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ASYMPTOTIC BEHAVIOR OF THE EDGE METRIC DIMENSION OF THE RANDOM GRAPH

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Abstract

Given a simple connected graph $G(V, E)$, the edge metric dimension, denoted $\operatorname{edim}(G)$, is the least size of a set $S \subseteq V$ that distinguishes every pair of edges of G, in the sense that the edges have pairwise different tuples of distances to the vertices of S. In this paper we prove that the edge metric dimension of the Erdős-Rényi random graph $G(n, p)$ with constant p is given by

$$\operatorname{edim}(G(n,p)) = (1 + o(1))\frac{4\log n}{\log(1/q)},$$

where $q = 1 - 2p(1-p)^2(2-p)$.

Keywords: random graph, edge dimension, Suen's inequality.

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1. Introduction

Let $G(V, E)$ be a finite, simple, connected graph, and define the *distance* $d(x, y)$ between two vertices $x, y \in V$ to be the length of the shortest path connecting x and y. The *metric dimension* of $G(V, E)$, denoted $\dim(G(V, E))$, is the minimal cardinality of a set $S \subseteq V$ such that for any distinct $x, y \in V$ there exists $v \in S$ which satisfies $d(v, x) \neq d(v, y)$.

The metric dimension was introduced by Slater [12] in 1975 in connection with the problem of uniquely recognizing the location of an intruder in a network, and independently by Harary and Melter in [4] a year later. Graphs with $\dim(G) = 1$ and 2 were characterized in [8], and graphs with $\dim(G) = |V| - 1$ and $|V| - 2$ were described in [3]. This graph invariant is useful in areas like robot navigation [8], image processing [10], and chemistry [2, 3, 6].

In [1], Bollobás, Mitsche and Pralat computed the asymptotic behavior at infinity of the metric dimension of the Erdős-Rényi random graph for a wide range of probabilities $p(n)$ (viewed as functions of n). For instance, for constant $p \in (0,1)$, it was shown that

$$\dim(G(n,p)) = (1+o(1))\frac{2\log n}{\log(1/Q)},$$

where $Q = p^2 + (1-p)^2$. In this paper we generalize those calculations to a variation on the metric dimension called the *edge metric dimension*, introduced by Kelenc, Tratnik and Yero in [7] in 2016. While the metric dimension is about uniquely identifying the vertices of a graph in terms of distances to a set, the edge metric dimension is about identifying the edges of a graph in the same way.

For an edge $e = xy \in E$ and a vertex $v \in V$, let $d(e,v) = \min\{d(x,v), d(y,v)\}$. The *edge metric dimension* (denoted edim) of a graph $G(V,E)$ is defined as the minimal cardinality of a set $S \subseteq V$ such that for any distinct $e_1, e_2 \in E$, there exists $v \in S$ satisfying $d(v,e_1) \neq d(v,e_2)$.

Kelenc, Tratnik and Yero computed the edge metric dimension of a range of families of graphs, showed $\operatorname{edim}(G)$ can be less, equal to, or more than $\dim(G)$, and showed computing $\operatorname{edim}(G)$ is NP-hard in general ([7]). Zubrilina ([13]) showed that the $\operatorname{edim}(G)/\dim(G)$ ratio is not bounded from above and classified graphs G with $\operatorname{edim}(G) = |V| - 1$. Kratica, Filipović and Kartelj studied the edge metric dimension of the generalized Petersen graph $GP(n,k)$ in [9]. In this paper, we prove the following theorem.

Theorem 1.1. *Let $G(n,p)$ be the Erdős-Rényi random graph with constant p. Then*

$$\operatorname{edim}(G(n,p)) = (1+o(1))\frac{4\log n}{\log(1/q)},$$

where $q = 1 - 2p(1-p)^2(2-p)$.

For a set $R = \{r_1, \dots, r_{|R|}\} \subseteq V$, we define the distance tuple $d_R : V \cup E \to \mathbb{N}^{|R|}$ via $(d_R(x))_i = d(x, r_i)$. We say R distinguishes $v_1, v_2 \in V$ if $d_R(v_1) \neq d_R(v_2)$, and similarly that R distinguishes $e_1, e_2 \in E$ if $d_R(e_1) \neq d_R(e_2)$. R is a *generating set* of G if it distinguishes any two distinct vertices, and an *edge generating set* if it distinguishes any two distinct edges of G.

