DECOMPOSING THE COMPLETE GRAPH INTO HAMILTONIAN PATHS (CYCLES) AND 3-STARS

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Abstract

Let $H$ be a graph. A decomposition of $H$ is a set of edge-disjoint subgraphs of $H$ whose union is $H$. A Hamiltonian path (respectively, cycle) of $H$ is a path (respectively, cycle) that contains every vertex of $H$ exactly once. A $k$-star, denoted by $S_k$, is a star with $k$ edges. In this paper, we give necessary and sufficient conditions for decomposing the complete graph into $\alpha$ copies of Hamiltonian path (cycle) and $\beta$ copies of $S_3$.

Keywords: decomposition, complete graph, Hamiltonian path, Hamiltonian cycle, star.

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1. Introduction

For positive integers $m$ and $n$, $K_n$ denotes the complete graph with $n$ vertices, and $K_{m,n}$ denotes the complete bipartite graph with parts of sizes $m$ and $n$. Let $k$ be a positive integer. A $k$-path, denoted by $P_k$, is a path on $k$ vertices. A $k$-cycle, denoted by $C_k$, is a cycle of length $k$. A $k$-star, denoted by $S_k$, is a star
with $k$ edges, i.e., $S_k = K_{1,k}$. Let $H$ be a graph. A \textit{spanning subgraph} of $H$ is a subgraph of $H$ containing every vertex of $H$. A spanning path (respectively, cycle) of $H$ is called a Hamiltonian path (respectively, cycle) of $H$. A \textit{1-factor} of $G$ is a spanning subgraph of $G$ in which each vertex is incident with exactly one edge.

Let $F$, $G$, and $H$ be graphs. A \textit{decomposition} of $H$ is a set of edge-disjoint subgraphs of $H$ whose union is $H$. If $H$ can be decomposed into $\alpha$ copies of $F$ and $\beta$ copies of $G$ for nonnegative integers $\alpha$ and $\beta$, then we say that $H$ has an $\{\alpha F, \beta G\}$-decomposition. Furthermore, if $\alpha \geq 1$ and $\beta \geq 1$, then we say that $H$ has an $(F,G)$-\textit{decomposition} or $H$ is $(F,G)$-\textit{decomposable}.

Study on the existence of an $(F,G)$-decomposition of a graph has attracted a fair share of interest. Abueida and Daven [3] investigated the problem of $(K_k, S_k)$-decomposition of the complete graph $K_n$. Abueida and Daven [4] investigated the problem of the $(C_4, E_2)$-decomposition of several graph products where $E_2$ denotes two vertex disjoint edges. Abueida and O’Neil [8] studied the existence problem for $(C_k, S_{k-1})$-decomposition of the complete multigraph $\lambda K_n$ for $k \in \{3, 4, 5\}$. Priyadharsini and Muthasamy [25, 26] investigated the existence of $(G,H)$-decompositions of $\lambda K_n$ and $\lambda K_{n,n}$ where $G, H \in \{C_n, P_n, S_{n-1}\}$. A graph-pair $(G, H)$ of order $m$ is a pair of non-isomorphic graphs $G$ and $H$ with $V(G) = V(H)$ such that both $G$ and $H$ contain no isolated vertices and $G \cup H$ is isomorphic to $K_m$. Abueida and Daven [2] and Abueida, Daven and Roblee [5] completely determined the values of $n$ for which $\lambda K_n$ admits a $(G, H)$-decomposition where $(G, H)$ is a graph-pair of order 4 or 5. Abueida, Clark and Leach [1] and Abueida and Hampson [6] considered the existence of decompositions of $K_n - F$ into the graph-pairs of order 4 and 5, respectively, where $F$ is a Hamiltonian cycle, a 1-factor, or an almost 1-factor. Lee [18, 19], Lee and Lin [22], and Lin [23] established necessary and sufficient conditions for the existence of $(C_k, S_k)$-decompositions of the complete bipartite graph, the complete bipartite multigraph, the complete bipartite graph with a 1-factor removed, and the multicrown, respectively. Abueida and Lian [7] and Beggas \textit{et al.} [10] investigated the problems of $(C_k, S_k)$-decompositions of the complete graph $K_n$ and $\lambda K_n$, giving some necessary or sufficient conditions for such decompositions to exist. Lee and Chen [20] completely settled the existence problem of $(P_{k+1}, S_k)$-decompositions of the complete multigraph $\lambda K_n$ and the balanced complete bipartite multigraph $\lambda K_{n,n}$.

Recently, the existence problem of an $\{\alpha F, \beta G\}$-decomposition of a graph where $\alpha$ and $\beta$ are essential is also studied. Shyu gave necessary and sufficient conditions for the decomposition of $K_n$ into paths and stars (both with 3 edges) [27], paths and cycles (both with $k$ edges where $k = 3, 4$) [28, 29], and cycles and stars (both with 4 edges) [30]. He [31] also gave necessary and sufficient conditions for the decomposition of $K_{m,n}$ into paths and stars both with 3 edges.
Decomposition Into Hamiltonian Paths (Cycles) and 3-Stars

Jeevadoss and Muthusamy [14,15] considered the $\{\alpha P_{k+1}, \ldots, 1, 2, \ldots, (n-1)/2, \}$, where the subscripts of $x$'s are taken modulo $n-1$ in the set of numbers $\{1, 2, \ldots, n-1\}$.

We first collect some needed terminology and notation. Let $G$ be a graph. The degree of a vertex $x$ of $G$, denoted by $\text{deg}_G x$, is the number of edges incident with $x$. For $k \geq 2$, the vertex of degree $k$ in $S_k$ is the center of $S_k$ and any vertex of degree 1 is a pendent vertex of $S_k$. Let $v_1v_2\cdots v_k$ denote the $k$-path through vertices $v_1, v_2, \ldots, v_k$ in order. The vertices $v_1$ and $v_k$ are referred to as its origin and terminus, respectively. In addition, $P_k(v_1, v_k)$ denotes a $k$-path with origin $v_1$ and terminus $v_k$. We use $\{v_1, v_2, \ldots, v_k\}$ to denote the $k$-cycle through vertices $v_1, v_2, \ldots, v_k$ in order, and $S(u; v_1, v_2, \ldots, v_k)$ to denote a star with center $u$ and pendent vertices $v_1, v_2, \ldots, v_k$. For $U, W \subseteq V(G)$ with $U \cap W = \emptyset$, we use $G[U]$ and $G[U, W]$ to denote the subgraph of $G$ induced by $U$, and the maximal bipartite subgraph of $G$ with bipartition $(U, W)$, respectively. When $G_1, G_2, \ldots, G_t$ are edge disjoint subgraphs of a graph, use $G_1 \cup G_2 \cup \cdots \cup G_t$ to denote the graph with vertex set $\bigcup_{i=1}^{t} V(G_i)$ and edge set $\bigcup_{i=1}^{t} E(G_i)$.

Before going into more details, we present some results which are useful for our discussions.

**Proposition 1** [11,13]. For an even integer $n$ and $V(K_n) = \{x_0, x_1, \ldots, x_{n-1}\}$, the complete graph $K_n$ can be decomposed into $n/2$ copies of $P_n$, $P(1), P(2), \ldots, P(n/2)$ with $P(i+1) = x_i x_{i-1} x_{i+1} x_{i-2} \cdots x_{\frac{i}{2}-2} x_{\frac{i}{2}+1} x_{\frac{i}{2}+\frac{1}{2}} x_{\frac{i}{2}-\frac{1}{2}} x_i$ for $0 \leq i \leq \frac{n}{2} - 1$, where the subscripts of $x$'s are taken modulo $n$ in the set of numbers $\{0, 1, 2, \ldots, n-1\}$.

The following results are attributed to Walecki, see [9].

