

GRAPHS WITH UNIQUE MAXIMUM PACKING OF CLOSED NEIGHBORHOODS

DRAGANA BOŽOVIĆ AND IZTOK PETERIN

University of Maribor
Faculty of Electrical Engineering and Computer Science
Koroška cesta 46, 2000 Maribor, Slovenia

e-mail: dragana.bozovic@um.si
iztok.peterin@um.si

Abstract

A packing of a graph G is a subset P of the vertex set of G such that the closed neighborhoods of any two distinct vertices of P do not intersect. We study graphs with a unique packing of the maximum cardinality. We present several general properties for such graphs. These properties are used to characterize the trees with a unique maximum packing. Two characterizations are presented where one of them is inductive based on five operations.

Keywords: unique maximum packing, closed neighborhoods, trees.

2010 Mathematics Subject Classification: 05C70, 05C05.

1. INTRODUCTION

The packing number of a graph G denoted by $\rho(G)$ is the maximum cardinality of closed neighborhoods that can be packed into a graph such that they have pairwise empty intersection. All vertices that are in the centers of mentioned neighborhoods form a packing set. Packing number has been studied as a natural lower bound for the domination number $\gamma(G)$. One of the first results of that type is from Meir and Moon [15], where it was shown that $\rho(T) = \gamma(T)$ for every tree T (in a different notation). It is easy to see that while the numbers are the same, the sets that yield both $\rho(T)$ and $\gamma(T)$ are often different.

The class of graphs with $\rho(G) = \gamma(G)$, where closed neighborhoods form a partition of $V(G)$, is called *efficient closed dominated graphs*. In such a case we call a minimum dominating set a *1-perfect code*. The study of perfect codes

in graphs was initiated by Biggs [1]. Later it was intensively studied and we recommend [14] for further information and references.

In the last decade the packing number became more interesting for itself not only in connection with the domination number. The relationship between the packing number and the maximal packing of minimum cardinality, also known as the lower packing number, is investigated in [17]. In [16] a connection between the packing number and the double domination in the form of an upper bound is presented. Graphs for which their packing number equals to the packing number of their complement are described in [4]. In [9] it was shown that the domination number can be also bounded from above by the packing number multiplied by the maximum degree of a graph. The inequality for the packing number of Vizing conjecture type was proven in [13].

A generalization of packing presented in [6] is the k -limited packing where every vertex can have at most k neighbors in a k -limited packing set S . A probabilistic approach to k -limited packings can be found in [5]. A further generalization, that is, the generalized limited packing of the k -limited packing, see [3], brings a dynamic approach with respect to the vertices of G , where $v \in V(G)$ can have a different number of neighbors k_v for every vertex v in a generalized limited packing. The problem of generalized limited packing is NP-complete, but solvable in polynomial time for P_4 -tidy graphs as shown in [3].

In this work we study the graphs with a unique maximum packing. In general one can have many maximum packings as shown in [12] where an asymptotic bounds for the maximum and the minimum number of packings in a graphs of fixed order are established. We present properties of graphs with a unique maximum packing. One can find sets with several different properties for which this (uniqueness of a set with minimum or maximum cardinality) was considered in the literature. For example, see [10] for graphs with a unique maximum independent set, [7] for graphs with a unique minimum dominating set and [8] for trees with a unique minimum total dominating set. In [2] graphs with a unique maximum open neighborhood packing are considered. In particular, it was shown that the classes of graphs with a unique maximum open packing, with a unique maximum packing and with a unique maximum independent set are polynomially equivalent from the recognition point of view. Another approach could also be to count the number of maximum/minimum cardinality sets of a certain type as in [11] for the number of maximum independent sets.

Throughout this work we consider finite undirected simple graphs. Given a vertex v of a graph G , $N(v)$ represents the *open neighborhood* of v , i.e., the set of all neighbors of v in G and the *degree* of v is $\deg(v) = |N(v)|$. The *closed neighborhood* of $v \in V(G)$ is $N[v] = N(v) \cup \{v\}$. For any two vertices u and v , the *distance* $d(u, v)$ between u and v is the minimum number of edges on a path between u and v . Given a subset of vertices S of G , we use $G - S$ to denote the

graph obtained from G by removing all the vertices from S and the edges incident with them. If $S = \{v\}$ for some vertex v , then we simply write $G - v$. Also, the subgraph of G induced by $D \subset V(G)$ will be denoted by $G[D]$.

A set $P \subset V(G)$ is a *packing* of G if $d(u, v) > 2$ for every pair of distinct vertices u and v from P . The *packing number* of G is the maximum cardinality of any packing of G and is denoted by $\rho(G)$. A $\rho(G)$ -*set* is a packing of cardinality $\rho(G)$. If there exists only one maximum packing P of a graph G , then G is a graph with a unique $\rho(G)$ -set.