We say $f(n) = \mathcal{O}(g(n))$ if there exists a constant $C > 0$ such that $|f(n)| \leq C\,|g(n)|$, and $f(n) = o(g(n))$ if $f = g \cdot o(1)$, where $o(1) \underset{n\to\infty}{\longrightarrow} 0$.

We say a property holds *asymptotically almost surely* (denoted a.a.s.) for the random graph if the probability that it holds for $G(n,p)$ goes to 1 as n goes to infinity. We denote probability with $\mathbb{P}$ and expected value with $\mathbb{E}$. All the graphs are assumed to be finite, simple, connected and undirected.

2. The Upper Bound

In this section we prove the following theorem.

Theorem 2.1. *For the random graph $G(n,p)$ with p constant, we have*

$$\mathrm{edim}(G(n,p)) \le (1+o(1))\frac{4\log n}{\log(1/q)},$$

where $q = 1-2p(1-p)^2(2-p)$.

In order to prove Theorem 2.1, we will need some lemmas.

Lemma 2.2. *Let $G = G(n,p)$ be the random graph, and let V, E denote its vertex and edge sets. Let $\omega \in \{1,\dots,n\}$ be such that for any two distinct edges $e_1, e_2 \in E$, a uniformly random subset $W \subseteq V$ of size $|W| = \omega$ satisfies*

$$\mathbb{P}(W \textit{ does not distinguish } e_1, e_2) \le 1/n^4p^2.$$

Then

$$\mathrm{edim}(G) \le \omega.$$

Proof. We use the probabilistic method. Note that

$$\mathbb{E}[|E|] = p\binom{n}{2} < pn^2/2,$$

so the expected number of distinct pairs of edges is no more than $\binom{pn^2/2}{2} \le p^2n^4/8$. Then by our hypothesis the expected number of pairs not distinguished by some $W \subseteq V$ with $|W| = \omega$ is less than $p^2n^4/8p^2n^4 = 1/8$. Since this is strictly less than 1, there must be at least one such set W that distinguishes all the pairs. ■

Lemma 2.3. *In $G(n,p)$, the probability that a vertex v doesn't distinguish two uniformly random edges e_1, e_2 is $(1+o(1))q$, where $q = 1-2p(1-p)^2(2-p)$.*

Proof. There are two types of distinct edge pairs.

1. ab, bc for some $a, b, c \in V$.
2. ab, cd for $a, b, c, d \in V$ and $\{a,b\} \cap \{c,d\} = \emptyset$.

Note that

$$\text{the expected number of type 2 pairs} = 3\binom{n}{4}p^2 = \frac{n^4p^2}{8}(1+o(1)),$$

and

$$\text{the expected number of type 1 pairs} \le n^3 = o\left(\frac{n^4p^2}{8}\right).$$

Thus, we can neglect the type 1 pairs. Let xy, zt be a type 2 pair and v a uniformly random vertex. Clearly, $\mathbb{P}(v \in \{x, y, z, t\}) = o\left(\frac{n^4p^2}{8}\right)$, so we can assume v is not a vertex of xy or zt. Since the random graph has diameter 2 a.a.s. (see [11]), v has distance 1 or 2 to x, y, z, t a.a.s.; moreover, $\mathbb{P}(d(v, x) = 1) = p$, so a.a.s. $\mathbb{P}(d(v, x) = 2) = 1 - p$. It is easy to see that v has distance 1 to xy and 2 to zt if and only if one of the following cases holds.

1. $(d(v, x), d(v, y), d(v, z), d(v, t)) = (1, 1, 2, 2)$ (with probability $p^2(1 - p)^2$).
2. $(d(v, x), d(v, y), d(v, z), d(v, t)) = (1, 2, 2, 2)$ (with probability $p(1 - p)^3$).
3. $(d(v, x), d(v, y), d(v, z), d(v, t)) = (2, 1, 2, 2)$ (with probability $p(1 - p)^3$).

The same probabilities hold for xy and zt switched. Thus, a.a.s.

$$\begin{aligned}\mathbb{P}(v \text{ distinguishes } xy, zt) &= (1 + o(1)) \cdot 2(p^2(1 - p)^2 + 2p(1 - p)^3) \\ &= (1 + o(1)) \cdot 2p(1 - p)^2(2 - p) = (1 + o(1))(1 - q).\end{aligned}$$

This gives us the desired result. ■

Lemma 2.4. *Let V, E be the vertex and edge sets of $G(n, p)$. Consider a uniformly random subset $W \subseteq V$ with*

$$|W| = (1 + o(1))\frac{4 \log n}{\log(1/q)}.$$

Then for uniformly random e_1 and $e_2 \in E$,

$$\mathbb{P}(W \textit{ does not distinguish } e_1, e_2) \leq (1 + o(1))/n^4p^2.$$

