Define the function $f$ by $f(w) = 0$ and $f(x) = 1$ for all $x \in V(G) \{w\}$. Clearly $f$ is a TR2DF on $G$ with weight $n - 1$, a contradiction. Thus, each vertex of $G$ is either a leaf or a support vertex. Now suppose that $G$ has a strong support vertex, say $u$. Let $u_1$ and $u_2$ be two leaves adjacent to $u$. Define the function $f$ by $f(u_1) = 2$, $f(u_2) = 0$ and $f(x) = 1$ otherwise. Since $n \geq 4$, $f$ is clearly a TR2DF on $G$ of weight $n - 1$, a contradiction too. Therefore, every vertex of $G$ is either a leaf or a weak support vertex as desired.

The converse is obvious.

5. Bounds

We present in this section some bounds on the total Roman $\{2\}$-domination number of ntc graphs in terms of the order, maximum and minimum degrees.

**Proposition 18.** Let $G$ be an ntc graph of order $n$ with girth $g \geq 6$ and minimum degree $\delta \geq 2$. Then $\gamma_{tR2}(G) \leq n + 2 - (\Delta + \delta)$.

**Proof.** Let $u$ be a vertex of $G$ of maximum degree and let $v$ be any neighbour of $u$. Define the function $f$ on $V(G)$ by $f(u) = 1$, $f(v) = 1$, $f(w) = 0$ for all $w \in N(\{u, v\}) \{u, v\}$ and $f(w) = 1$ otherwise. Since since $\delta \geq 2$ and $g \geq 6$,
set $A = V(G) \setminus N[\{u, v\}]$ is non-empty and no vertex of $A$ has two neighbors $N[\{u, v\}]$. Hence $f$ is well defined and is a TR2DF of weight $2 + n - (\Delta + \text{deg}_G(v))$, and thus

$$\gamma_{\text{TR2}}(G) \leq 2 + n - (\Delta + \text{deg}_G(v)) \leq n + 2 - (\Delta + \delta).$$

The sharpness of the previous bound can be seen by considering the cycles $C_6$ and $C_7$. Moreover, to see that the condition $\delta \geq 2$ is essential in the statement of Proposition 18, consider the star $K_{1,n-1}$ with $n \geq 3$, where $\gamma_{\text{TR2}}(K_{1,n-1}) = 3 > n + 2 - (\Delta + \delta) = 2$.

**Corollary 19.** Let $G$ be an ntc graph of order $n$ with girth $g \geq 6$ and minimum degree $\delta \geq 2$ such that $\gamma_{\text{TR2}}(G) = n + 2 - (\Delta + \delta)$. Then for every vertex $u$ of maximum degree we have that $d(v) = \delta(G)$ for all $v \in N(u)$.

**Proposition 20.** Let $G$ be an ntc graph. Then

$$\gamma_{\text{TR2}}(G) \geq \left\lfloor \frac{2n}{\Delta + 1} \right\rfloor.$$

If $\gamma_{\text{TR2}}(G) = \frac{2n}{\Delta + 1}$, then $V_2 = \emptyset$ for all $\gamma_{\text{TR2}}(G)$-function $f = (V_0, V_1, V_2)$.

**Proof.** Let $f = (V_0, V_1, V_2)$ be a $\gamma_{\text{TR2}}(G)$-function and let us denote by $V_{02} = \{w \in V_0 : N(w) \cap V_2 \neq \emptyset\}$ and by $V_{01} = V_0 \setminus V_{02}$. Thus $V(G) = V_{01} \cup V_{02} \cup V_1 \cup V_2$. Since any vertex $v \in V_2$ must have at least one neighbor in $V_1 \cup V_2$, we deduce that for each $v \in V_2$, $|N(v) \cap V_{02}| \leq \Delta - 1$ and thus $|V_{02}| \leq (\Delta - 1)|V_2|$. Analogously, $2|V_{01}| \leq (\Delta - 1)|V_1|$, because each vertex in $V_{01}$ must have at least two neighbors in $V_1$. Hence

$$n = |V_0| + |V_{02}| + |V_1| + |V_2| \leq \frac{\Delta - 1}{2}|V_1| + (\Delta - 1)|V_2| + |V_1| + |V_2|$$

$$= \frac{\Delta + 1}{2}|V_1| + \Delta|V_2| \leq \frac{\Delta + 1}{2}(|V_1| + 2|V_2|) = \frac{\Delta + 1}{2}\gamma_{\text{TR2}}(G),$$

which leads to the desired result. If $\gamma_{\text{TR2}}(G) = \frac{2n}{\Delta + 1}$, then all the previous inequalities become equalities and hence $|V_2| = 0$.

The sharpness of the bound in Proposition 20 can be shown for cycles.

6. **Total Roman $\{2\}$-Domination of Trees**

In this section, we present a lower and upper bounds on the total Roman $\{2\}$-domination number of trees. We start with a simple observation.