**Lemma 2.** For an odd integer $n$ and $V(K_n) = \{x_0, x_1, \ldots, x_{n-1}\}$, the complete graph $K_n$ can be decomposed into $(n-1)/2$ copies of $C_n$, $C(1), C(2), \ldots, C((n-1)/2)$ with $C(i) = \{x_0, x_1, x_1, x_{i-1}, x_1, x_{i-2}, \ldots, x_{i+(n-5)/2}, x_{i+(n+1)/2}, x_{i+(n-3)/2}, x_{i+(n-1)/2}\}$ for $i = 1, 2, \ldots, (n-1)/2$, where the subscripts of $x$'s are taken modulo $n-1$ in the set of numbers $\{1, 2, \ldots, n-1\}$. 
Lemma 3. For an even integer \( n \) and \( V(K_n) = \{ x_0, x_1, \ldots, x_{n-1} \} \), the complete graph \( K_n \) can be decomposed into \( n/2 - 1 \) copies of \( C_n \), \( C(1) \), \( C(2) \), \( \ldots \), \( C(n/2-1) \), and a 1-factor \( F \), where \( E(F) = \{ x_0x_{n-1}, x_1x_{n-2}, x_2x_{n-3}, \ldots, x_{n/2-2}x_{n/2+1}, x_{n/2-1}x_n/2 \} \) and \( C(i) = (x_0, x_i, x_{i-1}, x_{i+1}, x_{i-2}, \ldots, x_{i+n/2-1}, x_{i+n/2-2}, x_{i+n/2}, x_{i+n/2-1}) \) for \( i = 1, 2, \ldots, n/2 - 1 \), where the subscripts of \( x \)'s are taken modulo \( n - 1 \) in the set of numbers \( \{ 1, 2, \ldots, n - 1 \} \).

3. Decomposition of \( K_n \) into \( n \)-Paths and 3-Stars

In this section, we obtain necessary and sufficient conditions for decomposing \( K_n \) into \( \alpha \) copies of \( P_n \) and \( \beta \) copies of \( S_3 \).

Lemma 4. Let \( n \) be an odd integer and let \( \alpha \) be a nonnegative integer. If \( \binom{n}{2} - (n - 1)\alpha \) is a nonnegative integer and \( \binom{n}{2} - (n - 1)\alpha \equiv 0 \pmod{3} \), then

\[
\alpha \in \begin{cases} 
\{0, 1, \ldots, (n-1)/2\} & \text{if } n \equiv 1 \pmod{6}, \\
\{(n-3)/2 - 3t | t = 0, 1, \ldots, (n-3)/6\} & \text{if } n \equiv 3 \pmod{6}, \\
\{(n-3)/2 - 3t | t = 0, 1, \ldots, (n-5)/6\} & \text{if } n \equiv 5 \pmod{6}. 
\end{cases}
\]

Proof. Since \( \binom{n}{2} - (n - 1)\alpha \) is a nonnegative integer and \( n \) is odd, \( \alpha \leq \lfloor \binom{n}{2}/(n - 1) \rfloor = (n - 1)/2 \). Let \( \alpha = (n - 1)/2 - (3t + j) \) where \( t \) is a nonnegative integer and \( j \in \{0, 1, 2\} \). Since \( \binom{n}{2} - (n - 1)\alpha = n(n - 1)/2 - (n - 1)\alpha = (n - 2\alpha)(n - 1)/2 = (6t + 2j + 1)(n - 1)/2 \), \( \binom{n}{2} - (n - 1)\alpha \equiv (2j + 1)(n - 1)/2 \pmod{3} \). If \( n \equiv 1 \pmod{6} \), then \( (2j + 1)(n - 1)/2 \equiv 0 \pmod{3} \) for any integer \( j \). Hence \( \alpha \in \{0, 1, \ldots, (n-1)/2\} \) for \( n \equiv 1 \pmod{6} \). When \( n \equiv 3 \pmod{6} \) or \( n \equiv 5 \pmod{6} \), the condition \( (2j + 1)(n - 1)/2 \equiv 0 \pmod{3} \) holds if and only if \( j = 1 \). Thus \( \alpha = (n - 3)/2 - 3t \) for some integer \( t \) when \( n \equiv 3 \pmod{6} \) or \( n \equiv 5 \pmod{6} \). Since \( \alpha \) is a nonnegative integer, we have \( t \leq (n - 3)/6 \) for \( n \equiv 3 \pmod{6} \), and \( t \leq (n - 5)/6 \) for \( n \equiv 5 \pmod{6} \). This completes the proof.

Lemma 5. Let \( n \) be an even integer, and let \( \alpha \) be a nonnegative integer. If \( \binom{n}{2} - (n - 1)\alpha \equiv 0 \pmod{3} \), then

\[
\alpha \in \begin{cases} 
\{n/2 - 3t | t = 0, 1, \ldots, n/6\} & \text{if } n \equiv 0 \pmod{6}, \\
\{n/2 - 3t | t = 0, 1, \ldots, (n-2)/6\} & \text{if } n \equiv 2 \pmod{6}, \\
\{0, 1, \ldots, n/2\} & \text{if } n \equiv 4 \pmod{6}. 
\end{cases}
\]

Proof. Since \( \binom{n}{2} - (n - 1)\alpha \) is a nonnegative integer and \( n \) is even, \( \alpha \leq \lfloor \binom{n}{2}/(n - 1) \rfloor = n/2 \). Let \( \alpha = n/2 - (3t + j) \) where \( t \) is a nonnegative integer and \( j \in \{0, 1, 2\} \). Since \( \binom{n}{2} - (n - 1)\alpha = n(n - 1)/2 - (n - 1)\alpha = (n - 2\alpha)(n - 1)/2 = (3t + j)(n - 1) \), \( \binom{n}{2} - (n - 1)\alpha \equiv j(n - 1) \pmod{3} \). If \( n \equiv 4 \pmod{6} \), then \( j(n - 1) \equiv 0 \pmod{3} \) for any integer \( j \). Hence \( \alpha \in \{0, 1, \ldots, n/2\} \) for \( n \equiv 4 \pmod{6} \). When \( n \equiv 0 \)
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The following indecomposable case is trivial.

**Lemma 6.** The complete graph $K_4$ cannot be decomposed into

1. one copy of $P_4$ and one copy of $S_3$, nor
2. two copies of $S_3$.

In addition, we exclude the possibility $n = 5$.

**Lemma 7.** The complete graph $K_5$ cannot be decomposed into one copy of $P_5$ and two copies of $S_3$.

**Proof.** Suppose, on the contrary, that $K_5$ can be decomposed into one copy of $P_5$, say $P_5(x, y)$, and two copies of $S_3$, say $S$ and $T$. Note that the edge $xy$ must be in either $S$ or $T$. Without loss of generality, assume that $xy$ is in $S$. Since the degree of every vertex of $K_n - E(P_5(x, y) \cup S)$ is less than 3, we have a contradiction.

In the remainder of the paper, we assume that $V(K_n) = \{x_0, x_1, \ldots, x_{n-1}\}$.

**Lemma 8.** If $n$ is an odd integer with $n \geq 7$, then the following hold:

1. The complete graph $K_n$ can be decomposed into $(n-1)/2$ copies of $P_n$ and $(n-1)/6$ copies of $S_3$ when $n \equiv 1 \pmod{6}$.
2. The complete graph $K_n$ can be decomposed into $(n-3)/2$ copies of $P_n$ and $(n-1)/2$ copies of $S_3$.
3. The complete graph $K_n$ can be decomposed into $(n-5)/2$ copies of $P_n$ and $5(n-1)/6$ copies of $S_3$ when $n \equiv 1 \pmod{6}$.