Proof. Using Lemma 2.3, we see that

$$\begin{aligned}&\mathbb{P}(W \text{doesn't distinguish } e_1, e_2) \\ &\leq (1 + o(1))\mathbb{P}(\text{uniformly random vertex } v \text{ doesn't distinguish } e_1, e_2)^{|W|} \\ &\leq (1 + o(1))q^{(1+o(1))\frac{4 \log n}{\log(1/q)}} = (1 + o(1))q^{-\log_q(n^4)} \\ &= (1 + o(1))\frac{1}{n^4} \leq (1 + o(1))\frac{1}{p^2n^4}.\end{aligned}$$

■

***Proof of Theorem* 2.1.** Combining Lemmas 2.4 and 2.2, we see that $\mathrm{edim}(G(n, p))$ is at most

$$(1 + o(1))\frac{4 \log n}{\log(1/q)},$$

which concludes the proof of Theorem 2.1. ■

3. The Lower Bound

The goal of this section is to prove the following theorem.

Theorem 3.1. *For the random graph $G(n,p)$ with p constant, we have*

$$\operatorname{edim}(G(n,p)) \geq (1+o(1))\frac{4\log n}{\log(1/q)},$$

where $q = 1-2p(1-p)^2(2-p)$.

Let

$$\varepsilon := \frac{3\log\log n}{\log n} = o(1).$$

We will show that a.a.s. there is no edge generating set R of cardinality less than

$$r := \frac{(4-\varepsilon)\log n}{\log(1/q)}.$$

To do that we will use a theorem which is a version of Suen's inequality demonstrated by Janson in [5]. First we introduce some notation

- $\{I_i\}_{i\in\mathcal{I}}$ — a finite family of indicator random variables;
- Γ — the associated dependency graph ($\mathcal{I}$ is the set of vertices of Γ);
- For $i,j \in \mathcal{I}$, write $i \sim j$ if i,j are adjacent in Γ;
- $\mu := \sum_i \mathbb{P}(I_i = 1)$;
- $\Delta := \sum_{i\sim j} \mathbb{E}[I_i I_j]$;
- $\delta := \max_i \sum_{i\sim j} \mathbb{P}(I_j)$;
- $S := \sum_i I_i$.

Theorem 3.2 (Suen's inequality, Theorem 2 of [5])**.**

$$\mathbb{P}(S=0) \leq \exp\left(-\mu + \Delta\varepsilon^{2\delta}\right).$$

We now apply this theorem to our problem.

Let V, E be the vertex and edge sets of $G(n,p)$. Let $R \subseteq V$ with $|R| = r$. Let

$$\mathcal{I} := \{(xy,zt) \mid xy,zt \in E, xy \neq zt\}$$

be the set of pairs of distinct edges, and for any $(xy,zt)\in\mathcal{I}$ let $A_{xy,zt}$ be the event $d_R(xy) = d_R(zt)$ (with $I_{xy,zt}$ being the corresponding indicator function). Let $S = \sum_{(xy,zt)\in\mathcal{I}} I_{xy,zt}$. Then

$$\mathbb{P}(R \text{ is an edge generating set}) = \mathbb{P}(S=0).$$

The associated dependency graph has $\mathcal{I}$ as vertices and $(x_1y_1, z_1t_1) \sim (x_2y_2, z_2t_2)$ if and only if $\{x_1, y_1, z_1, t_1\} \cap \{x_2, y_2, z_2, t_2\} \neq \emptyset$ (here, again, $\sim$ denotes adjacency). Then by Theorem 3.2,

$$\mathbb{P}(S = 0) \leq \exp(-\mu + \Delta\varepsilon^{2\delta}), \tag{1}$$

where

$$\mu = \sum_{(e,f)\in\mathcal{I}} \mathbb{P}(A_{e,f}),$$
$$\Delta = \sum_{(e_1,f_1)\sim(e_2,f_2)} \mathbb{E}[I_{e_1f_1}I_{e_2f_2}],$$
$$\delta = \max_{(e_1,f_1)\in\mathcal{I}} \sum_{(e_2,f_2)\sim(e_1,f_1)} \mathbb{P}(A_{e_2,f_2}).$$

We now show the following estimate for μ.