Let G be a graph. A *leaf* of G is a vertex of degree one. A *support vertex* of G is a vertex of degree at least two adjacent to a leaf. A *strong support vertex* of G is a support vertex of G that is adjacent to at least two leaves. Let T be a tree, v an arbitrary vertex of degree k and v_1, \dots, v_k neighbors of v . We denote by T_1^v, \dots, T_k^v the trees in $T - v$ such that $v_i \in T_i^v$ for every $i \in \{1, \dots, k\}$. A *root* of a tree T is a special designated vertex of T . Let u and v be adjacent in T such that $d(u, r) > d(v, r)$ for a root r . In such a case we call u a *down-neighbor* of v and v is the *up-neighbor* of u .

2. BASIC PROPERTIES AND THE STRUCTURE OF GRAPHS WITH UNIQUE $\rho(G)$ -SET

We start with several basic properties of graphs with a unique $\rho(G)$ -set, that will be important later.

Lemma 1. *If a graph G has a unique $\rho(G)$ -set P , then every leaf of G belongs to P .*

Proof. Let P be a unique $\rho(G)$ -set. To prove the lemma assume there exists a leaf $\ell \notin P$. If the support vertex x of ℓ belongs to P , then $P' = (P - \{x\}) \cup \{\ell\}$ is a $\rho(G)$ -set of G that is different from P , which is a contradiction with the assumption. So $x \notin P$. If some neighbor of x , say y , is in P , then $P'' = (P - \{y\}) \cup \{\ell\}$ is a $\rho(G)$ -set of G that is different from P , the same contradiction again. Thus $N[x] \cap P = \emptyset$. This yields a contradiction with maximum cardinality of P because $P \cup \{\ell\}$ is a packing of bigger cardinality than P . Hence, all leaves must be in P . ■

Lemma 2. *If G is a graph with a unique $\rho(G)$ -set P , then G has no strong support vertex.*

Proof. Suppose that G has a strong support vertex v . That means v has at least two leaves ℓ_1 and ℓ_2 as its neighbors. Because all leaves of G belong to P by Lemma 1, we have $\ell_1, \ell_2 \in P$, such that $d(\ell_1, \ell_2) = 2$. This is a contradiction with the definition of a packing P . ■

By the definition, every pair of different vertices of a packing must be at distance that is at least three. However, vertices at distance three are obligatory in the case of a unique $\rho(G)$ -set as it can be seen from the following lemma.

Lemma 3. *Let G be a graph on at least two vertices. If G has a unique $\rho(G)$ -set P , then for every vertex $v \in P$ and its neighbor v' there exists a vertex $u \in P$ such that $d(v, u) = 3$ and v' is on a shortest path between v and u .*

Proof. Suppose there exists a vertex $v \in P$ and its neighbor v' such that for every vertex $u \in P - \{v\}$ either $d(v, u) > 3$ or $d(v, u) = 3$ and $d(v', u) > 2$. The set $(P - \{v\}) \cup \{v'\}$ is also a $\rho(G)$ -set which is a contradiction with the assumption that P is a unique $\rho(G)$ -set. ■

Notice that in the above lemma two or more neighbors of $v \in P$ can be on a shortest path to the same vertex $u \in P$ with $d(v, u) = 3$. A small example for this is presented by a graph G defined on the six-cycle $u_1u_2u_3u_4u_5u_6u_1$ together with edges u_2u_5 and u_3u_6 , where $P = \{u_1, u_4\}$ is the unique $\rho(G)$ -set. This cannot happen in the case of trees, because there is a unique shortest path between every pair of vertices.

Corollary 4. *Let T be a tree on at least two vertices. If T has a unique $\rho(T)$ -set P , then for every vertex $v \in P$ of degree k there exist different vertices $u_1, \dots, u_k \in P$, such that $d(v, u_i) = 3$ and $u_i \in V(T_i^v)$ for every $i \in \{1, \dots, k\}$.*

The above corollary guarantees that in a tree with a unique $\rho(T)$ -set P , for any $v \in P$ there exists a vertex in P at distance three from v in the direction of each of its neighbors. This is a key point in the first characterization of trees with a unique $\rho(T)$ -set, see Theorem 6.

Next we present the structure of a graph G with a unique $\rho(G)$ -set P (white squared vertices in Figure 1). The set $V(G) - P$ can be partitioned into sets Q (white vertices in Figure 1) and R (black vertices in Figure 1), where Q is the set of all neighbors of vertices from P , and $R = V(G) - (P \cup Q)$. Clearly, the set R may be empty, and Q is empty if and only if $G = \bigcup K_1$. From the definition of a unique $\rho(G)$ -set we infer the following properties:

- P is an independent set;
- for every vertex $v \in P$ there exists a vertex $u \in P$ with $d(v, u) = 3$ unless G has no edges (by Lemma 3);
- every vertex from Q has exactly one neighbor in P ;
- for every vertex $v' \in Q$ there exists a vertex $v \in P$ with $d(v, v') = 2$ (by Lemma 3);
- there are no edges between vertices from P and vertices from R ;

- every vertex u in R has at least two neighbors in Q (otherwise, if w is the only neighbor of u from Q , where $v \in P$ is a neighbor of w , then $(P - \{v\}) \cup \{u\}$ is also a $\rho(G)$ -set, a contradiction);
- $G[Q]$ and $G[R]$ are arbitrary, where $G[Q]$ has no isolated vertices (the latest follows by Lemma 3).

