ORIENTED CHROMATIC NUMBER OF CARTESIAN PRODUCTS $P_M \Box P_N$ AND $C_M \Box P_N$

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Abstract

We consider oriented chromatic number of Cartesian products of two paths $P_m \Box P_n$ and of Cartesian products of paths and cycles, $C_m \Box P_n$. We say that the oriented graph $\overrightarrow{G}$ is colored by an oriented graph $\overrightarrow{H}$ if there is a homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}$. In this paper we show that there exists an oriented tournament $\overrightarrow{H}_{10}$ with ten vertices which colors every orientation of $P_8 \Box P_n$ and every orientation of $C_m \Box P_n$, for $m = 3, 4, 5, 6, 7$ and $n \geq 1$. We also show that there exists an oriented graph $\overrightarrow{T}_{16}$ with sixteen vertices which colors every orientation of $C_m \Box P_n$.

Keywords: graphs, oriented coloring, oriented chromatic number.

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1. Introduction

An oriented graph is a digraph $\overrightarrow{G}$ obtained from an undirected graph $G$ by assigning to each edge one of two possible directions. We say that $\overrightarrow{G}$ is an orientation of $G$ and $G$ is the underlying graph of $\overrightarrow{G}$. A tournament $\overrightarrow{T}$ is an orientation of a complete graph. If there is a homomorphism $\phi : V(\overrightarrow{G}) \to V(\overrightarrow{T})$, then we say that $\overrightarrow{G}$ is colored by $\overrightarrow{T}$ or that $\overrightarrow{T}$ colors $\overrightarrow{G}$. We also say that $\overrightarrow{T}$ is a coloring graph (tournament). The oriented chromatic number of the oriented graph $\overrightarrow{G}$, denoted by $\overrightarrow{\chi}(\overrightarrow{G})$, is the smallest integer $k$ such that $\overrightarrow{G}$ is colored by a tournament with $k$ colors (vertices). The oriented chromatic number $\overrightarrow{\chi}(G)$ of an undirected graph $G$ is the maximal chromatic number over all possible orientations of $G$. The oriented chromatic number of a family of
graphs is the maximal oriented chromatic number over all possible graphs of the family. The upper oriented chromatic number $\overrightarrow{\chi}^+(G)$ of an undirected graph $G$ is the minimum order of an oriented graph $\overrightarrow{H}$ such that every orientation $\overrightarrow{G}$ of $G$ admits a homomorphism to $\overrightarrow{H}$.

It is easy to see that for every undirected graph $G$, $\chi(G) \leq \overrightarrow{\chi}(G) \leq \overrightarrow{\chi}^+(G)$, see [19]. The Cartesian product $G \Box H$ of two undirected graphs $G$ and $H$ is the graph with the vertex set $V(G) \times V(H)$, where two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. We use $P_k$ to denote the path on $k$ vertices. Sopena [19] considered upper oriented chromatic number of strong, Cartesian and direct products of graphs.

**Theorem 1** [19]. If $G$ and $H$ are two undirected graphs, then $\overrightarrow{\chi}^+(G \Box H) \leq \overrightarrow{\chi}^+(G) \cdot \overrightarrow{\chi}^+(H) \cdot \min\{\chi(G), \chi(H)\}$.

Oriented coloring has been studied in recent years [1, 2, 6, 8–10, 12, 14, 16–20, 22], see [15] for a survey of the main results. Several authors established or bounded chromatic numbers for some families of graphs, such as oriented planar graphs [12,14], outerplanar graphs [12,17,18], graphs with bounded degree three [10,17,20], k-trees [17], Halin graphs [5,9], graphs with given excess [8] or grids [3,4,6,13,22].

In this paper we focus on the oriented chromatic number of Cartesian products of two paths, called 2-dimensional grids $G_{m,n} = P_m \Box P_n$, and Cartesian products of cycles and paths, called stacked prism graphs $Y_{m,n} = C_m \Box P_n$.