**Proof.** By Lemma 2, $K_n$ can be decomposed into $(n-1)/2$ copies of $C_n$, $C(1)$, $C(2), \ldots, C((n-1)/2)$ with $C(i) = (x_0, x_i, x_{i-1}, x_{i+1}, x_{i-2}, \ldots, x_{i+(n-5)/2}, x_{i+(n+1)/2}, x_{i+(n-3)/2}, x_{i+(n-1)/2})$ for $i = 1, 2, \ldots, (n-1)/2$, where the subscripts of $x^i$s are taken modulo $n - 1$ in the set of numbers $\{1, 2, \ldots, n - 1\}$.

1. For $i = 1, 2, \ldots, (n-1)/2$, let $P(i) = C(i) - \{x_0x_i\}$. Clearly, $P(i)$ is an $n$-path. Let $G$ be the subgraph of $K_n$ which is induced by the set of edges $x_0x_1, x_0x_2, \ldots, x_0x_{(n-1)/2}$. Obviously, $G = S_{(n-1)/2}$. Since $n$ is odd and $n-1 \equiv 0 \pmod{3}$, the graph $G$ can be decomposed into $(n-1)/6$ copies of $S_3$. This settles (1).
(2) For \( n = 7 \), the complete graph \( K_7 \) can be decomposed into 2 copies of \( P_7 \) and 3 copies of \( S_3 \) as follows: \( x_6x_2x_5x_3x_4x_0x_1, x_6x_4x_5x_0x_3x_1x_2, (x_1; x_4, x_5, x_6), (x_2; x_0, x_3, x_4), (x_6; x_0, x_3, x_5) \).

Now we consider the case \( n \geq 9 \). For \( i \in \{1, 2, \ldots, (n - 1)/2\} \), let \( P(i) = C(i) - \{x_0x_i\} \). Note that \( x_{n-1}x_n \in E(C((n - 7)/2)) \) and \( P((n - 3)/2) = C((n - 3)/2) \) is a 3-star. Let \( P((n - 7)/2) = C((n - 7)/2) \) be decomposed into \( S_3 \). For \( k \in K \), let \( e''_k \) be an edge in \( C((k - 1)/3 + (n - 1)/3 - 1) \) incident with the center of \( S''(k) \).

Hence \( P(i) \) is an \( n \)-path for \( i = 1, 2, \ldots, (n - 1)/2 \). Moreover, \( P((n - 1)/2) = x_{n-1}/2x_6/n-3/2x_{n+1}/2x_{(n-5)/2} \) for \( i = 1, 2, \ldots, (n - 3)/2 \), let \( S(i) = (x_{n-1}/2; x_{n-1}/2+i-1, x_{n-1}/2+i) \) and \( S = (x_{n-1}; x_{n-2}, x_0) \). Obviously, \( S(i) \) and \( S \) are 2-stars, and \( P((n - 1)/2) \) can be decomposed into \( S(1), S(2), \ldots, S((n - 3)/2) \) and \( S \). Furthermore, let \( W(i) = S(i) \cup \{x_{0x_i}\} \) for \( i = 1, 2, \ldots, (n - 3)/2 \). \( W((n - 7)/2) = S((n - 7)/2) \) is a 3-star. This settles (2).

(3) We will remove one edge from \( C(i) \) to obtain an \( n \)-path for \( i \in \{1, 2, \ldots, (n-5)/2\} \), and use \( C((n-3)/2) \) and \( C((n-1)/2) \) together with the edges removed from \( C(i) \) to constitute 5\((n-1)/3\) copies of \( S_3 \).

Let \( S(i) = (x_{(n-1)/2+i-3}; x_{(n-1)/2-i+1}, x_{(n-1)/2-i-3}) \) for \( i = 1, 2, \ldots, (n - 1)/6 \), \( S''(i) = (x_{(n-1)/2+i-3}; x_{(n-1)/2-i+1}, x_{(n-1)/2+i-3}) \) for \( i = 1, 2, \ldots, (n - 7)/6 \), and \( S''((n - 1)/6) = (x_{n-2}; x_{n-3}, x_0) \). Obviously, \( S(i) \) and \( S''(i) \) are 2-stars. Let \( J = \{j | 2 \leq j \leq (n - 1)/2 \} \) and \( j \equiv 0 \pmod{3} \). For \( j \in J \), let

\[
e''_j = \begin{cases} x_{n-1}/2+jx_{(n-1)/2+j} & \text{if } j \equiv 0 \pmod{3}, \\
x_{n-1}/2+jx_{(n-1)/2+j} & \text{if } j \equiv 2 \pmod{3}, \end{cases}
\]

where the subscripts of \( x \)'s are taken modulo \( n-1 \) in the set of numbers \( \{1, 2, \ldots, n-1\} \). It is easy to see that \( \{S(i), S''(i) | i = 1, 2, \ldots, (n - 1)/6 \} \cup \{e''_j | j \in J \} \) is a decomposition of \( C((n-3)/2) - \{x_{(n-3)/2x_0}\} \).

Note that \( C((n - 1)/2) = (x_0, x_{(n-1)/2}, x_{(n-3)/2}, x_{(n+1)/2}, x_{(n-5)/2}, \ldots, x_{n-3}, x_1, x_{n-2}, x_{n-1}) \). Let \( S''(j) = (x_{(n-1)/2-j}; x_{(n-1)/2+j}, x_{(n+1)/2+j}) \) for \( j = 1, 2, \ldots, (n-3)/2 \) and \( S''((n - 1)/2) = (x_{n-1}; x_{n-2}, x_0) \) where the subscripts of \( x \)'s are taken modulo \( n-1 \) in the set of numbers \( \{1, 2, \ldots, n-1\} \). Obviously, \( S''(j) \) is a 2-star, and \( C((n-1)/2) - \{x_{(n-1)/2x_0}\} \) can be decomposed into \( S''(1), S''(2), \ldots, S''((n - 1)/2) \).

For \( i = 2, 3, \ldots, (n - 1)/6 \), let \( e_i \) be an edge in \( C(i - 1) \) incident with the center of \( S(i) \). Then \( C(i - 1) - \{e_i\} \) is an \( n \)-path and \( S(i) \cup \{e_i\} \) is a 3-star. For \( i = 1, 2, \ldots, (n - 1)/6 \), let \( e'_i \) be an edge in \( C((n - 1)/6 + i - 1) \) incident with the center of \( S''(i) \). Then \( C((n - 1)/6 + i - 1) - \{e'_i\} \) is an \( n \)-path and \( S''(i) \cup \{e'_i\} \) is a 3-star. Let \( K = \{k | 4 \leq k \leq (n - 5)/2 \} \) and \( k \equiv 1 \pmod{3} \). For \( k \in K \), let \( e''_k \) be an edge in \( C((k-1)/3 + (n-1)/3 - 1) \) incident with the center of \( S''(k) \).
Then $C((k - 1)/3 + (n - 1)/3 - 1) - \{e''_k\}$ is an $n$-path and $S''(i) \cup \{e''_k\}$ is a 3-star. For $j \in J$, $S''(j) \cup \{e''_j\}$ is a 3-star. Moreover, $S(1) \cup \{x_{(n-1)/2}\}$ and $S''(1) \cup \{x_{(n-3)/2}\}$ are also 3-stars. This completes the proof.

Lemma 9. If $n$ is an even integer with $n \geq 4$, then the following hold:

1. The complete graph $K_n$ can be decomposed into $n/2$ copies of $P_n$.
2. The complete graph $K_n$ can be decomposed into $(n/2 - 1)$ copies of $P_n$ and $(n-1)/3$ copies of $S_3$ when $n \equiv 4 \pmod{6}$ and $n \geq 10$.
3. The complete graph $K_n$ can be decomposed into $(n/2 - 2)$ copies of $P_n$ and $2(n-1)/3$ copies of $S_3$ when $n \equiv 4 \pmod{6}$ and $n \geq 10$.

Proof. By Proposition 1, we have (1).