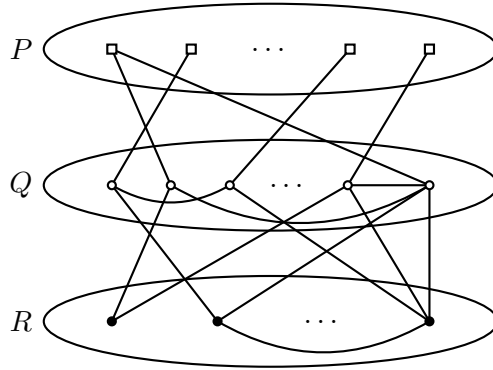


Figure 1. A partition of vertices of a graph G with a unique $\rho(G)$ -set P .

We continue with a lemma that describes what must be going on around a vertex from R in a graph with a unique maximum packing.

Lemma 5. *Let G be a graph which has a unique $\rho(G)$ -set P . If G contains a vertex $v \in R$, then there exist paths $xyzu$ and $x'y'z'u'$ such that $x, x', u, u' \in P$, $y, y', z, z' \in Q$ and $vy, vy' \in E(G)$ (see Figure 2). Moreover, $x \neq x'$ and $y \neq y'$.*

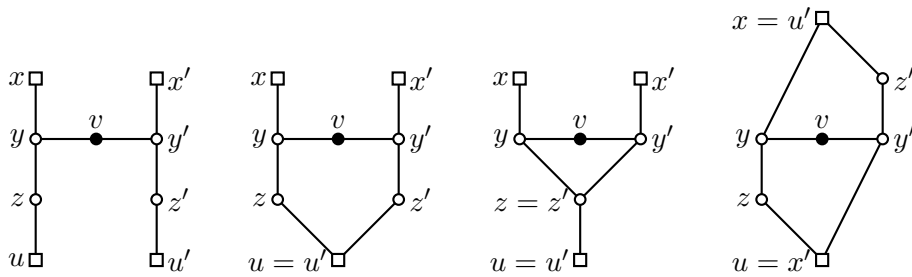


Figure 2. Possible subgraphs of a graph containing a vertex $v \in R$.

Proof. Let v be a vertex from the set R . If there is no vertex from P at distance 2 from v , then we have a contradiction with the maximality of P since $P \cup \{v\}$ is a packing. Let $x \in P$ be at distance two from v , and let y be a common neighbor of x and v . Clearly, $\deg(y) > 2$ by the comment in the fourth item of before mentioned properties. Because P is the unique $\rho(G)$ -set, $(P - \{x\}) \cup \{y\}$ is not a

packing. This implies that there exists a vertex $u \in P$ at distance two from y . Let the common neighbor of u and y be z . Clearly, $z \in Q$ and therefore different from v . If x is the only vertex from P at distance 2 from v , then $(P - \{x\}) \cup \{v\}$ is also a $\rho(G)$ -set, a contradiction with the uniqueness of P again. Therefore there exists $x' \in P$ so that $d(v, x') = 2$ and $x \neq x'$. Let y' be a common neighbor of v and x' . Clearly, $y' \in Q$ and $y \neq y'$ because otherwise $d(x, x') \leq 2$ which is not possible. Assume that there exist no $z' \in Q$ and $u' \in P$ such that $d(v, z') = 2 = d(x', z')$ and $d(v, u') = 3 = d(x', u')$. In this case $(P - \{x'\}) \cup \{y'\}$ is a $\rho(G)$ -set which is a contradiction with the uniqueness of P again. ■

The possibilities of Lemma 5 are presented in Figure 2. The four graphs are possible subgraphs of a graph with a unique maximum packing P . While the first three graphs are themselves graphs with a unique $\rho(G)$ -set P , the last one is not such. The minimum example of a graph that has a unique $\rho(G)$ -set P and contains the rightmost graph of Figure 2 as a subgraph can be obtained if we add four additional vertices t, t', s, s' together with edges $zt, tt', z's,$ and ss' , see Figure 3.

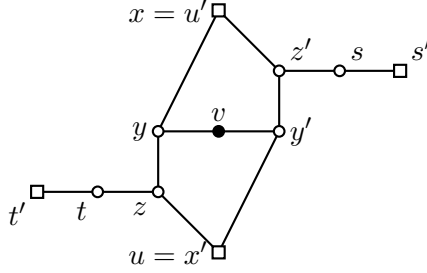


Figure 3. The minimum example of a graph that has a unique $\rho(G)$ -set P and contains the rightmost graph of Figure 2 as a subgraph.

3. TREES WITH A UNIQUE $\rho(T)$ -SET

In this section we limit ourselves to trees. We present two characterizations of trees with a unique $\rho(T)$ -set. First describes the properties of a unique $\rho(T)$ -set. For this, lemmas from the previous sections come in handy. In particular, the only possible outcome of Lemma 5 in the case of trees, is the leftmost tree of Figure 2.