**Observation 21.** Let $G$ be a graph without isolated vertices and $v \in V(G)$ a support vertex of $G$. 
For any total Roman \(\{2\}\)-dominating function \(f\) of \(G\), \(f(v) \geq 1\).

If \(v\) is a strong support vertex, then there exists a \(\gamma_{tR2}(G)\) function \(f\) such that \(f(v) = 2\).

**Theorem 22.** Let \(T\) be a tree of order \(n \geq 2\) with \(\ell(T)\) leaves. Then

\[
\gamma_{tR2}(T) \geq \left\lceil \frac{2(n - \ell(T) + 3)}{3} \right\rceil.
\]

This bound is sharp for paths, stars and double stars.

**Proof.** The proof is by induction on \(n\). Clearly for all nontrivial trees of order \(n \leq 4\) we have \(\gamma_{tR2}(T) > \left\lceil \frac{2(n - \ell(T) + 3)}{3} \right\rceil\). For the inductive hypothesis, let \(n \geq 5\) and assume that for every tree of order at least 2 and less than \(n\) the result is true. Let \(T\) be a tree of order \(n\). If \(\text{diam}(T) = 2\), then \(T\) is a star, which yields \(\gamma_{tR2}(T) = 3 = \left\lceil \frac{2(n - n + 1 + 3)}{3} \right\rceil\). If \(\text{diam}(T) = 3\), then \(T\) is a double star and we have \(\gamma_{tR2}(T) = 4 = \left\lceil \frac{2(n - n + 2 + 3)}{3} \right\rceil\). Henceforth we can assume \(\text{diam}(T) \geq 4\). Let \(f\) be a \(\gamma_{tR2}(T)\) function.

If \(T\) has a strong support vertex \(u\) with at least two leaves, say \(u_1\) and \(u_2\), then let \(T' = T - u_1\). By Observation 21, \(f(u) \geq 1\) and we may assume without loss of generality that \(f(u_2) \geq f(u_1)\). Now the function \(f\), restricted to \(T'\) is a TR2DF of \(T'\) and we deduce from the inductive hypothesis that

\[
\gamma_{tR2}(T) = \omega(f) \geq \gamma_{tR2}(T') \geq \left\lceil \frac{2((n - 1) - (\ell(T) - 1) + 3)}{3} \right\rceil = \left\lceil \frac{2(n - \ell(T) + 3)}{3} \right\rceil.
\]

Thus in the sequel, we can assume that \(T\) has no strong support vertex. Let \(v_1v_2\cdots v_k\) be a diametral path in \(T\) and root \(T\) in \(v_k\). Since \(T\) has no strong support vertex, any child of \(v_3\) is a leaf or a support vertex of degree 2. We consider the following cases.

Case 1. \(\deg_T(v_3) \geq 3\). First suppose \(v_3\) is a support vertex. By Observation 21, we may assume \(f(v_2) = f(v_3) = 2\). Let \(T' = T - v_1\) and define \(h : V(T') \to \{0, 1, 2\}\) by \(h(v_2) = 1\) and \(h(x) = f(x)\) for \(x \in V(T')\backslash\{v_2\}\). Clearly \(h\) is a TR2DF of \(T'\). It follows from the induction hypothesis that

\[
\gamma_{tR2}(T) = \omega(f) = \omega(h) + 1 \geq \gamma_{tR2}(T') + 1 \geq \left\lceil \frac{2((n - 1) - (\ell(T') + 3))}{3} \right\rceil + 1 > \left\lceil \frac{2(n - \ell(T) + 3)}{3} \right\rceil,
\]
as desired. Now suppose \(v_3\) is not a support vertex. Assume \(u_2\) is a child of \(v_3\) and \(u_1\) is a leaf adjacent to \(u_2\). Clearly \(f(u_1) + f(u_2) \geq 2\) and \(f(v_1) + f(v_2) \geq 2\). Assume without loss of generality that \(f(v_2) \geq f(u_2)\). Let \(T' = T - \{u_1, u_2\}\). If
\( f(v_3) \geq 1 \) or \( \deg(v_3) \geq 4 \), then clearly the function \( f \) restricted to \( T' \) is a \( \text{TR}_2 \text{RDF} \) of \( T \) and we conclude from the inductive hypothesis that

\[
\gamma_{\text{TR}_2}(T) = \omega(f) = \omega(f_{|T'}) + 2 \geq \gamma_{\text{TR}_2}(T') + 2 \\
\geq \left\lceil \frac{2(n-2-(\ell(T)-1)+3)}{3} \right\rceil + 2 > \left\lceil \frac{2(n-\ell(T)+3)}{3} \right\rceil,
\]

as desired. Hence assume that \( f(v_3) = 0 \) and \( \deg(v_3) = 3 \). Let \( T' = T - \{ u_1, u_2, v_3 \} \). Then the function \( g : V(T') \to \{ 0, 1, 2 \} \) defined by \( g(v_3) = 1 \) and \( g(x) = f(x) \) for \( x \in V(T') \setminus \{ v_3 \} \), is a \( \text{TR}_2 \text{RDF} \) of \( T' \) of weight \( \gamma_{\text{TR}_2}(T') - 2 \). By the inductive hypothesis we have

\[
\gamma_{\text{TR}_2}(T) = \omega(f) = \omega(g) + 2 \geq \gamma_{\text{TR}_2}(T') + 2 \\
\geq \left\lceil \frac{2(n-3-(\ell(T)-3)+3)}{3} \right\rceil + 2 > \left\lceil \frac{2(n-\ell(T)+3)}{3} \right\rceil.
\]

