ASYMPTOTIC BEHAVIOR OF THE EDGE METRIC DIMENSION OF THE RANDOM GRAPH

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Abstract

Given a simple connected graph $G(V,E)$, the edge metric dimension, denoted $edim(G)$, is the least size of a set $S \subseteq V$ that distinguishes every pair of edges of $G$, in the sense that the edges have pairwise different tuples of distances to the vertices of $S$. In this paper we prove that the edge metric dimension of the Erdős-Rényi random graph $G(n,p)$ with constant $p$ is given by

$$edim(G(n,p)) = (1 + o(1)) \frac{4 \log n}{\log(1/q)},$$

where $q = 1 - 2p(1-p)^2(2-p)$.

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1. Introduction

Let $G(V,E)$ be a finite, simple, connected graph, and define the distance $d(x,y)$ between two vertices $x, y \in V$ to be the length of the shortest path connecting $x$ and $y$. The metric dimension of $G(V,E)$, denoted $\text{dim}(G(V,E))$, is the minimal cardinality of a set $S \subseteq V$ such that for any distinct $x, y \in V$ there exists $v \in S$ which satisfies $d(v,x) \neq d(v,y)$.

The metric dimension was introduced by Slater [12] in 1975 in connection with the problem of uniquely recognizing the location of an intruder in a network, and independently by Harary and Melter in [4] a year later. Graphs with $\text{dim}(G) = 1$ and 2 were characterized in [8], and graphs with $\text{dim}(G) = |V| - 1$ and $|V| - 2$ were described in [3]. This graph invariant is useful in areas like robot navigation [8], image processing [10], and chemistry [2, 3, 6].
In [1], Bollobás, Mitsche and Pralat computed the asymptotic behavior at infinity of the metric dimension of the Erdős-Rényi random graph for a wide range of probabilities \( p(n) \) (viewed as functions of \( n \)). For instance, for constant \( p \in (0, 1) \), it was shown that

\[
\dim(G(n, p)) = (1 + o(1)) \frac{2 \log n}{\log(1/Q)},
\]

where \( Q = p^2 + (1 - p)^2 \). In this paper we generalize those calculations to a variation on the metric dimension called the edge metric dimension, introduced by Kelenc, Tratnik and Yero in [7] in 2016. While the metric dimension is about uniquely identifying the vertices of a graph in terms of distances to a set, the edge metric dimension is about identifying the edges of a graph in the same way.

For an edge \( e = xy \in E \) and a vertex \( v \in V \), let \( d(e, v) = \min\{d(x, v), d(y, v)\} \). The edge metric dimension (denoted edim) of a graph \( G(V, E) \) is defined as the minimal cardinality of a set \( S \subseteq V \) such that for any distinct \( e_1, e_2 \in E \), there exists \( v \in S \) satisfying \( d(v, e_1) \neq d(v, e_2) \).

Kelenc, Tratnik and Yero computed the edge metric dimension of a range of families of graphs, showed \( \text{edim}(G) \) can be less, equal to, or more than \( \dim(G) \), and showed computing \( \text{edim}(G) \) is NP-hard in general ([7]). Zubrilina ([13]) showed that the \( \text{edim}(G) / \dim(G) \) ratio is not bounded from above and classified graphs \( G \) with \( \text{edim}(G) = |V| - 1 \). Kratica, Filipović and Kartelj studied the edge metric dimension of the generalized Petersen graph \( GP(n,k) \) in [9]. In this paper, we prove the following theorem.

**Theorem 1.1.** Let \( G(n, p) \) be the Erdős-Rényi random graph with constant \( p \). Then

\[
\text{edim}(G(n, p)) = (1 + o(1)) \frac{4 \log n}{\log(1/q)},
\]

where \( q = 1 - 2p(1 - p)^2(2 - p) \).

For a set \( R = \{r_1, \ldots, r_{|R|}\} \subseteq V \), we define the distance tuple \( d_R : V \cup E \to \mathbb{N}^{|R|} \) via \( (d_R(x))_i = d(x, r_i) \). We say \( R \) distinguishes \( v_1, v_2 \in V \) if \( d_R(v_1) \neq d_R(v_2) \), and similarly that \( R \) distinguishes \( e_1, e_2 \in E \) if \( d_R(e_1) \neq d_R(e_2) \). \( R \) is a generating set of \( G \) if it distinguishes any two distinct vertices, and an edge generating set if it distinguishes any two distinct edges of \( G \).