Recall that P denotes a $\rho(T)$ -set, set Q contains all neighboring vertices of P and $R = V(T) - (P \cup Q)$.

Theorem 6. *Let T be a tree and let P be a $\rho(T)$ -set. A set P is a unique $\rho(T)$ -set if and only if P satisfies the following conditions.*

- (i) Every leaf is in P .
- (ii) For every $v \in P$ there exists at least one vertex from P at distance 3 from v in each T_i^v , $1 \leq i \leq \deg(v)$.
- (iii) For every $v \in R$ there exist vertex-disjoint paths $xyzu$ and $x'y'z'u'$ such that $x, x', u, u' \in P$, $y, y', z, z' \in Q$ and $vy, vy' \in E(T)$ (see the leftmost graph of Figure 3).

Proof. If a tree T has a unique $\rho(T)$ -set P , then (i) follows from Lemma 1, (ii) follows from Corollary 4 and (iii) from Lemma 5.

To prove the converse suppose that (i), (ii) and (iii) hold for a $\rho(T)$ -set P . We proceed by induction on the number of vertices n of T . If $n = 1$, then $T \cong K_1$ which is the smallest tree with a unique $\rho(T)$ -set and the base of the induction is clear. Similar holds when $T \cong P_4$ where P contains both leaves. We will denote by T' a tree obtained from T by deleting some vertices and by P' its $\rho(T')$ -set. Assume on the way to a contradiction that there exists a $\rho(T)$ -set $P_1 \neq P$ together with Q_1 and R_1 as usual. Let $r \in V(T)$ be a root of T . Choose a vertex $t \in (P - P_1) \cup (P_1 - P)$ at the maximum distance from r .

Suppose first that $t \in P_1 - P$. By (i) t is not a leaf and there exists a down-neighbor v of t . By the choice of t the sets $P \cap V(T_v^t)$ and $P_1 \cap V(T_v^t)$ must be the same. If $v \in Q$, then v has a down-neighbor w in P and by the choice of t also $w \in P_1$. Since $d(w, t) = 2$ we have a contradiction with $w, t \in P_1$. So $v \in R$.

Let first $t \in Q$. Every vertex from Q has a neighbor, say u , in P . If u is a down-neighbor of t , then $u \in P \cap P_1$ by the choice of t . This is a contradiction because we have adjacent vertices t and u in a $\rho(T)$ -set P_1 . Therefore u is the up-neighbor of t . By property (ii) there exists a vertex $z \in P$ such that $d(u, z) = 3$ and $d(t, z) = 2$. Clearly, z is a descendant of t and is in P_1 by the choice of t . Again a contradiction with P_1 being a $\rho(T)$ -set as $t, z \in P_1$.

Let now $t \in R$. By property (iii) there exist two paths P_4 and at least one of them is connected to t over a down-neighbor. Let this path be $xyzu$ where $x, u \in P$ and y is a down-neighbor of t . By the choice of t , we have $x \in P_1$, which yields the same contradiction again with $t, x \in P_1$ and $d(t, x) = 2$.

Hence there is no vertex in $P_1 - P$ at the maximum distance from r , which means that $t \in P - P_1$.

Case 1. t is not a leaf. If t is not a leaf there exists a down-neighbor v of t . By the choice of t the sets $P \cap V(T_v^t)$ and $P_1 \cap V(T_v^t)$ must be the same.

Let $T' = T - V(T_v^t)$ and let $P' = P \cap V(T')$. If t is not a leaf of T' , then (i) holds for P' in T' since (i) holds for all the leaves in T . Otherwise t is a leaf and $t \in P'$. So (i) holds for P' . Clearly, (ii) also holds for P' in T' since it holds for P in T . Let w be an arbitrary vertex from $R' = R \cap V(T')$. Notice that $d(t, w) \geq 2$ since $t \in P$. Clearly, (iii) holds for R', P' and $Q' = Q \cap V(T')$ in T' whenever $d(t, w) > 2$, since it holds for R, P and Q in T . So let $d(t, w) = 2$ and

let $xyzu$ and $x'y'z'u'$ be the paths in T such that $x, x', u, u' \in P$, $y, y', z, z' \in Q$ and $vy, vy' \in E(T)$. (Notice that they exist in T by (iii).) Because $t \in P$ and $d(t, w) \geq 2$ both mentioned paths must be in T' too, and (iii) holds for R', P' and Q' in T' as well.

By the induction hypothesis T' has a unique $\rho(T')$ -set P' . So $|P'| > |P'_1|$, where $P'_1 = P_1 \cap V(T')$. Because $|P \cap V(T_v^t)| = |P_1 \cap V(T_v^t)|$ it follows that $|P| > |P_1|$ which is a contradiction with P_1 being a $\rho(T)$ -set.