**Case 2.** \( \deg_T(v_3) = 2 \). As above we have \( f(v_1) + f(v_2) \geq 2 \). If \( f(v_3) \geq 1 \), then the function \( g : V(T-v_1) \to \{ 0, 1, 2 \} \) defined by \( g(v_2) = 1 \) and \( g(x) = f(x) \) for \( x \in V(T') \setminus \{ v_3 \} \), is a \( \text{TR}_2 \text{RDF} \) of \( T-v_1 \) of weight \( \gamma_{\text{TR}_2}(T) - 1 \) and by the inductive hypothesis we obtain

\[
\gamma_{\text{TR}_2}(T) = \omega(f) = \omega(g) + 1 \geq \gamma_{\text{TR}_2}(T-v_1) + 1 \\
\geq \left\lceil \frac{2(n-1-(\ell(T)+3)}{3} \right\rceil + 1 > \left\lceil \frac{2(n-\ell(T)+3)}{3} \right\rceil.
\]

Hence let \( f(v_3) = 0 \). If \( f(v_1) + f(v_2) \geq 3 \), then reassigning \( v_1, v_2, v_3 \) the value 1 provides a \( \gamma_{\text{TR}_2}(T) \)-function \( f' \) for which \( f'(v_3) \geq 1 \), and this situation was considered above. Therefore, we can assume that \( f(v_1) + f(v_2) = 2 \). More precisely, \( f(v_1) = f(v_2) = 1 \). It follows that \( f(v_4) \geq 1 \). Let \( T' = T - \{ v_1, v_2, v_3 \} \). Clearly \( T' \) is nontrivial since \( \text{diam}(T) \geq 4 \). Now if \( T' \) has order 2, then \( T \) is a path \( P_3 \) and \( \gamma_{\text{TR}_2}(P_3) = 4 \geq \left\lceil \frac{2(n-\ell(T)+3)}{3} \right\rceil \). Hence suppose that \( T' \) has order at least three. Note that \( \ell(T)-1 \leq \ell(T') \leq \ell(T) \). Also, the function \( f \) restricted to \( T' \) is a \( \text{T2RDF} \) of \( T' \) of weight \( \omega(f)-2 \). We deduce from the inductive hypothesis on \( T' \) that

\[
\gamma_{\text{TR}_2}(T) = \omega(f) = \omega(f_{|T'}) + 2 \geq \gamma_{\text{TR}_2}(T') + 2 \\
\geq \left\lceil \frac{2(n-3-(\ell(T)+3)}{3} \right\rceil + 2 > \left\lceil \frac{2(n-\ell(T)+3)}{3} \right\rceil,
\]

which completes the proof. \( \blacksquare \)

**Lemma 23.** If \( T \) is a tree obtained from a path \( v_1v_2\cdots v_k \) \( (k \geq 4) \) by adding a pendant path \( v_{k-1}w \), then \( \gamma_{\text{TR}_2}(T) < \frac{2(k+3)}{3} \).
**Proof.** If \( k \equiv 0 \pmod{3} \), then define the function \( f \) by \( f(v_{3i+1}) = f(v_{3i+2}) = 1 \)
for \( 0 \leq i \leq \frac{k}{3} - 2 \), \( f(v_{k-2}) = 1 \), \( f(v_{k-1}) = 2 \) and \( f(v) = 0 \) for any remaining vertex \( v \). If \( k \equiv 1 \pmod{3} \), then define the function \( f \) by \( f(v_{3i+1}) = f(v_{3i+2}) = 1 \)
for \( 0 \leq i \leq \frac{k-1}{3} - 1 \), \( f(v_{k-2}) = 2 \) and \( f(v) = 0 \) for any remaining vertex \( v \). If \( k \equiv 2 \pmod{3} \), then define the function \( f \) by \( f(v_{3i+1}) = f(v_{3i+2}) = 1 \)
for \( 0 \leq i \leq \frac{k-2}{3} - 1 \), \( f(v_{k-2}) = 1 \), \( f(v_{k-1}) = 2 \) and \( f(v) = 0 \) for any remaining vertex \( v \). Clearly \( f \) is an TR2DF of weight smaller than \( \frac{2(k+3)}{3} \).

**Theorem 24.** For every tree \( T \) of order \( n(T) \geq 4 \) with \( s(T) \) support vertices,

\[
\gamma_{tR2}(T) \leq \frac{3n(T) + 2s(T)}{4}
\]

with equality if and only if \( T \) is the corona of a tree.