We say \( f(n) = \mathcal{O}(g(n)) \) if there exists a constant \( C > 0 \) such that \( |f(n)| \leq C |g(n)| \), and \( f(n) = o(g(n)) \) if \( f = g \cdot o(1) \), where \( o(1) \xrightarrow{n \to \infty} 0 \).

We say a property holds asymptotically almost surely (denoted a.a.s.) for the random graph if the probability that it holds for \( G(n,p) \) goes to 1 as \( n \) goes to infinity. We denote probability with \( \mathbb{P} \) and expected value with \( \mathbb{E} \). All the graphs are assumed to be finite, simple, connected and undirected.
2. The Upper Bound

In this section we prove the following theorem.

**Theorem 2.1.** For the random graph $G(n, p)$ with $p$ constant, we have

$$\text{edim}(G(n, p)) \leq (1 + o(1)) \frac{4 \log n}{\log(1/q)},$$

where $q = 1 - 2p(1 - p)^2(2 - p)$.

In order to prove Theorem 2.1, we will need some lemmas.

**Lemma 2.2.** Let $G = G(n, p)$ be the random graph, and let $V,E$ denote its vertex and edge sets. Let $\omega \in \{1, \ldots, n\}$ be such that for any two distinct edges $e_1, e_2 \in E$, a uniformly random subset $W \subseteq V$ of size $|W| = \omega$ satisfies

$$P(W \text{ does not distinguish } e_1, e_2) \leq \frac{1}{n^4 p^2}.$$

Then

$$\text{edim}(G) \leq \omega.$$

**Proof.** We use the probabilistic method. Note that

$$\mathbb{E}[|E|] = p\binom{n}{2} < pn^2/2,$$

so the expected number of distinct pairs of edges is no more than $\binom{pn^2/2}{2} \leq p^2 n^4/8$. Then by our hypothesis the expected number of pairs not distinguished by some $W \subseteq V$ with $|W| = \omega$ is less than $p^2 n^4/8 p^2 n^4 = 1/8$. Since this is strictly less than 1, there must be at least one such set $W$ that distinguishes all the pairs.

**Lemma 2.3.** In $G(n, p)$, the probability that a vertex $v$ doesn’t distinguish two uniformly random edges $e_1, e_2$ is $(1 + o(1))q$, where $q = 1 - 2p(1 - p)^2(2 - p)$.

**Proof.** There are two types of distinct edge pairs.
1. $ab, bc$ for some $a, b, c \in V$.  
2. $ab, cd$ for $a, b, c, d \in V$ and $\{a, b\} \cap \{c, d\} = \emptyset$.

Note that

the expected number of type 2 pairs $= 3 \binom{n}{4} p^2 = \frac{n^4 p^2}{8}(1 + o(1))$,

and

the expected number of type 1 pairs $\leq n^3 = o\left(\frac{n^4 p^2}{8}\right)$.
Thus, we can neglect the type 1 pairs. Let $xy, zt$ be a type 2 pair and $v$ a uniformly random vertex. Clearly, $P(v \in \{x, y, z, t\}) = o\left(\frac{n^4p^2}{8}\right)$, so we can assume $v$ is not a vertex of $xy$ or $zt$. Since the random graph has diameter 2 a.a.s. (see [11]), $v$ has distance 1 or 2 to $x, y, z, t$ a.a.s.; moreover, $P(d(v, x) = 1) = p$, so a.a.s. $P(d(v, x) = 2) = 1 - p$. It is easy to see that $v$ has distance 1 to $xy$ and 2 to $zt$ if and only if one of the following cases holds.

1. $(d(v, x), d(v, y), d(v, z), d(v, t)) = (1, 1, 2, 2)$ (with probability $p^2(1 - p)^2$).
2. $(d(v, x), d(v, y), d(v, z), d(v, t)) = (1, 2, 2, 2)$ (with probability $p(1 - p)^3$).
3. $(d(v, x), d(v, y), d(v, z), d(v, t)) = (2, 1, 2, 2)$ (with probability $p(1 - p)^3$).