(2) For $n = 10$, the complete graph $K_{10}$ can be decomposed into 4 copies of $P_{10}$ and 3 copies of $S_3$ as follows: $x_8x_2x_7x_3x_6x_4x_5x_0x_9, x_1x_3x_8x_4x_7x_5x_6x_0x_2x_9, x_0x_3x_4x_1x_5x_8x_6x_7x_9, x_0x_7x_8x_9x_4x_3x_5x_2x_6x_1, (x_0; x_4, x_8, x_9), (x_1; x_2, x_7, x_8), (x_9; x_3, x_5, x_6)$.

Now we consider the case $n \geq 16$. Let $G = K_n[\{x_0, x_1, \ldots, x_{n-2}\}]$. Clearly $G$ is isomorphic to $K_{n-1}$. By Lemma 2, the graph $G$ can be decomposed into $n/2$ copies of $C_{n-1}$, $C(1), C(2), \ldots, C(n/2 - 1)$ with $C(i) = (x_0, x_i, x_{i-1}, x_{i+1}, x_{i-2}, \ldots, x_{i+n/2-3}, x_{i+n/2-2}, x_{i+n/2-1})$ for $i = 1, 2, \ldots, n/2 - 1$, where the subscripts of $x$'s are taken modulo $n - 2$ in the set of numbers $\{1, 2, \ldots, n - 2\}$. Note that $C(1)$ contains edges $x_1x_{n-2}$ and $x_{n/2}x_0$, $C(2)$ contains edges $x_1x_2$ and $x_{n/2+1}x_0$, and $C(3)$ contains the edge $x_4x_1$. Let $P(1) = C(1) \cup \{x_1x_{n-1}x_{n/2}\} - \{x_1x_{n-2}, x_{n/2}x_0\}$, $P(2) = C(2) \cup \{x_2x_{n-1}x_{n/2+1}\} - \{x_2x_1, x_{n/2+1}x_0\}$, and $P(3) = C(3) \cup \{x_4x_{n-1}\} - \{x_4x_1\}$. In addition, let $P(i) = C(i) \cup \{x_{i+n/2-1}x_{n-1}\} - \{x_{i+n/2-1}x_0\}$ for $i = 4, 5, \ldots, n/2 - 1$. Obviously, $P(i)$ is in an $n$-path for $i = 1, 2, \ldots, n/2 - 1$. Let $S(1) = (x_0; x_{n/2}, x_{n/2+1}, x_{n/2+3}, x_{n/2+4}, \ldots, x_{n-2})$ and $S(2) = (x_{n-1}; x_0, x_3, x_5, x_6, \ldots, x_{n/2-2}, x_{n/2-1}, x_{n/2+2})$. It is easy to see that $K_n - E\left(\bigcup_{i=1}^{\lfloor n/2 \rfloor} P(i)\right) = S(1) \cup S(2) \cup (x_1; x_2, x_4, x_{n-2})$. Note that $S(1)$ and $S(2)$ are $(n/2 - 2)$-stars. Since $n \equiv 4 \pmod{6}$, each of $S(1)$ and $S(2)$ can be decomposed into $(n-4)/6$ copies of $S_3$. This settles (2).

(3) By Lemma 3, $K_n$ can be decomposed into $(n/2 - 1)$ copies of $C_n, C(1), C(2), \ldots, C(n/2 - 1)$, and a 1-factor $F$, where $E(F) = \{x_0x_{n-1}, x_1x_{n-2}, x_2x_{n-3}, \ldots, x_{n/2}x_{n/2+1}, x_{n/2-1}x_{n/2}\}$ and $C(i) = (x_0, x_i, x_{i-1}, x_{i+1}, x_{i-2}, \ldots, x_{i+n/2-1}, x_{i+n/2-2}, x_{i+n/2}, x_{i+n/2+1})$ for $i = 1, 2, \ldots, n/2 - 1$, where the subscripts of $x$'s are taken modulo $n - 1$ in the set of numbers $\{1, 2, \ldots, n - 1\}$.

We obtain $n - 2$ copies of $P_n$ by removing one edge from each of $n$-cycles $C(1), C(2), \ldots, C(n/2 - 2)$. For $i = 1, 2, \ldots, n/2 - 3$, let $P(i) = C(i) - \{x_{2i}\}$. In addition, let $P(n/2 - 2) = C(n/2 - 2) - \{x_{n/2-3}x_{n/2-1}\}$. Trivially, $P(i)$ is an $n$-path for $i = 1, 2, \ldots, n/2 - 2$. 

DECOMPOSITION INTO HAMILTONIAN PATHS (CYCLES) AND 3-STATES
In the following, $2(n-1)/3$ copies of $S_3$ are constructed. We first obtain $n/2$ copies of $S_3$ by using all of the edges of $C(n/2 - 1)$ and $n/2 - 1$ edges of $F$ and the edge $x_{n/2-3}x_{n/2-1}$ removed from $C(n/2 - 2)$. Note that $C(n/2 - 1) = (x_0, x_{n/2-1}, x_{n/2-2}, x_{n/2-3}, \ldots, x_1, x_{n-3}, x_{n-1}, x_{n-2})$. For $i = 1, 2, \ldots, n/2 - 1$, let $S(i) = (x_{n/2-1+i}, x_{n/2-1-i}, x_{n/2-2-i})$ and $S = (x_{n/2-1}, x_{n/2-2}, x_0)$. Obviously, $S(i)$ and $S$ are 2-stars, and $C(n/2 - 1)$ is decomposable into $S(1), S(2), \ldots, S(n/2 - 1)$ and $S$. Let $W(i) = S(i) \cup \{x_{n/2-1+i}x_{n/2-1}\}$ for $i = 1, 2, \ldots, n/2 - 1$, and let $W(n/2) = S \cup \{x_{n/2-3}x_{n/2-1}\}$. Clearly, $W(i)$ is a 3-star.

Now we obtain $(n-4)/6$ copies of $S_3$ by using one edge of $F$ and the edges removed from $C(i)$'s in constructing $n$-paths for $i = 1, 2, \ldots, n/2 - 3$. Let $G$ be the subgraph of $K_n$ induced by the set of edges $x_0x_1, x_0x_2, \ldots, x_0x_{n/2-3}, x_0x_{n-1}$. Obviously, $G = S_{n/2-2}$. Since $n \equiv 4 \pmod{6}$, the graph $G$ can be decomposed into $(n - 4)/6$ copies of $S_3$. This settles (3) and completes the proof.

Lemma 10. Let $n$ and $t$ be positive integers. If $Q_1, Q_2, \ldots, Q_t$ are edge-disjoint Hamiltonian paths of $K_n$, then $\bigcup_{i=1}^t Q_i$ is $S_t$-decomposable.

Proof. Since each $Q_i$ is a Hamiltonian path of $K_n$, we have $V(Q_i) = V(K_n)$. For each $Q_i$, we orient the edges of $Q_i$ from $x_0$ along $Q_i$ to the end (or ends) of the path, and use $\overrightarrow{Q_i}$ to denote the digraph obtained from $Q_i$ for such an orientation. Note that there is exactly one arc directed into $x_j$ for each $j \in \{1, 2, \ldots, n-1\}$. Let $\overrightarrow{G} = \bigcup_{i=1}^t \overrightarrow{Q_i}$. It is easy to check that $\deg_G x_j = t$ for $j \neq 0$. Thus there exists an $S_t$-decomposition of $\bigcup_{i=1}^t Q_i$ such that $x_j$ is a center of a $t$-star for $j \neq 0$. This completes the proof.

By Lemma 10, the union of $3t$ edge-disjoint $n$-paths can be decomposed into $n - 1$ copies of $S_3$, in turn, each $S_3$ can be decomposed in to $t$ copies of $S_3$. Hence we have the following result.

Theorem 11. Let $n$, $p$ and $t$ be positive integers with $p \geq 3t$, and let $q$ be a nonnegative integer. If $K_n$ can be decomposed into $p$ copies of $P_n$ and $q$ copies of $S_3$, then $K_n$ can be decomposed into $p - 3t$ copies of $P_n$ and $q + (n-1)t$ copies of $S_3$.