**Theorem 2** [16,21]. Let $G$ be an undirected graph. Then:

(a) If $G$ is a forest with at least three vertices, then $\overrightarrow{\chi}^+(G) = 3$.

(b) $\overrightarrow{\chi}^+(C_5) = 5$. Moreover, every orientation of $C_5$ can be colored by $\overrightarrow{H}_2$ (see Figure 1(b)).

(c) For each $k \leq 3$, $k \neq 5$, we have $\overrightarrow{\chi}^+(C_k) = 4$. Moreover, every orientation of a cycle $C_k$ with $k \leq 3$ and $k \neq 5$ can be colored by $\overrightarrow{H}_1$ (see Figure 1(a)).

Theorems 1 and 2 imply that $\overrightarrow{\chi}^+(P_m \Box P_n) \leq 3 \cdot 3 \cdot 2 = 18$. Furthermore, we know that

- $\overrightarrow{\chi}(P_m \Box P_n) \leq 11$, for every $m, n \geq 2$ [6],
- there exists an orientation of $P_4 \Box P_5$ which requires 7 colors for oriented coloring [6],
- there exists an orientation of $P_4 \Box P_5$ which requires 8 colors for oriented coloring [3],
- $\overrightarrow{\chi}(P_2 \Box P_2) = 4$, $\overrightarrow{\chi}(P_2 \Box P_3) = 5$ and $\overrightarrow{\chi}(P_2 \Box P_n) = 6$, for $n \geq 6$ [6],
- $\overrightarrow{\chi}(P_3 \Box P_n) = 6$, for every $3 \leq n \leq 6$, and $\overrightarrow{\chi}(P_3 \Box P_n) = 7$, for every $n \geq 7$ [6,22],
Oriented Chromatic Number of Cartesian Products $P_m \square P_n$ and ...

Since $\chi(P_5 \square P_n) \leq 3 \cdot 4 = 24$, for $m \neq 5$, $n \geq 3$.

2. Coloring Graphs

2.1. Paley tournament

Let $p$ be a prime number such that $p \equiv 3 \mod 4$, and let $\mathbb{Z}_p = \{0, \ldots, p - 1\}$ be the ring of integers modulo $p$. We denote by $QR_p = \{r : r \neq 0, r = s^2, \text{for some} \ s \in \mathbb{Z}_p\}$ — the set of nonzero quadratic residues of $\mathbb{Z}_p$. All arithmetic operation in this section are made in the ring $\mathbb{Z}_p$.

**Definition 3.** The directed graph $T_p$ with the set of vertices $V(T_p) = \mathbb{Z}_p$ and the set of arcs $A(T_p) = \{(x, y) : x, y \in V(T_p) \text{ and } y - x \in QR_p\}$ is called the *Paley tournament* of order $p$. Observe that $T_p$ is a tournament.
Lemma 4. If \( a \in \text{QR}_p \) and \( b \in \mathbb{Z}_p \), then the mapping \( f : \overrightarrow{T}_p \to \overrightarrow{T}_p \) defined by \( f(x) = a \cdot x + b \) is an automorphism.

Lemma 5 [7]. The Paley tournament \( \overrightarrow{T}_p \) is arc-transitive; i.e., for any two pairs of arcs \((u, v), (x, y) \in A(\overrightarrow{T}_p)\), there exists an automorphism \( h \) such that \( h(u) = x \) and \( h(v) = y \).

Lemma 6. The Paley tournament \( \overrightarrow{T}_p \) is self-converse; i.e., \( \overrightarrow{T}_p \) and its converse \( \overrightarrow{T}_p^R \) are isomorphic.

Proof. Consider the function \( f : \overrightarrow{T}_p^R \to \overrightarrow{T}_p \) defined by \( f(x) = -x \). Then \((x, y) \in A(\overrightarrow{T}_p^R)\) if and only if \((-x, -y) \in A(\overrightarrow{T}_p)\).