**Proof.** If \( T \) is the corona of a tree \( T' \), then \( \gamma_{tR2}(T) = n(T) = \frac{3n(T) + 2s(T)}{4} \). To prove that if \( T \) is a tree of order \( n(T) \geq 4 \) with \( s(T) \) support vertices, then \( \gamma_{tR2}(T) \leq \frac{3n(T) + 2s(T)}{4} \) with equality only if \( T \) is the corona of a tree, we proceed by induction on the order \( n(T) \). If \( n(T) = 4 \), then \( T \) is either a star \( K_{1,3} \), where \( \gamma_{tR2}(K_{1,3}) = 3 < \frac{3n(T) + 2s(T)}{4} \) or a path \( P_4 \) where \( \gamma_{tR2}(P_4) = 4 = \frac{3n(T) + 2s(T)}{4} \) and \( P_4 \) is the corona of the path \( P_2 \). Let \( n(T) \geq 5 \) and assume that every \( T' \) of order \( n(T') < n(T) \) with \( s(T') \) support vertices satisfies \( \gamma_{tR2}(T') \leq \frac{3n(T') + 2s(T')}{4} \) with equality only if \( T' \) is the corona of a tree. Let \( T \) be a tree of order \( n(T) \). If \( T \) is a star, then \( \gamma_{tR2}(T) = 3 < \frac{3n(T) + 2s(T)}{4} \). Likewise, if \( T \) is a double star, then \( \gamma_{tR2}(T) = 4 < \frac{3n(T) + 2s(T)}{4} \) (since \( n(T) \geq 5 \)). Henceforth, we can assume that \( T \) has diameter at least 4. Denote by \( T_x \) the subtree induced by a vertex \( x \) and its descendants in the rooted tree \( T \).

If \( T \) has a strong support vertex \( u \) with at least three leaves, then let \( T' \) be the tree obtained from \( T \) by removing a leaf neighbor \( w \) of \( u \). Let \( f \) be a \( \gamma_{tR2}(T') \)-function \( f \) such that \( f(u) = 2 \), \( f(v) \geq 1 \) for some \( v \in N_T(u) \). Clearly, \( f \) can be extended to TR2D-function of \( T \) by assigning a 0 to \( w \), and thus \( \gamma_{tR2}(T) \leq \gamma_{tR2}(T') \). Now using the induction on \( T' \) and the fact that \( n(T') = n(T) - 1 \) and \( s(T') = s(T) \), we obtain the desired result. Henceforth, we can assume that every support vertex of \( T \) is adjacent to at most two leaves.

Let \( v_1v_2 \cdots v_k \) be a diametral path in \( T \) such that \( \deg_T(v_2) \) is as large as possible and root \( T \) at \( v_k \). Clearly \( \deg_T(v_2) \in \{2,3\} \). We consider the following cases.

**Case 1.** \( \deg_T(v_2) = 3 \). We distinguish the following subcases.

**Subcase 1.1.** \( \deg_T(v_3) \geq 3 \). If \( v_3 \) is a support vertex or \( v_3 \) has a child with degree 3 other than \( v_2 \), then any \( \gamma_{tR2}(T - T_{v_2}) \)-function can be extended to a TR2D-function of \( T \) by assigning 2 to \( v_2 \) and 0 to the leaf neighbors of \( v_2 \) and
so $\gamma_{TR^2}(T) \leq \gamma_{TR^2}(T - T_{v_2}) + 2$. Since $T - T_{v_2}$ is a tree of order at least four, by induction on $T - T_{v_2}$ and using the facts $n(T - T_{v_2}) = n(T) - 3$ and $s(T - T_{v_2}) = s(T) - 1$, we obtain $\gamma_{TR^2}(T) \leq \gamma_{TR^2}(T') + 2 \leq \frac{3n(T') + 2s(T')}{4} + 2 < \frac{3n(T) + 2s(T)}{4}$. Hence assume that every child of $v_3$ except $v_2$ is of degree 2. Let $w_2$ be a child of $v_3$ besides $v_2$ and let $w_1$ be the leaf neighbor of $w_2$. Clearly any $\gamma_{TR^2}(T - T_{w_2})$-function can be extended to a TR2D-function of $T$ by assigning 1 to $w_1$, $w_2$ and so $\gamma_{TR^2}(T) \leq \gamma_{TR^2}(T - T_{w_2}) + 2$. Note that $T - T_{w_2}$ is a tree of order at least four with $n(T - T_{w_2}) = n(T) - 2$ and $s(T - T_{w_2}) = s(T) - 1$. Using the induction on $T - T_{w_2}$, we obtain $\gamma_{TR^2}(T) \leq \gamma_{TR^2}(T') + 2 < \frac{3n(T') + 2s(T')}{4} + 2 < \frac{3n(T) + 2s(T)}{4}$.