Obviously, if $K_n$ can be decomposed into $\alpha$ copies of $n$-paths and $\beta$ copies of $S_3$, then $\binom{n}{2} = (n-1)\alpha + 3\beta$. Using Theorem 11 together with Lemmas 4 to 9, we have the main result of this section.

Theorem 12. Let $n$ be a positive integer with $n \geq 4$, and let $\alpha$ and $\beta$ be nonnegative integers. The complete graph $K_n$ can be decomposed into $\alpha$ copies of $P_n$ and $\beta$ copies of $S_3$ if and only if $\binom{n}{2} = (n-1)\alpha + 3\beta$ and $(n, \alpha, \beta) \notin \{(4, 1, 1), (4, 0, 2), (5, 1, 2)\}$. 


4. Decomposition of $K_n$ into $n$-Cycles and 3-Stars

In this section, we obtain necessary and sufficient conditions for decomposing $K_n$ into $\alpha$ copies of $C_n$ and $\beta$ copies of $S_3$. The first two lemmas in the following are from [17] and [32], respectively.

**Lemma 13.** For an odd integer $n$ and $V(K_{n,n}) = \{x_0, \ldots, x_{n-1}\} \cup \{y_0, \ldots, y_{n-1}\}$, the complete bipartite graph $K_{n,n}$ can be decomposed into $(n-1)/2$ copies of $C_{2n}$, $C(0)$, $C(1), \ldots, C((n-3)/2)$, and a 1-factor $F$, where $E(F) = \{x_0y_{n-1}, x_1y_{n-2}, \ldots, x_{n-1}y_0\}$ and $C(i) = (y_{2i}, x_0, y_{2i+1}, x_1, \ldots, y_{2i+(n-2)}, x_{n-2}, y_{2i+(n-1)}, x_{n-1})$ for $i = 0, 1, \ldots, (n-3)/2$.

**Lemma 14.** For an even integer $n$ and $V(K_{n,n}) = \{x_0, \ldots, x_{n-1}\} \cup \{y_0, \ldots, y_{n-1}\}$, the complete bipartite graph $K_{n,n}$ can be decomposed into $n/2$ copies of $C_{2n}$, $C(0)$, $C(1), \ldots, C(n/2-1)$, where $C(i) = (y_{2i}, x_0, y_{2i+1}, x_1, \ldots, y_{2i+(n-2)}, x_{n-2}, y_{2i+(n-1)}, x_{n-1})$ for $i = 0, 1, \ldots, n/2 - 1$.

**Lemma 15.** Let $n$ be an odd integer and let $\alpha$ be a nonnegative integer. If \(^n\binom{3}{2} - n\alpha\) is a nonnegative integer and \(^n\binom{3}{2} - n\alpha \equiv 0 \pmod{3}\), then

$$\alpha \in \begin{cases} 0, 1, \ldots, (n-1)/2 & \text{if } n \equiv 0 \pmod{3}, \\ \{(n-1)/2 - 3t \mid t = 0, 1, \ldots, [(n-1)/6]\} & \text{otherwise}. \end{cases}$$

**Proof.** Since \(^n\binom{3}{2} - n\alpha\) is a nonnegative integer and $n$ is odd, $\alpha \leq \lfloor \binom{n}{2}/n \rfloor = (n-1)/2$. Let $\alpha = (n-1)/2 - (3t+i)$, where $t$ is a nonnegative integer and $i \in \{0, 1, 2\}$. Since \(^n\binom{3}{2} - n\alpha = n(n-1)/2 - n\alpha = n(n-1-2\alpha)/2 = n(3t+i), \) \(^n\binom{3}{2} - n\alpha \equiv n \pmod{3}\). If $n$ is a multiple of 3, then $n \equiv 0 \pmod{3}$ holds for any $i \in \{0, 1, 2\}$. Hence $\alpha \in \{0, 1, \ldots, (n-1)/2\}$ for $n \equiv 0 \pmod{3}$. Otherwise, the condition $n \equiv 0 \pmod{3}$ holds if and only if $i = 0$. This implies $\alpha = (n-1)/2 - 3t$. Moreover, $t \leq [(n-1)/6]$ since $\alpha$ is a nonnegative integer. This completes the proof.

**Lemma 16.** Let $n$ be an even integer and let $\alpha$ be a nonnegative integer. If \(^n\binom{3}{2} - n\alpha\) is a nonnegative integer and \(^n\binom{3}{2} - n\alpha \equiv 0 \pmod{3}\), then

$$\alpha \in \begin{cases} 0, 1, \ldots, n/2-1 & \text{if } n \equiv 0 \pmod{3}, \\ n/2 - 3t - 2 & t = 0, 1, \ldots, [(n-4)/6] & \text{otherwise}. \end{cases}$$

**Proof.** Since \(^n\binom{3}{2} - n\alpha\) is a nonnegative integer and $n$ is even, $\alpha \leq \lfloor \binom{n}{2}/n \rfloor = n/2-1$. Let $\alpha = n/2-1-(3t+i)$, where $t$ is a nonnegative integer and $i \in \{0, 1, 2\}$. Since \(^n\binom{3}{2} - n\alpha = n(n-1-2\alpha)/2 = n(6t+2i+1)/2, \) \(^n\binom{3}{2} - n\alpha \equiv n(2i+1)/2 \pmod{3}\). If $n \equiv 0 \pmod{3}$, then $n/2 \equiv 0 \pmod{3}$, this implies that $n(2i+1)/2 \equiv 0 \pmod{3}$ holds for any $i \in \{0, 1, 2\}$. Hence $\alpha \in \{0, 1, \ldots, n/2-1\}$ for $n \equiv 0 \pmod{3}$.
Let $m = (n-3)/2$ for odd $n$ and $m = (n-2)/2$ for even $n$. Let $C(1), C(2), \ldots, C(m)$ be edge-disjoint $n$-cycles in $K_n$, and let $G = K_n - \bigcup_{i=1}^{m} E(C(i))$. Since $\deg_G x = n - 1 - 2m \leq 2$ for each vertex $x$, $G$ has no $S_3$-decomposition. Thus we have the following result.

**Lemma 17.** Let $n \equiv 0 \pmod{3}$. The complete graph $K_n$ cannot be decomposed into $(n-3)/2$ copies of $C_n$ and $n/3$ copies of $S_3$ for odd $n$, and cannot be decomposed into $(n-2)/2$ copies of $C_n$ and $n/6$ copies of $S_3$ for even $n$.

**Lemma 18.** If $n$ is an odd integer with $n \geq 5$, then the following hold:

1. The complete graph $K_n$ can be decomposed into $(n-1)/2$ copies of $C_n$.
2. The complete graph $K_n$ can be decomposed into $(n-5)/2$ copies of $C_n$ and $2n/3$ copies of $S_3$ when $n \equiv 3 \pmod{6}$ and $n \geq 9$.
3. The complete graph $K_n$ can be decomposed into $(n-9)/2$ copies of $C_n$ and $4n/3$ copies of $S_3$ when $n \equiv 3 \pmod{6}$ and $n \geq 9$.

**Proof.** By Lemma 2, the complete graph $K_n$ can be decomposed into $(n-1)/2$ copies of $C_n$, $C(1), C(2), \ldots, C((n-1)/2)$ with $C(i) = (x_0, x_i, x_{i-1}, x_{i+1}, \ldots, x_i+(n-5)/2, x_i+(n-3)/2, x_i+(n-1)/2)$ for $i = 1, 2, \ldots, (n-1)/2$, where the subscripts of $x$’s are taken modulo $n - 1$ in the set of numbers $\{1, 2, \ldots, n-1\}$. Hence we have (1).