Case 2. t is a leaf. Denote by v the up-neighbor of t . Clearly, $v \neq r$ by the choice of t and let w be the up-neighbor of v . By (ii) there exists a vertex $x \in P$ such that $d(t, x) = 3$ and $d(v, x) = 2$. Denote by y the common neighbor of x and v . Notice that one of v, w must be in P_1 , since otherwise $P_1 \cup \{t\}$ is a packing of cardinality greater than P . (Here a down-neighbor of v cannot be in P_1 because this down-neighbor would be in $P_1 - P$ at the same distance from r as t , which is not possible.)

Subcase 2.1. $w = y$ and x is the up-neighbor of w . Let $T' = T - V(T_w^x)$ and let $P' = P \cap V(T')$, see Figure 4. Properties (i), (ii) and (iii) hold for P' together with $Q' = Q \cap V(T')$ and $R' = R \cap V(T')$ by the same reasons as in Case 1. By the induction hypothesis T' has a unique $\rho(T')$ -set P' . So $|P'| > |P'_1|$, where $P'_1 = P_1 \cap V(T')$. There is no other vertex from P_1 in T_w^x than w or v at the same distance from r as w or v . There is also no other vertex from $P_1 - P$ at the same distance to r as t by the choice of t . Moreover P and P_1 equals for vertices that are farther away from r as t . Therefore, $|P \cap V(T_w^x)| \geq |P_1 \cap V(T_w^x)|$. All together we have $|P| > |P_1|$ which is a contradiction with P_1 being a $\rho(T)$ -set.

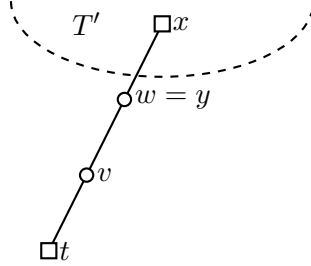


Figure 4. The case when t is a leaf, $w = y$ and x is the up-neighbor of w .

Subcase 2.2. $w = y$ and x is a down-neighbor of w . In this case let z be the up-neighbor of w and we set $T' = T - V(T_z^w)$, see Figure 5. Notice that we did the cut in somewhat reversed order because now $r \notin V(T')$. Again let $P' = P \cap V(T')$, $Q' = Q \cap V(T')$ and $R' = R \cap V(T')$. Property (i) holds for P' in T' since w has two down-neighbors and is not a leaf in T' and since (i) holds for P in T . Properties (ii) and (iii) hold for P', Q' and R' by the same reason as before. By the induction hypothesis T' has a unique $\rho(T')$ -set P' . So

$|P'| > |P'_1|$, where $P'_1 = P_1 \cap V(T')$. Again either v or w is in P_1 . Hence a vertex from P_1 is closer or at the same distance to z as $x \in P$ and with this to T_z^w . Therefore $P_1 \cap V(T_z^w)$ cannot have more vertices than $|P \cap V(T_z^w)|$ and we have $|P \cap V(T_z^w)| \geq |P_1 \cap V(T_z^w)|$. It follows that $|P| > |P_1|$ which is a contradiction with P_1 being a $\rho(T)$ -set.

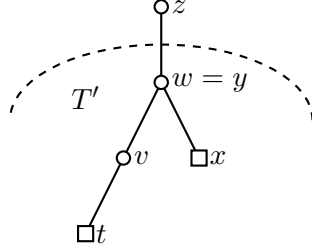
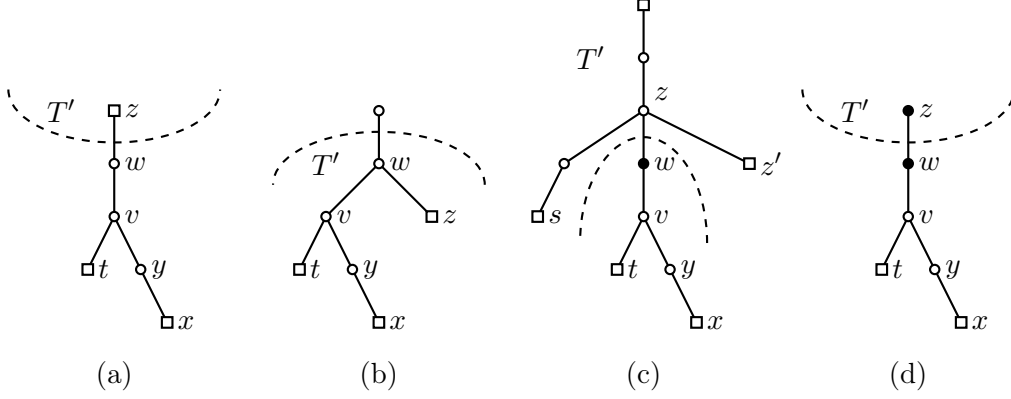


Figure 5. The case when t is a leaf, $w = y$ and x is a down-neighbor of w .