Lemma 3.3 (Evaluation of μ)**.**

$$\mu = (1 + o(1))p^2n^\varepsilon/8.$$

Proof. Using Lemma 2.3, we can derive that that

$$\mathbb{P}(A_{e,f}) = (1 + o(1))q^r,$$

so, since the expected number of pairs is $(1 + o(1))(n^4p^2/8)$, we indeed get

$$\mu = (1 + o(1))n^4p^2q^r/8.$$

Since $r = \frac{(4-\varepsilon)\log n}{\log(1/q)}$,

$$q^r = q^{-(4-\varepsilon)\log_q(n)} = n^{\varepsilon-4}. \tag{2}$$

Thus,

$$(1 + o(1))n^4p^2q^r/8 = (1 + o(1))n^4p^2n^{\varepsilon-4}/8 = (1 + o(1))p^2n^\varepsilon/8.$$

This means that, indeed,

$$\mu = (1 + o(1))p^2n^\varepsilon/8.$$

∎

Now we estimate Δ and show the following.

Lemma 3.4 (Evaluation of Δ)**.**

$$\Delta = o(\mu).$$

Proof.

Claim 3.5. *In calculating Δ, we may only consider the adjacent pairs*

$$(x_1y_1, z_1t_1), (x_2y_2, z_2t_2) \in \mathcal{I}$$

for which

$$|\{x_1, y_1, z_1, t_1\} \cap \{x_2, y_2, z_2, t_2\}| = 1.$$

Proof. Consider two adjacent elements of $\mathcal{I}$: $(x_1y_1, z_1t_1) \sim (x_2y_2, z_2t_2)$. Suppose $|\{x_1, y_1, z_1, t_1, x_2, y_2, z_2, t_2\}| = 7$. The expected number of such pairs is

$$p^4 \frac{n!}{4 \cdot (n-7)!} = (1 + o(1))p^4 n^7/4.$$

Now consider two adjacent elements of $\mathcal{I}$ with $|\{x_1, y_1, z_1, t_1, x_2, y_2, z_2, t_2\}| \leq 6$. There are no more than

$$n^6 = o(p^4 n^7)$$

such pairs of pairs. □

Thus we can and will only consider pairs of elements of $\mathcal{I}$ with only one vertex in common.

We will now compute the probability that $I_{(x_1y_1,z_1t_1)} I_{(x_1y_2,z_2t_2)} = 1$. Consider a uniformly random vertex v. We can neglect the case when $v \in \{x_1,\, y_1,\, z_1,\, t_1,\, y_2,\, z_2,\, t_2\}$ because it happens with probability $o(1)$. Since the random graph has diameter 2 a.a.s., $I_{(x_1y_1,z_1t_1)} I_{(x_1y_2,z_2t_2)} = 1$ in the following cases.

Case 1. $d_v(x_1) = 1$. Then v has to have distance 1 to all four edges. v has distance 1 to z_1t_1 (or z_2t_2) with probability $p^2 + 2p(1-p) = p(2-p)$, and the distances from v to y_1, y_2 don't affect anything, so

$$\mathbb{P}\left(I_{(x_1y_1,z_1t_1)} I_{(x_1y_2,z_2t_2)} = 1 | \text{ Case 1 holds}\right) = p^3(2-p)^2.$$

Case 2. $d_v(x_1) = 2$. Then v has distance 2 to both x_1y_1 and z_1t_1 with probability $(1-p)^3$ and distance 1 to both x_1y_1 and z_1t_1 with probability $p^2(2-p)$. So v is equidistant from the two edges with probability $(1-p)^3 + p^2(2-p)$. Thus,

$$\mathbb{P}\left(I_{(x_1y_1,z_1t_1)} I_{(x_1y_2,z_2t_2)} = 1 | \text{ Case 2 holds}\right) = (1-p)((1-p)^3 + p^2(2-p))^2.$$

Hence the total probability

$$\mathbb{P}\left(I_{(x_1y_1,z_1t_1)} I_{(x_1y_2,z_2t_2)} = 1\right) = (1-p)((1-p)^3 + p^2(2-p))^2 + p^3(2-p)^2.$$

We will henceforth refer to this constant as s_p.

$$s_p := (1-p)((1-p)^3 + p^2(2-p))^2 + p^3(2-p)^2.$$

It follows that

$$\Delta = (1 + o(1))p^4 n^7 s_p^r/4.$$

Using (2), we get

$$\begin{aligned}\Delta &= (1 + o(1))p^4 n^7 s_p^r/4 = (1 + o(1))p^4 n^3 n^\varepsilon n^{4-\varepsilon} s_p^r/4 \\ &= (1 + o(1))2p^2 n^3 \left(\frac{s_p}{q}\right)^r \frac{p^2 n^\varepsilon}{8} = (1 + o(1))2p^2 n^3 \left(\frac{s_p}{q}\right)^r \mu.\end{aligned}$$