(2) If there exist $s$ and $t$ $(1 \leq s < t \leq (n-1)/2)$ such that $C(s) \cup C(t)$ can be decomposed into $2n/3$ copies of $S_3$, then we have the result. Consider the case $s = (n+3)/6$ and $t = n/3$. Note that $C((n+3)/6) = (x_0, x_{(n+3)/6}, x_{(n-3)/6}, x_{(n+9)/6}, x_{(n-9)/6}, \ldots, x_{n/3-3}, x_1, x_{n/3}, x_{n-1}, \ldots, x_{2n/3-2}, x_{2n/3+1}, x_{2n/3-1}, x_{2n/3})$. For $i = 1, 2, \ldots, n/3 - 1$, let $S_2(i) = (x_{n-i}, x_{n/3+i}, x_{n/3+i})$ and $S_2(n/3) = (x_{2n/3}, x_{2n/3-1}, x_0)$. For $j = 1, 2, \ldots, (n-3)/6$, let $P_2(j) = x_j x_{n/3+1-j}$. For $j = (n+3)/6, (n+9)/6, \ldots, n/3 - 1$, let $P_2(j) = x_j x_{n/3-j}$. In addition, let $P_2(0) = x_0 x_{(n+3)/6}$. Obviously, $S_2(i)$ is a 2-star for $i = 1, 2, \ldots, n/3$, and $P_2(j)$ is a 2-path for $j = 0, 1, \ldots, n/3 - 1$. One can see that $C((n+3)/6)$ can be decomposed into $S_2(1), S_2(2), \ldots, S_2(n/3)$ and $P_2(0), P_2(1), \ldots, P_2(n/3 - 1)$.

On the other hand, $C(n/3) = (x_0, x_{n/3}, x_{n/3-1}, x_{n/3+1}, x_{n/3-2}, x_{n/3+2}, \ldots, x_{2n/3-2}, x_1, x_{2n/3-1}, x_{n-1}, \ldots, x_{5n-15}/6, x_{5n+15}/6, x_{5n-9}/6, x_{5n+9}/6)$. For $j = 1, 2, \ldots, n/3 - 1$, let $S'_2(j) = (x_j x_{2n/3-1-j}, x_{2n/3-j})$. For $i = 1, 2, \ldots, (n+3)/6$, let $P'_2(i) = x_{n-i} x_{2n/3-2-i}$. For $i = (n+9)/6, (n+12)/6, \ldots, n/3$, let $P'_2(i) = x_n - x_{2n/3-1+i}$. In addition, let $P'_2(0) = x_0 x_{n/3}$ and $P'_2(0) = x_0 x_{(5n-3)/6}$. Obviously, $S'_2(j)$ is a 2-star for $i = 1, 2, \ldots, n/3 - 1$, and $P'_2(0)$ and $P'_2(0)$ are 2-paths for $i = 0, 1, \ldots, n/3$. One can see that $C(n/3)$ can be decomposed into $S'_2(1), S'_2(2), \ldots, S'_2(n/3 - 1)$ and $P'_2(0), P'_2(1), \ldots, P'_2(n/3)$ as well as $P'_2(0)$. Otherwise, the condition $n(2i+1)/2 \equiv 0 \pmod{3}$ holds if and only if $i = 1$. This implies $\alpha = n/2 - 3t - 2$. Moreover, $t \leq [(n-4)/6]$ since $\alpha$ is a nonnegative integer. This completes the proof. \hfill \blacksquare
Decomposition Into Hamiltonian Paths (Cycles) and 3-Stars

For $i = 1, 2, \ldots, n/3$, let $S_3(i) = S_2(i) \cup P_2'(i)$. For $j = 1, 2, \ldots, n/3 - 1$, let $S_3'(j) = S_2'(j) \cup P_3(j)$. Clearly, $S_3(i)$ and $S_3'(j)$ are 3-stars. In addition, $P_2(0) \cup P_2'(0) \cup P_2''(0)$ is also a 3-star. Hence $C((n+3)/6) \cup C(n/3)$ can be decomposed into $2n/3$ copies of $S_3$. This settles (2).

(3) According to the proof of (2), the result holds if there exist $s'$ and $t'$ ($s', t' \notin \{(n+3)/6, n/3\}$) such that $C(s') \cup C(t')$ can be decomposed into $2n/3$ copies of $S_3$. Consider the case $s' = (n + 9)/6$ and $t' = n/3 + 1$. Note that $C((n+9)/6) = (x_0, x_{(n+9)/6}, x_{(n+9)/6}, x_{(n+9)/6}, x_{(n+9)/6}, x_{n/3+1}, x_{n/3+2}, x_{n/3+3}, \ldots, x_{2n/3-1}, x_{2n/3+2}, x_{2n/3+3}, x_{2n/3+4})$. For $i = 1, 2, \ldots, n/3 - 1$, let $S_2(i) = \{x_{n/3-1}; x_{n/3+i}, x_{n/3+1+i}\}$ with $x_n = x_1$ and $S_2(n/3) = \{x_{2n/3+1}, x_{2n/3+2}, x_{2n/3+3}, x_{2n/3+4}\}$. For $j = 2, 3, \ldots, (n + 3)/6$, let $P_2(j) = x_j x_{n/3+3-j}$. For $j = (n + 9)/6, (n + 15)/6, \ldots, n/3$, let $P_2(j) = x_j x_{n/3+3-j}$. In addition, let $P_2(0) = x_0 x_{(n+9)/6}$. Obviously, $S_2(i)$ is a 2-star for $i = 1, 2, \ldots, n/3$, and $P_2(j)$ is a 2-path for $j = 0, 2, 3, \ldots, n/3$. One can see that $C((n+3)/6)$ can be decomposed into $S_2(1), S_2(2), \ldots, S_2(n/3)$ and $P_2(0), P_2(2), P_2(3), \ldots, P_2(n/3)$.

On the other hand, $C(n/3+1) = (x_0, x_{n/3+1}, x_{n/3}, x_{n/3+2}, x_{n/3-1}, \ldots, x_{2n/3}, x_1, x_{2n/3+1}, x_{n-1}, x_{2n/3+2}, x_{n-2}, \ldots, x_{(5n-9)/6}, x_{(5n+9)/6}, x_{(5n-3)/6}, x_{(5n+3)/6})$. For $j = 2, 3, \ldots, n/3$, let $S_2'(j) = \{x_j; x_{2n/3+1-j}; x_{2n/3+2-j}\}$. For $i = 1, 2, \ldots, (n + 3)/6$, let $P_2'(i) = x_{n+1-j} x_{2n/3-1+i}$, and for $i = (n + 9)/6, (n + 12)/6, \ldots, n/3$, let $P_2'(i) = x_{n+1-i} x_{2n/3+i}$ with $x_n = x_1$. In addition, let $P_2''(0) = x_0 x_{n/3+1}$ and $P_2''(0) = x_0 x_{(5n+3)/6}$. Obviously, $S_2'(j)$ is a 2-star for $i = 2, 3, \ldots, n/3$, and $P_2''(0)$ and $P_2''(0)$ are 2-paths for $i = 0, 1, \ldots, n/3$. One can see that $C(n/3+1)$ can be decomposed into $S_2'(2), S_2'(3), \ldots, S_2'(n/3)$ and $P_2''(0), P_2''(1), \ldots, P_2''(n/3)$ as well as $P_2''(0)$.

For $i = 1, 2, \ldots, n/3$, let $S_3(i) = S_2(i) \cup P_2''(i)$. For $j = 2, 3, \ldots, n/3$, let $S_3'(j) = S_2'(j) \cup P_3(j)$. Clearly, $S_3(i)$ and $S_3'(j)$ are 3-stars. In addition, $P_2(0) \cup P_2''(0) \cup P_2''(0)$ is also a 3-star. Hence $C((n+9)/6) \cup C(n/3+1)$ can be decomposed into $2n/3$ copies of $S_3$. This settles (3).