Subcase 2.3. $w \neq y$. With this y is a down-neighbor of v and x is a down-neighbor of y . First, if the up-neighbor z of w is in P , then we rename z to x and w to y and we obtain Subcase 2.1, see Figure 6(a). Similar holds if w has a down-neighbor z in P and we obtain Subcase 2.2 by renaming $z = x$ and $w = y$, see Figure 6(b). Therefore w must be in R and let z be its up-neighbor. We set $T' = T - V(T_w^z)$. Clearly, $z \notin P$. If $z \in Q$, then it has a neighbor z' in $P \cap V(T')$ and there exists $s \in P$ with $d(z', s) = 3$ and $d(z, s) = 2$ by the property (ii), see Figure 6(c). The common neighbor of s and z together with z' assures that z is not a leaf in T' . If $z \in R$, then it has at least two neighbors in $Q \cap V(T')$ by property (iii) and z is again not a leaf, see Figure 6(d). Therefore property (i) holds for T' . Properties (ii) and (iii) again hold for T by the same reason as before. By the induction hypothesis T' has a unique $\rho(T')$ -set P' and $|P'| > |P'_1|$ for $P'_1 = P_1 \cap V(T')$. As mentioned either v or w must be in P_1 and by the same reason as in Subcase 2.1 we have $|P \cap V(T_z^w)| \geq |P_1 \cap V(T_z^w)|$. Hence $|P| > |P_1|$ follows, which is the final contradiction with P_1 being a $\rho(T)$ -set. ■

The conditions (i), (ii) and (iii) from the last theorem can be checked in polynomial time for a given set P . This implies that the problem of recognizing trees with a unique $\rho(T)$ -set is in the class NP . Moreover, by checking all three possibilities (either $r \in P$ or $r \in Q$ or $r \in R$) one can find the packing number of T and also count the number of maximum-sized solutions using dynamic programming. So, one can decide in polynomial time whether a given tree has a unique maximum packing.

We continue with an inductive characterization. Let T' be a tree with a unique $\rho(T')$ -set P' . We introduce five operations to construct from T' a larger tree T with a unique $\rho(T)$ -set. We will prove that every tree T with a unique

Figure 6. The cases when $w \neq y$.

$\rho(T)$ -set can be constructed from K_1 with these operations. The operations are illustrated in Figure 7 where we use the notation introduced in Section 2: a vertex from P' (or P) is white squared, a vertex from Q' (or Q) is white circled and a vertex from R' (or R) is black circled.

- (O_1) For $y \in P'$ we obtain T from T' by adding a path xuv and an edge xy .
- (O_2) For $x \in Q'$ we obtain T from T' by adding a path uv and an edge ux .
- (O_3) For a path $lkjzj'k'\ell'$ and vertices i and i' together with edges ij and $i'j'$, where $i, i', \ell, \ell' \in P'$, $j, j', k, k' \in Q'$ and $z \in R'$, we obtain T from T' by adding a path $vuxy$ and an edge xz .
- (O_4) For a path $v'u'x'y'$, where $u', x' \in Q'$ and $v', y' \in P'$, we obtain T from T' by adding a vertex z , a path $vuxy$ and edges $x'z$ and zx .
- (O_5) For a vertex $w \in R'$ we obtain T from T' by adding a path $vuxzx'u'v'$ and vertices y and y' together with edges xy , $x'y'$ and zw .

It is easy to see that no operation $O_1 - O_5$ can be replaced by a combination of other operations, so these five operations are independent. The paths P_{3k+1} can be obtained from K_1 only by k times applying the operation O_1 , so operation O_1 is needed. The sequence of operations O_1, O_2 gives us a tree with a unique maximum packing that cannot be obtained in any other way, see Figure 8(a). Also the sequence O_1, O_4 results in the leftmost tree of Figure 2, so operation O_4 is needed. In addition, if to the previous sequence we add either O_3 to get O_1, O_4, O_3 or O_5 to get O_1, O_4, O_5 we again obtain two trees that cannot be obtained in any other way, see Figure 8(b) and 8(c). Hence we need all five operation and since the mentioned sequences are unique, the operations $O_1 - O_5$ are independent.

Theorem 7. *A tree T has a unique $\rho(T)$ -set P if and only if T can be obtained from K_1 by a sequence of operations $O_1 - O_5$.*

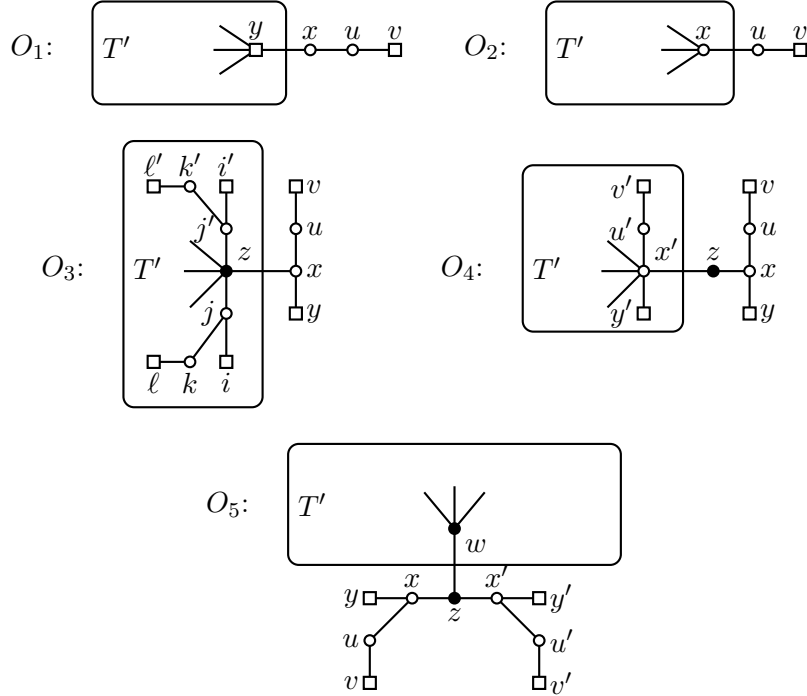


Figure 7. Operations $O_1 - O_5$.