The same probabilities hold for $xy$ and $zt$ switched. Thus, a.a.s.

\[
P(v \text{ distinguishes } xy, zt) = (1 + o(1)) \cdot 2(p^2(1 - p)^2 + 2p(1 - p)^3) = (1 + o(1)) \cdot 2p(1 - p)^2(2 - p) = (1 + o(1))(1 - q).
\]

This gives us the desired result.

\begin{lemma}
Let $V, E$ be the vertex and edge sets of $G(n, p)$. Consider a uniformly random subset $W \subseteq V$ with $|W| = (1 + o(1)) \frac{4 \log n}{\log(1/q)}$. Then for uniformly random $e_1$ and $e_2 \in E$,

\[
P(W \text{ does not distinguish } e_1, e_2) \leq (1 + o(1))/n^4p^2.
\]

\begin{proof}
Using Lemma 2.3, we see that

\[
P(W \text{doesn’t distinguish } e_1, e_2) \\
\leq (1 + o(1))P(\text{uniformly random vertex } v \text{ doesn’t distinguish } e_1, e_2)^{|W|} \\
\leq (1 + o(1))q^{1+o(1)}\frac{4\log n}{\log(1/q)} = (1 + o(1))q^{-\log_q(n^4)} \\
= (1 + o(1))\frac{1}{n^4} \leq (1 + o(1))\frac{1}{p^2n^4}.
\]

\end{proof}

\begin{proof}[of Theorem 2.1] Combining Lemmas 2.4 and 2.2, we see that $\text{edim}(G(n, p))$ is at most

\[
(1 + o(1))\frac{4 \log n}{\log(1/q)}.
\]

which concludes the proof of Theorem 2.1.

\end{proof}
3. The Lower Bound

The goal of this section is to prove the following theorem.

**Theorem 3.1.** For the random graph $G(n, p)$ with $p$ constant, we have

$$\text{edim}(G(n, p)) \geq (1 + o(1)) \frac{4 \log n}{\log(1/q)},$$

where $q = 1 - 2p(1 - p)^2(2 - p)$.

Let

$$\varepsilon := 3 \log \log n \log n = o(1).$$

We will show that a.a.s. there is no edge generating set $R$ of cardinality less than

$$r := \left(4 - \varepsilon\right) \frac{\log n}{\log(1/q)}.$$

To do that we will use a theorem which is a version of Suen’s inequality demonstrated by Janson in [5]. First we introduce some notation

- $\{I_i\}_{i \in \mathcal{I}}$ — a finite family of indicator random variables;
- $\Gamma$ — the associated dependency graph ($\mathcal{I}$ is the set of vertices of $\Gamma$);
- For $i, j \in \mathcal{I}$, write $i \sim j$ if $i, j$ are adjacent in $\Gamma$;
- $\mu := \sum_i \mathbb{P}(I_i = 1)$;
- $\Delta := \sum_{i \sim j} \mathbb{E}[I_i I_j]$;
- $\delta := \max_i \sum_{i \sim j} \mathbb{P}(I_j)$;
- $S := \sum_i I_i$.

**Theorem 3.2** (Suen’s inequality, Theorem 2 of [5]).

$$\mathbb{P}(S = 0) \leq \exp \left(-\mu + \Delta \varepsilon^2\right).$$

We now apply this theorem to our problem.

Let $V, E$ be the vertex and edge sets of $G(n, p)$. Let $R \subseteq V$ with $|R| = r$. Let

$$\mathcal{I} := \{(xy, zt) \mid xy, zt \in E, xy \neq zt\}$$

be the set of pairs of distinct edges, and for any $(xy, zt) \in \mathcal{I}$ let $A_{xy,zt}$ be the event $d_R(xy) = d_R(zt)$ (with $I_{xy,zt}$ being the corresponding indicator function). Let $S = \sum_{(xy, zt) \in \mathcal{I}} I_{xy,zt}$. Then

$$\mathbb{P}(R \text{ is an edge generating set}) = \mathbb{P}(S = 0).$$
The associated dependency graph has \( I \) as vertices and \((x_1y_1, z_1t_1) \sim (x_2y_2, z_2t_2)\) if and only if \( \{x_1, y_1, z_1, t_1\} \cap \{x_2, y_2, z_2, t_2\} \neq \emptyset \) (here, again, \( \sim \) denotes adjacency). Then by Theorem 3.2,

\[
P(S = 0) \leq \exp(-\mu + \Delta \varepsilon^2),
\]

(1)

where

\[
\mu = \sum_{(e,f) \in I} P(A_{e,f}),
\]

\[
\Delta = \sum_{(e_1,f_1) \sim (e_2,f_2)} E[I_{e_1}I_{f_1}I_{e_2}I_{f_2}],
\]

\[
\delta = \max_{(e_1,f_1) \in I} \sum_{(e_2,f_2) \sim (e_1,f_1)} P(A_{e_2,f_2}).
\]

We now show the following estimate for \( \mu \).