For positive integers $l$ and $n$ with $1 \leq l \leq n$, the $(n,l)$-crown $C_{n,l}$ is the bipartite graph with bipartition $(X, Y)$, where $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $B = \{y_0, y_1, \ldots, y_{n-1}\}$, and edge set $\{x_i y_j : i = 0, 1, \ldots, n-1, j \equiv i+1, i+2, \ldots, i+l \mod l\}$.

**Proposition 19** [24]. $X C_{n,l}$ is $S_k$-decomposable if and only if $k \leq l$ and $\lambda n l \equiv 0 \pmod{k}$.

**Lemma 20.** If $n$ is an even integer $n \geq 6$, then the following hold:

1. The complete graph $K_n$ can be decomposed into $n/2 - 2$ copies of $C_n$ and $n/2$ copies of $S_3$.
2. The complete graph $K_n$ can be decomposed into $n/2 - 3$ copies of $C_n$ and $5n/6$ copies of $S_3$ when $n \equiv 0 \pmod{6}$.
The complete graph $K_n$ can be decomposed into $n/2 - 4$ copies of $C_n$ and $7n/6$ copies of $S_3$ when $n \equiv 0 \pmod{6}$ and $n \geq 12$.

**Proof.** Let $V(K_n) = X \cup Y$, where $X = \{x_0, \ldots, x_{n/2-1}\}$ and $Y = \{y_0, \ldots, y_{n/2-1}\}$. Note that $K_n = K_n[X] \cup K_n[Y] \cup K_n[X, Y]$ where $K_n[X]$ and $K_n[Y]$ are isomorphic to $K_{n/2}$ and $K_n[X, Y]$ is isomorphic to $K_{n/2, n/2}$. We distinguish two cases: Case 1. $n \equiv 0 \pmod{4}$ and Case 2. $n \equiv 2 \pmod{4}$.

**Case 1.** $n \equiv 0 \pmod{4}$. By Lemma 14, $K_n[X, Y]$ can be decomposed into $n/4$ copies of $C_n$, $C(0), C(1), \ldots, C(n/4 - 1)$, where $C(i) = (y_{2i}, x_0, y_{2i+1}, x_1, \ldots, y_{2i+(n/2-2)}, x_{n/2-2}, y_{2i+(n-2)}, x_{n/2-1})$ for $i = 0, 1, 2, \ldots, n/4 - 1$. By Proposition 1, we have the following results. $K_n[X]$ can be decomposed into the following $n/4$ copies of $P_{n/2}$: $P_{n/2}(x_0, x_{n/4}), P_{n/2}(x_1, x_{1+n/4}), \ldots, P_{n/2}(x_{n/4}, y_{n/2-1})$, and $K_n[Y]$ can be decomposed into the following $n/4$ copies of $P_{n/2}: P_{n/2}(y_0, y_{n/4}), P_{n/2}(y_1, y_{1+n/4}), \ldots, P_{n/2}(y_{n/4-1}, y_{n/2-1})$.

For $i = 0, 1, \ldots, n/4 - 1$, let $Q(i) = P_{n/2}(x_i, x_{i+n/4}) \cup P_{n/2}(y_i, y_{i+n/4}) \cup \{y_ix_i, y_{i+n/4}x_{i+n/4}\}$. Clearly, $Q(i)$ is an $n$-cycle, and $y_ix_i, y_{i+n/4}x_{i+n/4} \in E(C(0))$ for $i = 0, 1, \ldots, n/4 - 1$. For $1 \leq t \leq n/4 - 1$, let

$$R(t) = \bigcup_{i=0}^{t} C(i) - \{y_ix_i, y_{i+n/4}x_{i+n/4} \mid 0 \leq i \leq n/4 - 1\}.$$ 

It is easy to see that $R(t)$ is isomorphic to the crown $C_{n/2, 2t+1}$. Therefore, $K_n$ can be decomposed into $n/2 - (t + 1)$ copies of $C_n$, $Q(0), Q(1), \ldots, Q(n/4 - 1)$ and $C(t+1), C(t+2), \ldots, C(n/4 - 1)$, and one copy of $(n/2, 2t+1)$-crown $R(t)$.

Note that $2t + 1 \geq 3$ and $|E(R(t))| = |E(C_{n/2, 2t+1})| = (2t + 1)n/2$. If $(2t + 1)n/2 \equiv 0 \pmod{3}$, then $R(t)$ can be decomposed into $(2t + 1)n/2$ copies of $S_3$ by Proposition 19. Hence for $n \equiv 0 \pmod{4}$, we have the following.

If $t = 1$, then $(2t + 1)n/2 = 3n/2 \equiv 0 \pmod{3}$ for each $n$. Thus $K_n$ can be decomposed into $n/2 - 2$ copies of $C_n$ and $n/2$ copies of $S_3$.

If $t = 2$, then $(2t + 1)n/2 = 5n/2 \equiv 0 \pmod{6}$ for $n \equiv 0 \pmod{6}$. Thus $K_n$ can be decomposed into $n/2 - 3$ copies of $C_n$ and $5n/6$ copies of $S_3$.

If $t = 3$, then $(2t + 1)n/2 = 7n/2 \equiv 0 \pmod{3}$ for $n \equiv 0 \pmod{6}$. Thus $K_n$ can be decomposed into $n/2 - 4$ copies of $C_n$ and $7n/6$ copies of $S_3$. This settles Case 1.

**Case 2.** $n \equiv 2 \pmod{4}$. Since $n \equiv 2 \pmod{4}$, $n/2$ is odd. By Lemma 13, $K_n[X, Y]$ can be decomposed into $(n - 2)/4$ copies of $C_n$, $C(0), C(1), \ldots, C((n - 6)/4)$, and a 1-factor $F$, where $E(F) = \{x_0y_{n/2-1}, x_1y_0, \ldots, x_{n/2-1}y_{n/2-2}\}$ and $C(i) = (y_{2i}, x_0, y_{2i+1}, x_1, \ldots, y_{2i+(n-2)}, x_{n/2-1})$ for $i = 0, 1, \ldots, (n - 6)/4$.

Now we consider $K_n[X]$ and $K_n[Y]$. By Lemma 2, we have the following results. $K_n[X]$ can be decomposed into $(n - 2)/4$ copies of $C_{n/2}$, $W(1), W(2), \ldots, W((n - 2)/4)$ with $W(i) = (x_0, x_i, x_{i-1}, x_{i+1}, x_{i-2}, \ldots, x_{i+(n-10)/4}, x_{i+(n+2)/4}, \ldots$
Decomposition Into Hamiltonian Paths (Cycles) and 3-Stars

$x_i + (n-6)/4, x_i + (n-2)/4$, and \( K_n[Y] \) can be decomposed into \((n-2)/4\) copies of \( C_{n/2}, W'(1), W'(2), \ldots, W'(n-2)/4\) with \( W'(i) = \{y_0, y_1, y_{i-1}, y_{i+1}, y_{i-2}, \ldots, y_{i+(n-10)/4}, y_{i+(n-2)/4}, y_{i+(n-6)/4}, y_{i+(n-2)/4}\} \) for \( i = 1, 2, \ldots, (n-2)/4\), where the subscripts of \( x \)'s and \( y \)'s are taken modulo \((n-2)/2\) in the set of numbers \( \{1, 2, \ldots, (n-2)/2\} \). For \( i = 1, 2, \ldots, (n-2)/4 \), let

\[
e(i) = \begin{cases} 
  x_0 x_1 & \text{if } i = 1, \\
  x_i x_{i-1} & \text{if } i \text{ is odd and } i \geq 3, \\
  x_i + (n-6)/4 x_i + (n-2)/4 & \text{if } i \text{ is even},
\end{cases}
\]

and let

\[
e'(i) = \begin{cases} 
  y_0 y_1 & \text{if } i = 1, \\
  y_i y_{i-1} & \text{if } i \text{ is odd and } i \geq 3, \\
  y_{i+(n-6)/4} y_{i+(n-2)/4} & \text{if } i \text{ is even}.
\end{cases}
\]