Subcase 1.2. $\deg_T(v_3) = 2$ and $\deg_T(v_4) \geq 3$. If $T - T_{v_3} = P_3$, then clearly $\gamma_{TR^2}(T) = 5 < \frac{3n(T) + 2s(T)}{4}$, Hence assume that $T - T_{v_3}$ has order at least four. Clearly any $\gamma_{TR^2}(T - T_{v_3})$-function can be extended to a TR2D-function of $T$ by assigning a 2 to $v_2$, a 1 to $v_1$ and a 0 to the leaves of $v_2$. It follows from the induction hypothesis on $T - T_{v_3}$ and the facts $n(T - T_{v_3}) = n - 4$ and $s(T - T_{v_3}) = s(T) - 1$ that

$$\gamma_{TR^2}(T) \leq \gamma_{TR^2}(T - T_{v_2}) + 3 \leq \frac{3n(T - T_{v_2}) + 2s(T - T_{v_2})}{4} + 3 \leq \frac{3(n(T) - 4) + 2(s(T) - 1) + 3}{4} < \frac{3n(T) + 2s(T)}{4}.$$ 

Subcase 1.3. $\deg_T(v_3) = 2$ and $\deg_T(v_4) = 2$. First let $\deg_T(v_5) \geq 3$. Hence $T - T_{v_4}$ has order at least three. If $T - T_{v_4} = P_3$, then it is easy to see that $\gamma_{TR^2}(T) = 6 < \frac{3n(T) + 2s(T)}{4}$. Thus let $T - T_{v_4} \neq P_3$. Then any $\gamma_{TR^2}(T - T_{v_4})$-function can be extended to a TR2D-function of $T$ by assigning a 2 to $v_2$ and $v_3$, and a 0 to other vertices in $T_{v_4}$. Using the induction hypothesis on $T - T_{v_4}$ and the facts $n(T - T_{v_4}) = n(T) - 5$ and $s(T - T_{v_4}) = s(T) - 1$ we obtain

$$\gamma_{TR^2}(T) \leq \frac{3n(T - T_{v_4}) + 2s(T - T_{v_4})}{4} + 4 < \frac{3n(T) + 2s(T)}{4}.$$ 

Assume now that $\deg_T(v_5) = 2$. If $\deg(v_i) \leq 2$ for each $i \geq 5$, then the result follows from Lemma 23. Hence let $t$ be the smallest integer such that $\deg(v_i) \geq 3$ for some $t \geq 6$. Let $T' = T - T_{v_{t-1}}$. Note that $T'$ has order at least three. Suppose that $n(T') = 3$, that is $T' = P_3$. Then any $\gamma_{TR^2}(T_{v_{t-1}})$-function as defined in Lemma 23 can be extended to a TR2DF of $T$ by assigning a 2 to $v_t$ and a 0 to other vertices of $T'$, and clearly we have $\gamma_{TR^2}(T) < \frac{3n(T') + 2s(T')}{4}$. Suppose now that $n(T') \geq 4$. If $t \equiv 1 \pmod{3}$, then any $\gamma_{TR^2}(T - T_{v_{t-1}})$-function can be extended to a TR2D-function of $T$ by assigning a 2 to $v_2$, a 1 to $v_3$, $v_{3i+2}$, $v_{3i+3}$ for $1 \leq i < \frac{t-1}{3} - 1$ and a 0 to the remaining vertices of $T_{v_{t-1}}$. Using the induction on $T - T_{v_{t-1}}$ and the fact $\frac{2(t-1)}{3}$ can be rewritten $\frac{3(t-1)}{4} - \frac{t-1}{12}$, we have
\[\gamma_{tR2}(T) \leq \gamma_{tR2}(T - T_{v_{t-1}}) + \frac{2(t - 1)}{3} + 1\]
\[\leq 3n(T - T_{v_4}) + 2s(T - T_{v_4}) + \frac{3(t - 1)}{4} - t + 1 = 3n(T) + 2s(T) + 1 + 1 < 3n(T) + 2s(T).\]

Assume now that \(t \equiv 2 \pmod{3}\). Then any \(\gamma_{tR2}(T - T_{v_{t-1}})\)-function can be extended to a TR2D-function of \(T\) by assigning a 2 to \(v_2\), a 1 to \(v_{3i}, v_{3i+1}\) for \(1 \leq i \leq \frac{t-2}{3}\) and a 0 to the remaining vertices of \(T_{v_{t-1}}\). By the induction hypothesis on \(T - T_{v_{t-1}}\) we obtain
\[\gamma_{tR2}(T) \leq \gamma_{tR2}(T - T_{v_{t-1}}) + \frac{2(t - 2)}{3} + 2\]
\[\leq 3n(T - T_{v_4}) + 2s(T - T_{v_4}) + \frac{3(t - 2)}{4} - t + 2 = 3n(T) + 2s(T) + 2 < 3n(T) + 2s(T).\]