**Lemma 3.3** (Evaluation of \( \mu \)).

\[
\mu = (1 + o(1))p^2n^\varepsilon/8.
\]

**Proof.** Using Lemma 2.3, we can derive that that

\[
P(A_{e,f}) = (1 + o(1))q^r,
\]

so, since the expected number of pairs is \((1 + o(1))(n^4p^2/8)\), we indeed get

\[
\mu = (1 + o(1))n^4p^2q^r/8.
\]

Since \( r = \frac{(4-\varepsilon)\log n}{\log(1/q)} \),

\[
q^r = q^{-(4-\varepsilon)\log_q(n)} = n^{-\varepsilon^4}.
\]

Thus,

\[
(1 + o(1))n^4p^2q^r/8 = (1 + o(1))n^4p^2n^{-\varepsilon^4}/8 = (1 + o(1))p^2n^\varepsilon/8.
\]

This means that, indeed,

\[
\mu = (1 + o(1))p^2n^\varepsilon/8.
\]

\[\blacksquare\]

Now we estimate \( \Delta \) and show the following.

**Lemma 3.4** (Evaluation of \( \Delta \)).

\[
\Delta = o(\mu).
\]
Proof. 

Claim 3.5. In calculating $\Delta$, we may only consider the adjacent pairs 

$$(x_1 y_1, z_1 t_1), (x_2 y_2, z_2 t_2) \in \mathcal{I}$$

for which

$$|\{(x_1, y_1, z_1, t_1) \cap (x_2, y_2, z_2, t_2)\} = 1.$$

Proof. Consider two adjacent elements of $\mathcal{I}$: $(x_1 y_1, z_1 t_1) \sim (x_2 y_2, z_2 t_2)$. Suppose $|\{(x_1, y_1, z_1, t_1, x_2, y_2, z_2, t_2)\} = 7$. The expected number of such pairs is

$$p^4 \frac{n!}{4 \cdot (n-7)!} = (1 + o(1))p^4 n^7 / 4.$$ 

Now consider two adjacent elements of $\mathcal{I}$ with $|\{(x_1, y_1, z_1, t_1, x_2, y_2, z_2, t_2)\} \leq 6$. There are no more than

$$n^6 = o(p^4 n^7)$$

such pairs of pairs. 

Thus we can and will only consider pairs of elements of $\mathcal{I}$ with only one vertex in common.

We will now compute the probability that $I_{(x_1 y_1, z_1 t_1)} I_{(x_1 y_2, z_2 t_2)} = 1$. Consider a uniformly random vertex $v$. We can neglect the case when $v \in \{x_1, y_1, z_1, t_1, y_2, z_2, t_2\}$ because it happens with probability $o(1)$. Since the random graph has diameter 2 a.a.s., $I_{(x_1 y_1, z_1 t_1)} I_{(x_1 y_2, z_2 t_2)} = 1$ in the following cases.

Case 1. $d_v(x_1) = 1$. Then $v$ has to have distance 1 to all four edges. $v$ has distance 1 to $z_1 t_1$ (or $z_2 t_2$) with probability $p^2 + 2p(1 - p) = p(2 - p)$, and the distances from $v$ to $y_1, y_2$ don’t affect anything, so

$$\mathbb{P}(I_{(x_1 y_1, z_1 t_1)} I_{(x_1 y_2, z_2 t_2)} = 1 \mid \text{Case 1 holds}) = p^3(2 - p)^2.$$ 

Case 2. $d_v(x_1) = 2$. Then $v$ has distance 2 to both $x_1 y_1$ and $z_1 t_1$ with probability $(1 - p)^3$ and distance 1 to both $x_1 y_1$ and $z_1 t_1$ with probability $p^2(2 - p)$. So $v$ is equidistant from the two edges with probability $(1 - p)^3 + p^2(2 - p)$. Thus,

$$\mathbb{P}(I_{(x_1 y_1, z_1 t_1)} I_{(x_1 y_2, z_2 t_2)} = 1 \mid \text{Case 2 holds}) = (1 - p)((1 - p)^3 + p^2(2 - p))^2.$$ 

Hence the total probability

$$\mathbb{P}(I_{(x_1 y_1, z_1 t_1)} I_{(x_1 y_2, z_2 t_2)} = 1) = (1 - p)((1 - p)^3 + p^2(2 - p))^2 + p^3(2 - p)^2.$$ 

We will henceforth refer to this constant as $s_p$.

$$s_p := (1 - p)((1 - p)^3 + p^2(2 - p))^2 + p^3(2 - p)^2.$$
It follows that
\[ \Delta = (1 + o(1))p^4n^7s_p^r/4. \]