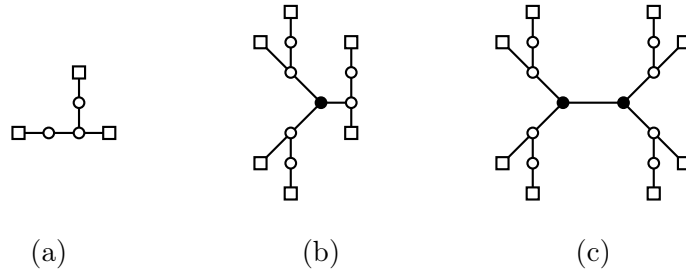


Figure 8. Trees obtained from K_1 by applying operations O_1 and O_2 (a), O_1 , O_4 and O_3 (b) and O_1 , O_4 and O_5 (c).

Proof. Assume first that T is a tree obtained from K_1 by a sequence of operations $O_1 - O_5$. We will show that T is a tree with a unique $\rho(T)$ -set by induction on the length k of the mentioned sequence. If $k = 0$, then $T \cong K_1$ which is a tree with a unique $\rho(T)$ -set. Let now $k > 0$ and let T' be a tree obtained from K_1 by using the same sequence as for T , but without including the last step. By the induction hypothesis, T' is a tree with a unique $\rho(T')$ -set P' . We will use the notation presented in Figure 7.

Let T be a tree obtained from T' by operation O_1 . Clearly, $P = P' \cup \{v\}$ is a maximum packing of T because P' is a maximum packing of T' . Let P_1 be a packing of T such that either $u \in P_1$ or $x \in P_1$. We have $P_1 \cap V(T') \neq P'$, because $y \notin P_1$. By the uniqueness of P' we have $|P_1 \cap V(T')| < |P'|$. Therefore $|P_1| < |P|$ and P_1 is not a $\rho(T)$ -set. Hence P is the unique $\rho(T)$ -set.

Assume now T is obtained from T' by operation O_2 . We will prove that T has a unique $\rho(T)$ -set $P = P' \cup \{v\}$. Because $x \in Q$ there exists $y \in P'$ which is a neighbor of x . Let P_1 be a packing of T such that $u \in P_1$. Clearly, $P_1 \cap V(T') \neq P'$, because $y \notin P_1$. By the uniqueness of P' we have $|P_1 \cap V(T')| < |P'|$. Therefore $|P_1| < |P|$ and P_1 is not a $\rho(T)$ -set. Hence P is the unique $\rho(T)$ -set.

Suppose next that we apply operation O_3 on T' to get T . If $u \in P$ (or $x \in P$), then $P' \cup \{u\}$ (or $P' \cup \{x\}$) has $|P'| + 1$ elements. But setting $P = P' \cup \{v, y\}$ we get a packing of T with $|P'| + 2$ elements, so $P' \cup \{u\}$ (or $P' \cup \{x\}$) is not a $\rho(T)$ -set. Notice also that every packing P_1 of T with $|P_1 \cap V(T')| < |P'|$ has fewer than $|P'| + 2$ elements. Meaning that T has a unique $\rho(T)$ -set $P = P' \cup \{v, y\}$.

If operation O_4 is applied to get T from T' , then u and x do not belong to a $\rho(T)$ -set by the same argument as in the case of O_3 . Suppose that there exists a packing P_1 of T such that $z \in P_1$. Clearly, $y, y' \notin P_1$ and $|P_1 \cap V(T')| < |P'|$ because P' is the unique maximum packing of T' . This implies that $|P_1| < |P|$ for $P = P' \cup \{v, y\}$. Therefore, P_1 is not a maximum packing of T and P is the unique $\rho(T)$ -set.

Finally suppose that T is obtained from T' by operation O_5 . If a packing of T' contains x (or x'), then $P' \cup \{x, y', v'\}$ (or $P' \cup \{x', y, v\}$) has $|P'| + 3$ elements and no packing of T that contains x (or x') contains more elements. Similarly, if $u \in P$ (or $u' \in P$), then $P' \cup \{u, y', v'\}$ (or $P' \cup \{u', y, v\}$) contains $|P'| + 3$ elements, which is again best possible in this case. If $z \in P$, then $P' \cup \{z, v, v'\}$ has also $|P'| + 3$ elements and more is not possible. But setting $P = P' \cup \{v, y, v', y'\}$ we get a packing of T with $|P'| + 4$ elements, so any packing with $|P'| + 3$ elements or fewer is not a $\rho(T)$ -set. Meaning that T has a unique $\rho(T)$ -set $P = P' \cup \{v, y, v', y'\}$.