2.2. Coloring graph \( \overrightarrow{H}_{10} \)

Consider the coloring graph \( \overrightarrow{H}_{10} \) obtained from the Paley tournament \( \overrightarrow{T}_{11} \) by removing the vertex 0, i.e., \( V(\overrightarrow{H}_{10}) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \) and \((u, v) \in A(\overrightarrow{H}_{10})\) if \((v - u) \in \{1, 3, 4, 5, 9\}\), see Figure 2.

![Figure 2. Coloring graph \( \overrightarrow{H}_{10} \).](image-url)
Lemma 7. (a) For every $a \in \{1, 3, 4, 5, 9\}$, the function $h_a(x) = ax \pmod{11}$ is an automorphism of $\overrightarrow{H}_{10}$.
(b) For every $x \in \{1, 3, 4, 5, 9\}$ there is an automorphism $h_a$ such that $h_a(x) = 1$.
(c) For every $x \in \{2, 6, 7, 8, 10\}$ there is an automorphism $h_a$ such that $h_a(x) = 10$.

Lemma 8. Let $\overrightarrow{G}$ be an orientation of a grid and let $v$ be one of its vertex. Then the following two statements are equivalent.
(a) There exists an oriented coloring (homomorphism) $c : \overrightarrow{G} \rightarrow \overrightarrow{H}_{10}$.
(b) There exists an oriented coloring (homomorphism) $c' : \overrightarrow{G} \rightarrow \overrightarrow{H}_{10}$ such that $c'(v) \in \{1, 10\}$.

2.3. Tromp graph

Definition 9. Let $\overrightarrow{G}$ be an oriented graph. We build the Tromp graph $\overrightarrow{T}(\overrightarrow{G})$ in the following way.
- Let $\overrightarrow{G'}$ be an isomorphic copy of $\overrightarrow{G}$.
- $\infty, \infty'$ be two additional vertices.
- Let $t : V(\overrightarrow{G}) \cup \{\infty\} \rightarrow V(\overrightarrow{G'}) \cup \{\infty'\}$ be an isomorphism with $t(\infty) = \infty'$. For every $u \in V(\overrightarrow{G}) \cup \{\infty\}$ by $u'$ we denote $t(u)$ and for every $u \in V(\overrightarrow{G'}) \cup \{\infty'\}$ by $u'$ we denote $t^{-1}(u)$. The pair $(u, u')$ is called a pair of twin vertices.
- The set of vertices $V(\overrightarrow{T}(\overrightarrow{G})) = V(\overrightarrow{G}) \cup V(\overrightarrow{G'}) \cup \{\infty, \infty'\}$.
- The set of arcs is defined by
  $$\forall_{u \in V(\overrightarrow{G})}(\infty, u), (\infty, u'), (u', \infty'), (\infty', u) \in A(\overrightarrow{T}(\overrightarrow{G})), \forall_{u, v \in V(\overrightarrow{G}), (u, v) \in A(\overrightarrow{G})}(u, v), (u', v'), (v, u'), (v', u) \in A(\overrightarrow{T}(\overrightarrow{G})).$$

Let $\overrightarrow{T}_{16} = \overrightarrow{T}(\overrightarrow{T}_7)$ be the Tromp graph on sixteen vertices obtained from the Paley tournament $\overrightarrow{T}_7$, see Figure 3.