Finally, assume that \(t \equiv 0 \pmod{3}\). Then any \(\gamma_{tR2}(T - T_{v_{t-1}})\)-function can be extended to a TR2D-function of \(T\) by assigning a 2 to \(v_2\), a 1 to \(v_3, v_{3i+1}, v_{3i+2}\) for \(1 \leq i \leq \frac{t}{3} - 1\) and a 0 to the remaining vertices of \(T_{v_{t-1}}\). By the induction hypothesis on \(T - T_{v_{t-1}}\) we have
\[\gamma_{tR2}(T) \leq \gamma_{tR2}(T - T_{v_{t-1}}) + \frac{2t}{3} + 1\]
\[\leq 3n(T - T_{v_4}) + 2s(T - T_{v_4}) + \frac{3t}{4} - t + 1 = 3n(T) + 2s(T) + \frac{6 - t}{12} \leq 3n(T) + 2s(T).\]

If further \(\gamma_{tR2}(T) = \frac{3n(T) + 2s(T)}{4}\), then we have equality throughout the previous inequality chain. In particular, we have \(t = 6\) and \(\gamma_{tR2}(T - T_{v_{t-1}}) = \frac{3n(T) - 2s(T)}{4}\). It follows from the induction on \(T - T_{v_{t-1}}\) that \(T - T_{v_{t-1}}\) is the corona of a tree and \(v_6\) is support vertex (since \(\deg_T(v_6) \geq 3\)). It follows that for any \(\gamma_{tR2}(T - T_{v_{t-1}})\)-function \(g, g(v_6) \geq 1\) and clearly \(g\) can be extended to a TR2D-function of \(T\) by assigning a 2 to \(v_2\), a 1 to \(v_3, v_5\) and a 0 to other vertices in \(T_{v_5}\). By the induction hypothesis we obtain \(\gamma_{tR2}(T) \leq \gamma_{tR2}(T - T_{v_{t-1}}) + 4 < \frac{3n(T) + 2s(T)}{4}\).

Case 2. \(\deg_T(v_2) = 2\). By the choice of the diametral path, we deduce that every child of \(v_3\) with depth one has degree two. Consider the following subcases.
Subcase 2.1. $\deg_T(v_3) \geq 3$. Suppose first that $v_3$ is a strong support vertex, and let $u, w$ be two leaves of $v_3$. Let $T' = T - \{u, v_1, v_2\}$. Clearly $T'$ is a tree of order $n(T') = n(T) - 3 \geq 4$ with $s(T') = s(T) - 1$ support vertices. Let $g$ be a $\gamma_{tR2}(T')$-function. Then we extend $g$ to a TR2D-function of $T$ by assigning a 1 to $v_1, v_2$ and a 0 to $u$. In addition if $g(v_3) \neq 2$, then we reassign $v_3$ and $w$ the values 2 and 0 instead of 1 to both. Now using the induction hypothesis on $T'$, we get

$$\gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 2 \leq \frac{3n(T') + 2s(T')}{4} + 2$$

$$= \frac{3(n(T) - 3) + 2(s(T) - 1)}{4} + 2 < \frac{3n(T) + 2s(T)}{4}.$$

Now, suppose that $v_3$ is not support vertex. Recall that every child of $v_3$ is a support vertex of degree two. Let $T' = T - v_3$. Clearly $T_v$ has order $2\deg_T(v_3) - 1$ and $T'$ has order $n(T') \geq 2$ (since diam($T') \geq 4$). If $n(T') = 2$, then $\gamma_{tR2}(T) = 2\deg_T(v_3) < \frac{3n(T) + 2s(T)}{4}$, and if $n(T') = 3$, then $\gamma_{tR2}(T) = 2\deg_T(v_3) + 1 < \frac{3n(T) + 2s(T)}{4}$. Hence we assume that $n(T') \geq 4$, and thus by induction on $T'$, $\gamma_{tR2}(T') \leq \frac{3n(T') + 2s(T')}{4}$. Since any $\gamma_{tR2}(T')$-function can be extended to a TR2D-function of $T$ by assigning a 0 to $v_3$ and a 1 to each of the remaining vertices of $T_v$, $\gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 2(\deg_T(v_3) - 1)$. Using the fact that $s(T') \leq s(T) - \deg_T(v_3) + 2$, we obtain

$$\gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 2(\deg_T(v_3) - 1) = \frac{3n(T') + 2s(T')}{4} + 2(\deg_T(v_3) - 1)$$

$$\leq \frac{3(n(T) - 2\deg_T(v_3) + 1) + 2(s(T) - \deg_T(v_3) + 2) + 2(\deg_T(v_3) - 1)}{4}$$

$$< \frac{3n(T) + 2s(T)}{4}.$$

Next we can assume that $v_3$ is a support vertex with $\deg_T(v_3) = 3$. Let $T' = T - \{v_1, v_2\}$. As above we can easily see that

$$\gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 2 \leq \frac{3n(T') + 2s(T')}{4} + 2 \leq \frac{3n(T) + 2s(T)}{4}.$$

If further $\gamma_{tR2}(T) = \frac{3n(T) + 2s(T)}{4}$, then we have equality throughout the previous inequality chain. In particular, $\gamma_{tR2}(T - \{v_1, v_2\}) = \frac{3(n(T) - 2) + 2(s(T) - 1)}{4}$. It follows from the induction on $T - \{v_1, v_2\}$ that $T - \{v_1, v_2\}$ is the corona of some tree, implying that $T$ is the corona of a tree.