Notice that

$$\begin{aligned}\left(\frac{s_p}{q}\right)^r = \left(\frac{s_p}{q}\right)^{((4-\varepsilon)\log n)/\log(1/q)} &= n^{(4-\varepsilon)\log\left(\frac{s_p}{q}\right)/\log(1/q)} \\ = n^{(4-\varepsilon)\left(\frac{-\log(s_p)}{\log(q)}+1\right)} = n^{(4-\varepsilon)(-\log_q s_p+1)} &\le n^{\varepsilon-4}\end{aligned}$$

(since $q, s_p \le 1$). Thus,

$$(1 + o(1))2p^2 n^3 \left(\frac{s_p}{q}\right)^r \mu \le (1 + o(1))2p^2 n^3 n^{\varepsilon-4} \mu = o(\mu).$$

This concludes the proof that

$$\Delta = o(\mu).$$

■

Finally, we estimate δ and show the following.

Lemma 3.6 (Evaluation of δ)**.**

$$\delta = o(1).$$

Proof. Note that for fixed f_1, e_1,

$$\begin{aligned}&\mathbb{P}\left(A_{e_2,f_2} \mid (e_2, f_2) \text{ uniformly random}, (e_2, f_2) \sim (e_1, f_1)\right) \\ &= \mathbb{P}(A_{e,f} \mid e, f \text{ uniformly random}).\end{aligned}$$

Thus, the maximum for δ is achieved for (e_1, f_1) with the largest possible number of adjacent edge pairs (e_2, f_2). Clearly, this number is the greatest when e_1 and f_1 don't share vertices. The expected number of adjacent edge pairs in this case is $(1 + o(1))2n^3 p^2$. Since $q^r = \mathbb{P}(A_{e,f})$ for uniformly random edges e, f we have

$$2\delta = (1 + o(1))2n^3 p^2 q^r.$$

Using (2), we get

$$\delta = (1 + o(1))2p^2 n^{\varepsilon-1} = o(1).$$

■

We are now ready to prove Theorem 3.1.

***Proof of Theorem* 3.1.** Substituting the results of Lemmas 3.3, 3.4, 3.6 into inequality (1), we obtain

$$\log(\mathbb{P}(S=0)) \leq (1+o(1))\left(-\mu + o(\mu)e^{o(1)}\right) \leq (1+o(1))\,(-\mu+o(\mu))$$
$$\leq -(1+o(1))\mu \leq -(1+o(1))p^2n^{\varepsilon}/8 \leq -p^2n^{\varepsilon}/16$$

for sufficiently large n. Then the expected number of edge generating sets of cardinality r is no greater than

$$\binom{n}{r}\exp(-p^2n^{\varepsilon}/16) \leq n^r\exp(-p^2n^{\varepsilon}/16)$$
$$= \mathcal{O}\left(\exp[(4-\varepsilon)\log^2(n)/\log(1/q) - p^2n^{\varepsilon}/16]\right)$$
$$\leq \mathcal{O}\left(\exp[\log^2(n) - \log^3(n)p^2/16]\right) = o(1).$$

This concludes the proof of Theorem 3.1, and together with Theorem 2.1, this proves the main result, Theorem 1.1. ■

4. Concluding Remarks

We have shown that

$$\operatorname{edim}(G(n,p)) = (1+o(1))\frac{4\log n}{\log(1/q)},$$

where

$$q = 1-2p(1-p)^2(2-p).$$

As demonstrated by Bollobas *et al.* in [1],

$$\dim(G(n,p)) = (1+o(1))\frac{2\log n}{\log(1/Q)},$$

where $Q = p^2+(1-p)^2$. Since $2/\log(1/Q) < 4/\log(1/q)$, this means that

$$\dim(G(n,p)) < \operatorname{edim}(G(n,p))$$

a.a.s. for all $p \in (0,1)$.

While random graphs with constant edge probability don't help in resolving the problem of finding more examples of graphs G for which $\operatorname{edim}(G) < \dim(G)$ posed in [7], perhaps this problem could be addressed with random graphs of non-constant probability $p(n)$. Because of this it would be interesting to calculate $\mathrm{edim}(G(n, p(n))$ for non-constant $p(n)$.The analogous results for $\dim(G(n, p(n)))$ can be found in [1].

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