Suppose that $i$ and $j$ are integers such that $i \geq 1$ and $j \geq 1$. Consider the star $K_{1,i}$ with the set of vertices $V(K_{1,i}) = \{x, v_1, v_2, \ldots, v_l\}$ and edges of the form $\{x, v_k\}$ for $1 \leq k \leq i$; and a Tromp graph $\overrightarrow{T}(\overrightarrow{G})$. Let $\overrightarrow{K}$ be an orientation of the star $K_{1,i}$ and $c : \overrightarrow{K} \rightarrow \overrightarrow{T}(\overrightarrow{G})$ be a homomorphism. We say that the sequence of colors $(c(v_1), c(v_2), \ldots, c(v_l))$ chosen for leaves of the star is compatible with orientation $\overrightarrow{K}$ if for every pair of vertices $v_k, v_l$ it holds:
- $c(v_k) \neq c(v_l)$ if $(v_k, x)$ and $(x, v_l) \in \overrightarrow{K}$ or if $(v_l, x)$ and $(x, v_k) \in \overrightarrow{K}$, and
- $c(v_k) \neq c(v_l)'$ if $(v_k, x)$ and $(v_l, x) \in \overrightarrow{K}$ or if $(x, v_l)$ and $(x, v_k) \in \overrightarrow{K}$. 

Figure 3. Coloring graph \( \overrightarrow{T}_{16} = \overrightarrow{T}_r(\overrightarrow{T}_7) \).

**Definition 10.** We say that the Tromp graph \( \overrightarrow{T} \) has the property \( P_{c}(i, j) \) if \( |V(\overrightarrow{T})| \geq i \) and for every orientation \( \overrightarrow{K} \) of the star \( K_{1, i} \) and every sequence of colors \( (c(v_1), c(v_2), \ldots, c(v_k)) \) chosen for leaves compatible with \( \overrightarrow{K} \) we can choose \( j \) different ways to color \( x \), the central vertex of the star.

**Lemma 11** [11]. The Tromp graph \( \overrightarrow{T}_{16} \) has the properties \( P_{c}(1, 7) \), \( P_{c}(2, 3) \) and \( P_{c}(3, 1) \).

3. **Grids** \( G_{8,n} = P_8 \Box P_n \)

**Definition 12.** The comb \( R_8 \) is an undirected graph with the set of vertices \( V(R_8) = \{(1,1), \ldots, (8,1), (1,2), \ldots, (8,2)\} \) and edges of the form \( \{(i,1), (i,2)\} \) for \( 1 \leq i \leq 8 \), or \( \{(i,2), (i+1,2)\} \) for \( 1 \leq i < 8 \); see Figure 4. The vertices \( (1,1), \ldots, (8,1) \) form the first column of the comb \( R_8 \), while \( (1,2), \ldots, (8,2) \) form the second column.

**Definition 13.** A set \( S \subseteq (V(\overrightarrow{H}_{10}))^8 \) is closed under extension if
Oriented Chromatic Number of Cartesian Products $P_m \square P_n$ and ...

(a) for every orientation $\vec{P}$ of the path $P_8 = (v_1, \ldots, v_8)$, there exists a coloring $c : \vec{P} \to \vec{H}_{10}$ such that $(c(v_1), \ldots, c(v_8)) \in S$.

(b) for every orientation $\vec{R}$ of the comb $R_8$ and for every sequence $(c_1, \ldots, c_8) \in S$, there exists a coloring $c : \vec{R} \to \vec{H}_{10}$ and an automorphism $h_a$ of $\vec{H}_{10}$ such that

1. $(c(1,1), \ldots, c(8,1)) = (c_1, \ldots, c_8)$, and
2. $h_a(c(1,2), \ldots, c(8,2)) \in S$.

Lemma 14. There exists a set $S \subseteq (V(\vec{H}_{10}))^8$ which is closed under extension.