Subcase 2.2. $\deg_T(v_3) = 2$ and $\deg_T(v_4) \geq 3$. If $T' = T - v_3 = P_3$, then clearly $\gamma_{tR2}(T) = 5 < \frac{3n(T) + 2s(T)}{4}$. Hence assume that $T' \neq P_3$. If $v_4$ is support
vertex or has a child with depth 1 and degree at least 3, then clearly there exists a γtR2(T′)-function that assigns a non-zero positive value to v4 and such a γtR2(T′)-function can be extended to a TR2D-function of T by assigning a 1 to v1, v2 and a 0 to v3. It follows from the induction hypothesis on T′ that

\[ \gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 2 \leq \frac{3n(T') + 2s(T')} {4} + 2 < \frac{3n(T) + 2s(T)} {4}. \]

Now let v4 have child w2 with depth 1 and degree two, and let w1 be the leaf neighbor of w2. Let T′ = T − {w2, w1}. Clearly, γtR2(T′) ≤ γtR2(T′) + 2. By the inductive hypothesis on T′ and since T′ is not a corona, γtR2(T′) < \frac{3n(T') + 2s(T')}{4}. Hence assume that

\[ \gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 2 < \frac{3n(T') + 2s(T')} {4} + 2 \leq \frac{3n(T) + 2s(T)} {4}. \]

Henceforth we assume that any child of v4 is of depth 2. Thus Tv4 is a tree obtain from a star by subdividing every edge twice. Let w1iw2iw3iv4 be paths in T where w3 is a child of v4 for each i ∈ {1, 2, . . . , t} and w3 = v3. If t ≥ 3, then any γtR2(T−Tv4)-function can be extended to a TR2D-function of T by assigning 1 to v1, v2, v3, v4, w3i for i ≥ 2. Now we deduce from the induction hypothesis on T′ and the facts n(T′) = n(T) − 3t − 1 and s(T′) ≤ s(T) − t + 1 that

\[ \gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 2t + 2 \leq \frac{3(n(T) − 3t − 1) + 2(s(T) − t + 1)} {4} + 2t + 2 < \frac{3n(T) + 2s(T)} {4}. \]

Hence assume that t = 2. If deg(v5) ≥ 3, then let T′ = T − Tv3. Then γtR2(T) ≤ γtR2(T − Tv3) + 6. By the induction hypothesis on T′ and the facts n(T − Tv3) = n(T) − 7 and s(T − Tv3) = s(T) − 2 we obtain

\[ \gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 6 \leq \frac{3(n(T) − 7) + 2(s(T) − 2)} {4} + 6 < \frac{3n(T) + 2s(T)} {4}. \]

Thus let deg(v5) = 2 and let T′ = T − Tv3. Note that T′ has order n(T′) ≥ 2. If n(T′) ∈ {2, 3}, then one can check that γtR2(T) < \frac{3n(T) + 2s(T)} {4}. Hence we assume that n(T′) ≥ 4. Then γtR2(T) ≤ γtR2(T′) + 6. It follows from the induction hypothesis on T − Tv3 and the facts n(T′) = n(T) − 8 and s(T′) ≤ s(T) − 1 that

\[ \gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 6 \leq \frac{3(n(T) − 8) + 2(s(T) − 1)} {4} + 6 < \frac{3n(T) + 2s(T)} {4}. \]

Subcase 2.3. degT(v3) = deg(v4) = 2. First let degT(v5) ≥ 3. If T′ = T − Tv4 = P3, then it is easy to see that γtR2(T) = 5 < \frac{3n(T) + 2s(T)} {4}. Hence assume that
If \( v_5 \) is a support vertex, then \( v_5 \) is assigned a non-zero positive value under any \( \gamma_{TR2}(T') \)-set and thus one can easily see that \( \gamma_{TR2}(T) \leq \gamma_{TR2}(T') + 3 \). Using the induction hypothesis on \( T' \) and the facts \( n(T') = n(T) - 4 \) and \( s(T') = s(T) - 1 \) we obtain

\[
\gamma_{TR2}(T) \leq \frac{3n(T') + 2s(T')}{4} + 3 < \frac{3n(T) + 2s(T)}{4}.
\]