Using (2), we get
\[ \Delta = (1 + o(1))p^4n^7s_p^r/4 = (1 + o(1))p^4n^3n^r s_p^{4-r}/4 \]
\[ = (1 + o(1))2p^2n^3 \left( \frac{sp}{q} \right)^r \frac{p^2n^r}{8} = (1 + o(1))2p^2n^3 \left( \frac{sp}{q} \right)^r \mu. \]

Notice that
\[ \left( \frac{sp}{q} \right)^r = \left( \frac{sp}{q} \right)^{(4-\varepsilon) \log n/\log(1/q)} = n^{(4-\varepsilon) \log \left( \frac{sp}{q} \right)/\log(1/q)} \]
\[ = n^{(4-\varepsilon) \left( -\log(q/s_p) + 1 \right)} = n^{(4-\varepsilon)(-\log q + s_p + 1)} \leq n^{\varepsilon - 4} \]
(since \( q, s_p \leq 1 \)). Thus,
\[ (1 + o(1))2p^2n^3 \left( \frac{sp}{q} \right)^r \mu \leq (1 + o(1))2p^2n^3n^{\varepsilon - 4} \mu = o(\mu). \]

This concludes the proof that
\[ \Delta = o(\mu). \]

Finally, we estimate \( \delta \) and show the following.

**Lemma 3.6** (Evaluation of \( \delta \)).
\[ \delta = o(1). \]

**Proof.** Note that for fixed \( f_1, e_1, \)
\[ \mathbb{P}(A_{e_2,f_2} \mid (e_2, f_2) \text{ uniformly random}, (e_2, f_2) \sim (e_1, f_1)) \]
\[ = \mathbb{P}(A_{e,f} \mid e, f \text{ uniformly random}). \]

Thus, the maximum for \( \delta \) is achieved for \( (e_1, f_1) \) with the largest possible number of adjacent edge pairs \( (e_2, f_2) \). Clearly, this number is the greatest when \( e_1 \) and \( f_1 \) don’t share vertices. The expected number of adjacent edge pairs in this case is \( (1 + o(1))2n^3p^2 \). Since \( q^r = \mathbb{P}(A_{e,f}) \) for uniformly random edges \( e, f \) we have
\[ 2\delta = (1 + o(1))2n^3p^2 q^r. \]

Using (2), we get
\[ \delta = (1 + o(1))2p^2n^{\varepsilon - 1} = o(1). \]
We are now ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** Substituting the results of Lemmas 3.3, 3.4, 3.6 into inequality (1), we obtain

\[
\log (\mathbb{P}(S = 0)) \leq (1 + o(1)) \left( -\mu + o(\mu) e^{o(1)} \right) \leq (1 + o(1)) ( -\mu + o(\mu)) \\
\leq -(1 + o(1)) \mu \leq -(1 + o(1)) p^2 n^\varepsilon /8 \leq -p^2 n^\varepsilon /16
\]

for sufficiently large \( n \). Then the expected number of edge generating sets of cardinality \( r \) is no greater than

\[
\binom{n}{r} \exp(-p^2 n^\varepsilon /16) \leq n^r \exp(-p^2 n^\varepsilon /16) \\
= \mathcal{O}(\exp(4 - \varepsilon) \log^2 n / \log(1/q) - p^2 n^\varepsilon /16) \\
\leq \mathcal{O}(\exp[\log^2 n - \log^3(n)p^2 /16]) = o(1).
\]

This concludes the proof of Theorem 3.1, and together with Theorem 2.1, this proves the main result, Theorem 1.1.

\[\Box\]

4. **Concluding Remarks**

We have shown that

\[\text{edim}(G(n,p)) = (1 + o(1)) \frac{4 \log n}{\log(1/q)},\]

where

\[q = 1 - 2p(1-p)^2(2-p).\]

As demonstrated by Bollobas et al. in [1],

\[\text{dim}(G(n,p)) = (1 + o(1)) \frac{2 \log n}{\log(1/Q)},\]

where \( Q = p^2 + (1-p)^2 \). Since \( 2/\log(1/Q) < 4/\log(1/q) \), this means that

\[\text{dim}(G(n,p)) < \text{edim}(G(n,p))\]

a.a.s. for all \( p \in (0,1) \).

While random graphs with constant edge probability don’t help in resolving the problem of finding more examples of graphs \( G \) for which \( \text{edim}(G) < \text{dim}(G) \) posed in [7], perhaps this problem could be addressed with random graphs of
non-constant probability $p(n)$. Because of this it would be interesting to calculate $\text{edim}(G(n, p(n)))$ for non-constant $p(n)$. The analogous results for $\dim(G(n, p(n)))$ can be found in [1].

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