Proof. In order to prove the lemma we use a computer. We have designed an algorithm that finds a proper set $S$. Let

$S_{\text{max}}(P_8) = \{(c_1, \ldots, c_8) : c_1 \in \{1, 10\}, \text{ and } \forall_{2 \leq i \leq 8} c_i \in V(\vec{H}_{10}), \text{ and } c_{i-1} \neq c_i \}.$

Note, that for every sequence $t = (t_1, \ldots, t_8) \in S_{\text{max}}(P_8)$, there exists an orientation $\vec{P}$ of the path $P_8 = (v_1, \ldots, v_8)$ and a coloring $c : \vec{P} \to \vec{H}_{10}$ such that $(c(v_1), \ldots, c(v_8)) = t$. For a set $T$, a sequence $t = (t_1, \ldots, t_8) \in T$, and an orientation $\vec{R}$ of the comb $R_8$, we say that $t$ can be extended in $T$ on $\vec{R}$ if there exists a coloring $c : \vec{R} \to \vec{H}_{10}$ and a homomorphism $h_a$ such that

- $(c(1,1), \ldots, c(8,1)) = t$, and
- $h_a(c(1,2), \ldots, c(8,2)) \in S$.

The algorithm starts with $T = S_{\text{max}}(P_8)$. In the while loop, for each sequence $t \in T$ and for each orientation $\vec{R}$ of the comb $R_8$, the algorithm checks if $t$ can be extended in $T$ on $\vec{R}$. If the sequence $t$ can not be extended, then $t$ is removed from $T$. After the while loop, the set $T$ satisfies the condition (b) of Definition 13. It is easy to see that if $T$ is not empty, then it also satisfies the condition (a). In this case $S = T$ is returned. If $T$ is empty, then the algorithm returns NO.
Algorithm ComputeSet $S$

OUTPUT: a set $S \subset (V(\overrightarrow{H}_{10}))^8$ closed under extension or NO if such a set does not exist.

1. compute the set $S_{\text{max}}(P_8)$
2. $T := S_{\text{max}}(P_8)$
3. SetIsReady := false
4. while not SetIsReady
5. SetIsReady := true
6. for every sequence $t = (t_1, \ldots, t_8) \in T$
7. color the first column of the comb $R_8$
8. by setting $c(i, 1) = t_i$, for $1 \leq i \leq 8$
9. SeqCanBeExtended := true
10. for every orientation $\overrightarrow{R}$ of the comb $R_8$
11. if $t$ cannot be extended on $\overrightarrow{R}$
12. SeqCanBeExtended := false
13. if not SeqCanBeExtended
14. $T := T - t$
15. SetIsReady := false
16. if $T = \emptyset$
17. return NO
18. else
19. $S := T$
20. return the set $S$

Using Algorithm ComputeSet $S$ we have found a nonempty set $S$ closed under extension. The set $S$ is posted on the website https://inf.ug.edu.pl/grids/.

Theorem 15. Every orientation of every grid with eight rows can be colored by the coloring graph $\overrightarrow{H}_{10}$.

Proof. For a given orientation $\overrightarrow{G}$ of $G(8, n)$ and $i \leq n$, by $\overrightarrow{G}(i)$ we denote the induced subgraph of $\overrightarrow{G}$ formed by the first $i$ columns of $\overrightarrow{G}$. It is easy to show by induction that, for every $i$, there is a coloring $c : \overrightarrow{G}(i) \rightarrow \overrightarrow{H}_{10}$ such that $c(\text{i}th \text{ column}) \in S$.

4. Stacked Prism Graphs $Y_{m,n} = C_m \square P_n$

Theorem 16. Every orientation of $C_m \square P_n$ with $m \geq 3$ and $n \geq 1$ can be colored by the Tromp graph $\overrightarrow{T}_{16}$.
Oriented Chromatic Number of Cartesian Products $P_m \Box P_n$ and ...

Figure 5. Stacked prism graph $Y_{m,n}$.