If \( v_5 \) has child \( w \) with depth one, then since there is a \( \gamma_{TR2}(T - T_w) \)-function that assigns a non-zero positive value to \( v_5 \), such a \( \gamma_{TR2}(T - T_w) \)-function can be extended to a TR2D-function of \( T \) by assigning a 2 to \( w \) and 0 to other vertices in \( T_w \). By the inductive hypothesis on \( T - T_w \) and since \( T - T_w \) is not a corona, \( \gamma_{TR2}(T - T_w) < \frac{3n(T - T_w) + 2s(T - T_w)}{4} \). Moreover, we have \( n(T - T_w) = n(T) - 2 \) and \( s(T - T_w) = n(T) - 1 \), and thus

\[
\gamma_{TR2}(T) \leq \frac{3n(T) - 2 + 2(s(T) - 1)}{4} + 2 = \frac{3n(T) + 2s(T)}{4}.
\]

Suppose now that \( v_5 \) has child \( w \) with depth two. Let \( w \) have \( t_3 \) leaves, \( t_2 \) children with depth one and degree at least three and \( t_1 \) children with depth one and degree two. Let \( T' = T - T_w \). Then any \( \gamma_{TR2}(T') \)-function can be extended to a TR2D-function of \( T \) by assigning a 2 to every child of \( w \) with depth one, 1 + \( t \) to \( w \) and 0 to other vertices in \( T_w \), where \( t = 0 \) if \( t_3 = 0 \) and \( t = 1 \) if \( t_3 \geq 1 \). Clearly by the inductive hypothesis on \( T' \) and since \( T' \) is not a corona, \( \gamma_{TR2}(T') < \frac{3n(T') + 2s(T')}{4} \). Moreover, we know that \( n(T') \leq n(T) - 3t_2 - 2t_1 - t_3 - 1 \) and \( s(T') = s(T) - t_1 - t_2 - t \). Now

- Assume that \( t_2 \neq 0 \) or \( t_3 \neq 0 \). Then we have

\[
\gamma_{TR2}(T) \leq \frac{3n(T') + 2s(T')}{4} + 2t_2 + 2t_1 + 1 + t < \frac{3n(T') + 2s(T')}{4} + 2t_2 + 2t_1 + 1 + t
\]

\[
\leq \frac{3n(T) - 3t_2 - 2t_1 - t_3 - 1 + 2(s(T) - t_1 - t_2 - t)}{4} + 2t_2 + 2t_1 + 1 + t
\]

\[
\leq \frac{3n(T) + 2s(T)}{4}.
\]

- Assume that \( t_2 = 0 \) and \( t_3 = 0 \). Thus \( t_3 \geq 1 \). Using the fact that there is a \( \gamma_{TR2}(T') \)-function that assigns a non-zero positive value to \( v_5 \), clearly then such a \( \gamma_{TR2}(T') \)-function can be extended to a TR2D-function of \( T \) by assigning a 0 to \( w \) and 1 to the remaining vertices of \( T_w \). It follows that

\[
\gamma_{TR2}(T) \leq \frac{3n(T') + 2s(T')}{4} + 2t_1 < \frac{3n(T) - 2t_1 - 1 + 2(s(T) - t_1)}{4} + 2t_1 < \frac{3n(T) + 2s(T)}{4}.
\]
Assume that \( v_5 \) has child with depth three and let \( w_1w_2w_3w_4v_5 \) be a path in \( T \) where \( w_4 \) is a child of \( v_5 \) different from \( v_4 \). Considering the above cases and subcases we may assume that \( \deg(w_i) = 2 \) for \( i \in \{1,2,3,4\} \). Clearly \( T - T_{w_4} \) has a \( \gamma_{tR^2}(T - T_{w_4}) \)-function \( f \) such that \( f(v_5) \geq 1 \), and \( f \) can be extended to a TR2D-function of \( T \) by assigning a 1 to \( w_3, w_2, w_1 \) and 0 to \( v_4 \). Using the induction hypothesis on \( T - T_{w_4} \) we obtain

\[
\gamma_{tR^2}(T) \leq \gamma_{tR^2}(T - T_{w_4}) + 3 < \frac{3n(T - T_{w_4}) + 2s(T - T_{w_4})}{4} + 3 < \frac{3n(T) + 2s(T)}{4}.
\]

Finally, assume that \( \deg_T(v_5) = 2 \).

Let \( f \) be \( \gamma_{tR^2}(T - T_{v_3}) \)-function such that \( f(v_4) \) is as large as possible. It is easy to see that \( f(v_4) \geq 1 \) and \( f \) can be extended to a TR2D-function of \( T \) by assigning a 1 to \( v_1, v_2 \) and a 0 to \( v_3 \). Using the induction hypothesis on \( T - T_{v_3} \) we obtain

\[
\gamma_{tR^2}(T) \leq \gamma_{tR^2}(T - T_{v_3}) + 2 < \frac{3n(T - T_{v_3}) + 2s(T)}{4} + 2 < \frac{3n(T) + 2s(T)}{4}.
\]

We conclude this section with two open problems.

**Problem 1.** Is the problem of deciding whether \( \gamma_{tR^2}(G) = 3\gamma(G) \) for a given graph \( G \) NP-hard.

**Problem 2.** Characterize all graphs \( G \) such that \( \gamma_{tR^2}(G) = 3\gamma(G) \).

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**References**


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