Let \( P(i) = W(i) - \{e(i)\} \) and \( P'(i) = W'(i) - \{e'(i)\} \). Trivially, \( P(i) \) and \( P'(i) \) are \((n/2)\)-paths. Let \( M = \{e(i)\} | 1 \leq i \leq (n-2)/4 \) and \( M' = \{e'(i)\} | 1 \leq i \leq (n-2)/4 \). If \( n \equiv 2 \pmod{8} \), then \((n-2)/4\) is even. Hence \( M = \{x_0 x_1, x_2 x_3, \ldots, x_{n-10}/4 x_{n-6}/4, x_{n-2}/4 x_{n+6}/4, \ldots, x_{n-2}/4 x_{n+6}/4\} \) and \( M' = \{y_0 y_1, y_2 y_3, \ldots, y_{n-10}/4 y_{n-6}/4, y_{n+2} y_{n+6}/4, \ldots, y_{n-2}/4 y_{n+6}/4\} \). If \( n \equiv 6 \pmod{8} \), then \((n-2)/4\) is odd. Hence \( M = \{x_0 x_1, x_2 x_3, \ldots, x_{n-2}/4 x_{n+6}/4\} \) and \( M' = \{y_0 y_1, y_2 y_3, \ldots, y_{n-2}/4 y_{n+6}/4\} \). Let \( H \) be the subgraph of \( K_n[X] \) induced by \( M \), and let \( H' \) be the subgraph of \( K_n[Y] \) induced by \( M' \). Clearly, \( K_n[X] \) can be decomposed into \( H \) and \((n-2)/4\) copies of \( P_{n/2}, P(1), P(2), \ldots, P((n-2)/4) \), and \( K_n[Y] \) can be decomposed into \( H' \) and \((n-2)/4\) copies of \( P_{n/2}, P'(1), P'(2), \ldots, P'((n-2)/4) \).

Let \( Z = \{y_0 x_0, y_1 x_1\} \cup \{y_{i-1} x_{i-1}, y_i x_i\} \) if \( i \) is odd and \( i \geq 3 \} \cup \{y_{i+(n-6)/4} x_{i+(n-6)/4}, y_{i+(n-2)/4} x_{i+(n-2)/4}\} \) if \( i \) is even. Obviously, \( Z \subseteq E(C(0)) \). For \( i = 1, 2, \ldots, (n-2)/4 \), let \( K = \{y_{i+(n-6)/4} x_{i+(n-6)/4}, y_{i+(n-2)/4} x_{i+(n-2)/4}\} \)

\[
Q(i) = \begin{cases} 
  P(1) \cup P'(1) \cup \{y_0 x_0, y_1 x_1\} & \text{if } i = 1, \\
  P(i) \cup P'(i) \cup \{y_{i-1} x_{i-1}, y_i x_i\} & \text{if } i \text{ is odd and } i \geq 3, \\
  P(i) \cup P'(i) \cup K & \text{if } i \text{ is even},
\end{cases}
\]

and let \( Q((n+2)/4) = H \cup H' \cup C(0) - Z \). One can see that each \( Q(i) \) is an \( n \)-cycle. Thus \( K_n[X] \cup K_n[Y] \cup C(0) \) can be decomposed into \((n+2)/4\) copies of \( C_n \). For \( 1 \leq t \leq (n-6)/4 \), let

\[
R(t) = \left( \bigcup_{i=1}^{t} C((n-6)/4 - i + 1) \right) \cup F.
\]

It is easy to see that \( R(t) \) is isomorphic to the crown \( C_{n/2,2t+1} \). Hence \( K_n[X,Y] \) can be decomposed into \( n/2 - (t+1) \) copies of \( C_n, Q(1), Q(2), \ldots, Q((n+2)/4) \).
and $C(1), C(2), \ldots, C((n - 6)/4 - t)$, and one copy of $(n/2, 2t + 1)$-crown $R(t)$.

Note that $2t + 1 \geq 3$ and $|E(R(t))| = |E(C_{n/2,2t+1})| = (2t + 1)n/2$. If $(2t + 1)n/2 \equiv 0 \pmod{3}$, then $R(t)$ can be decomposed into $(2t + 1)n/6$ copies of $S_3$ by Proposition 19.

If $t = 1$, then $(2t + 1)n/2 = 3n/2 \equiv 0 \pmod{3}$ for each $n$. Thus $K_n$ can be decomposed into $n/2 - 2$ copies of $C_n$ and $n/2$ copies of $S_3$.

If $t = 2$, then $(2t + 1)n/2 = 5n/2 \equiv 0 \pmod{3}$ for $n \equiv 0 \pmod{6}$. Thus $K_n$ can be decomposed into $n/2 - 3$ copies of $C_n$ and $5n/6$ copies of $S_3$.

If $t = 3$, then $(2t + 1)n/2 = 7n/2 \equiv 0 \pmod{3}$ for $n \equiv 0 \pmod{6}$. Thus $K_n$ can be decomposed into $n/2 - 4$ copies of $C_n$ and $7n/6$ copies of $S_3$. This settles Case 2.

Let $x$ and $y$ be distinct vertices of a multigraph $G$. We use $e_G(x, y)$ to denote the number of edges joining $x$ and $y$. A star decomposition of $G$ is center balanced if every vertex of $G$ is the center of the same number of members in the decomposition.

**Proposition 21** [21]. Let $G$ be an $r$-regular multigraph with $r \geq 1$. Then $G$ has a center balanced $S_t$-decomposition if and only if $r \equiv 0 \pmod{2t}$ and $e_G(x, y) \leq r/t$ for all $x, y \in V(G)$ with $x \neq y$.

**Lemma 22.** Let $n$ and $t$ be positive integers. If $Q_1, Q_2, \ldots, Q_t$ are edge-disjoint Hamiltonian cycles of $K_n$, then $\bigcup_{i=1}^t Q_i$ is $S_t$-decomposable.

**Proof.** Since each $Q(i)$ is 2-regular and $V(Q(i)) = V(Q(j))$ for $i, j \in \{1, 2, \ldots, t\}$, $\bigcup_{i=1}^t Q_i$ is 2t-regular. Since $2t \equiv 0 \pmod{2t}$ and $e_{\bigcup_{i=1}^t Q_i}(x, y) \leq 1 < (2t)/t$ for all $x, y \in V(\bigcup_{i=1}^t Q_i)$ with $x \neq y$, the result follows from Proposition 21.

By Lemma 22, the union of $3t$ copies of edge-disjoint $n$-cycles can be decomposed into $n$ copies of $S_{3t}$, in turn, each $S_{3t}$ can be decomposed into $t$ copies of $S_3$. Hence we have the following result.

**Theorem 23.** Let $n$, $p$ and $t$ be positive integers with $p \geq 3t$, and let $q$ be a nonnegative integer. If $K_n$ can be decomposed into $p$ copies of $C_n$ and $q$ copies of $S_3$, then $K_n$ can be decomposed into $p - 3t$ copies of $C_n$ and $q + nt$ copies of $S_3$.

Obviously, if $K_n$ can be decomposed into $\alpha$ copies of $C_n$ and $\beta$ copies of $S_3$, then $\binom{n}{3} = n\alpha + 3\beta$. Using Theorem 23 together with Lemmas 15 to 20, we have the main result of this section.

**Theorem 24.** Let $n$, $\alpha$ and $\beta$ be positive integers. The complete graph $K_n$ can be decomposed into $\alpha$ copies of $C_n$ and $\beta$ copies of $S_3$ if and only if $\binom{n}{3} = n\alpha + 3\beta$ and $\alpha \neq (n - 3)/2$ for $n \equiv 3 \pmod{6}$ and $\alpha \neq (n - 2)/2$ for $n \equiv 0 \pmod{6}$. 
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