**Proof.** Let $\vec{Y}$ be any orientation of stacked prism graph $Y_{m,n} = C_m \Box P_n$. We identify each vertex $u \in \vec{Y}$ with the pair of its coordinates $(i, j)$, $1 \leq i \leq m$, $1 \leq j \leq n$. We shall show that $\vec{Y}$ can be colored by $\vec{T}_{16}$. We color the vertices of $\vec{Y}$ row by row. For the first row, clearly, it is always possible to color any oriented cycle by homomorphism to $\vec{T}_{16}$, because $\vec{T}_{16}$ has the properties $P_c(2,3)$ and $P_c(1,7)$. Now, suppose that $i > 1$ and the rows from 1 to $i - 1$ are already colored. To color the vertex $(1, i)$ we choose a color which is compatible

- with the color of vertex $(2, i - 1)$ in the star $\{(2, i), (1, i), (2, i - 1)\}$,
- with the color of vertex $(m, i - 1)$ in the star $\{(m, i), (1, i), (m, i - 1)\}$,

which is always possible using the property $P_c(1,7)$. Using the property $P_c(2,3)$ it is always possible to color vertex $(2, i)$ by the color compatible with color of the vertex $(3, i - 1)$ in the star $\{(3, i), (2, i), (3, i - 1)\}$. Then we continue this method to color vertices $(3, i), \ldots, (m - 2, i)$. To color the vertex $(m - 1, i)$ we choose a color which is compatible with the colors of vertices $(m, i - 1)$ and $(1, i)$ in the star $\{(m, i), (1, i), (m, i - 1), (m - 1, i)\}$. This is possible, because the colors of vertices $(1, i)$ and $(m, i - 1)$ are compatible in the star $\{(m, i), (1, i), (m, i - 1)\}$.

Finally we color the vertex $(m, i)$ using the property $P_c(3,1)$. Similarly we can color the following rows.

**Theorem 17.** Every orientation of stacked prism graph $Y_{m,n} = C_m \Box P_n$ with $3 \leq m \leq 7$ can be colored by the coloring graph $\vec{H}_{10}$. 

**Proof.** The proof of the theorem is similar to the proof of Theorem 15 and follows from Lemma 20.

**Definition 18.** For \( m \geq 3 \), the \( m \)-sunlet graph \( \text{Sun}_m \) is an undirected graph with the set of vertices \( V(\text{Sun}_m) = \{(1,1),\ldots,(m,1),(1,2),\ldots,(m,2)\} \) and edges of the form \( \{(i,1),(i,2)\} \) for \( 1 \leq i \leq m \), or \( \{(i,2),(i+1,2)\} \) for \( 1 \leq i < m \), or \( \{(m,2),(1,2)\} \); see Figure 6.

![Figure 6. m-sunlet graph.](image)

**Definition 19.** A set \( S \subseteq (V(\vec{H}_{10}))^m \) is cycle-closed under extension if

(a) for every orientation \( \vec{C} \) of the cycle \( C_m = (v_1,\ldots,v_m) \), there exists a coloring \( c : \vec{C} \to \vec{H}_{10} \) such that \( (c(v_1),\ldots,c(v_m)) \in S \),

(b) for every orientation \( \vec{\text{Sun}} \) of the \( m \)-sunlet graph \( \text{Sun}_m \) and for every sequence \( (c_1,\ldots,c_m) \in S \), there exists a coloring \( c : \vec{\text{Sun}} \to \vec{H}_{10} \) and an automorphism \( h_a \) of \( \vec{H}_{10} \) such that

1. \( (c(1,1),\ldots,c(m,1)) = (c_1,\ldots,c_m) \), and
2. \( h_a(c(1,2),\ldots,c(m,2)) \in S \).

**Lemma 20.** For each \( m = 3,4,5,6,7 \), there exists a nonempty set \( S_m \subseteq (V(\vec{H}_{10}))^m \), which is cycle-closed under extension.

**Proof.** In order to proof the lemma we use a computer. We have designed an algorithm, similar to the Algorithm ComputeSetS, that finds a set cycle-closed under extension. The algorithm, for a given \( m \), uses the \( m \)-sunlet \( \text{Sun}_m \) instead of a comb \( R_8 \). Using the algorithm we have found that for each \( m = 3,\ldots,7 \), there exists a nonempty set cycle-closed under extension.
Oriented Chromatic Number of Cartesian Products $P_m \square P_n$ and …

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