To prove the converse, let T be a tree with a unique $\rho(T)$ -set P . Let $r \in V(T)$ be a vertex of T and consider T as a rooted tree with the root r . We proceed by induction on the number of vertices of T . If $T \cong K_1$, then T is the smallest tree with a unique $\rho(T)$ -set, hence the base of the induction is clear. Let v be a leaf of T that is at the maximum distance from r . Obviously $v \neq r$ and let u be the support vertex of v . Clearly, $u \neq r$ because T has a unique $\rho(T)$ -set. Since v is a leaf of T that is at the maximum distance from r we know that $\deg(u) = 2$ by Lemma 2. Let x be the up-neighbor of u . Clearly, $x \notin P$ because $d(x, v) = 2$. Assume $x \in R$. By Lemma 5 we immediately get a contradiction since v is a leaf at the largest distance from r . Therefore $x \in Q$. Let y be a neighbor of x that is in P . If T has only four vertices, then $T \cong P_4 = vuxy$ and by deleting v, u and x we obtain $K_1 = y$. Clearly, we can obtain T from K_1 by operation O_1

and we are done. So we may assume that z is the up-neighbor of x and let w be the up-neighbor of z (if it exists). If x has a neighbor in R , then it must be the up-neighbor of x by Lemma 5 again. Denote by T' a tree obtained from T by deleting some vertices and by P' its $\rho(T')$ -set. We distinguish the following cases.

Case 1. $z \notin R$.

Subcase 1.1. $\deg(x) = 2$. Notice that in this case $z = y$. We obtain a tree T' from T by deleting vertices x , u and v , see Figure 9. Assume that T' has two $\rho(T')$ -sets P_1 and P_2 . Then $P_1 \cup \{v\}$ and $P_2 \cup \{v\}$ are both $\rho(T)$ -sets which is a contradiction with the uniqueness of the $\rho(T)$ -set. So T' has a unique $\rho(T')$ -set. By the induction hypothesis T' can be built from K_1 by a sequence of operations $O_1 - O_5$. If we add the operation O_1 at the end of this sequence, then we obtain T from K_1 by a sequence of operations $O_1 - O_5$.

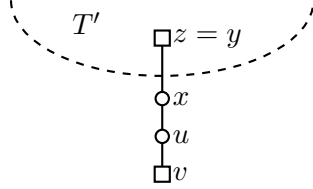
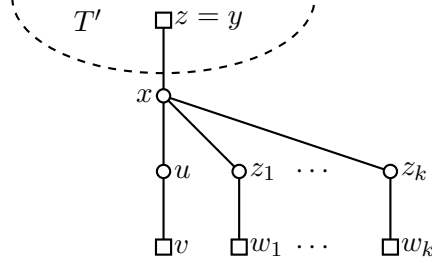


Figure 9. The case when $z \notin R$ and $\deg(x) = 2$.

Subcase 1.2. $\deg(x) \geq 3$. Let z_1, \dots, z_k be down-neighbors of x (different from u and different from y in the case that y is a down-neighbor of x). Again every z_i , $1 \leq i \leq k$, is not in R by Lemma 5 and the choice of v and is therefore in Q . Notice that every down-neighbor of z_i , $1 \leq i \leq k$, must be a leaf by the choice of v . By Lemma 2 every z_i , $1 \leq i \leq k$ has exactly one down-neighbor w_i which is in P since $y \in P$.

If $z = y$, then we obtain a tree T' from T by deleting a subtree rooted at x , see Figure 10. If there exist two $\rho(T')$ -sets P_1 and P_2 with $|P| - 1 - k$ elements, then $P_1 \cup \{v, w_1, \dots, w_k\}$ and $P_2 \cup \{v, w_1, \dots, w_k\}$ are both $\rho(T)$ -sets which is a contradiction with $\rho(T)$ -set being unique. So T' has a unique $\rho(T')$ -set. By the induction hypothesis T' can be built from K_1 by a sequence of operations $O_1 - O_5$. If we add the operation O_1 for x , u and v and k times operation O_2 for z_i and w_i , $1 \leq i \leq k$, at the end of this sequence, then we obtain T from K_1 by a sequence of operations $O_1 - O_5$.

If $z \neq y$ (so y is a down-neighbor of x), then $z \in Q$ as $d(y, z) = 2$. So z has one neighbor t which is in P . Denote with a_1, \dots, a_ℓ down-neighbors of z different from t (if t is a down-neighbor). Every a_i , $1 \leq i \leq \ell$, is not in R by Lemma 5 and by the choice of v . Also every a_i , $1 \leq i \leq \ell$ is not in P because $d(t, a_i) = 2$ and must therefore be in Q . Every vertex from Q has exactly one neighbor in P and let b_i be such a neighbor of a_i , $1 \leq i \leq \ell$. Because z is not in P , b_i is

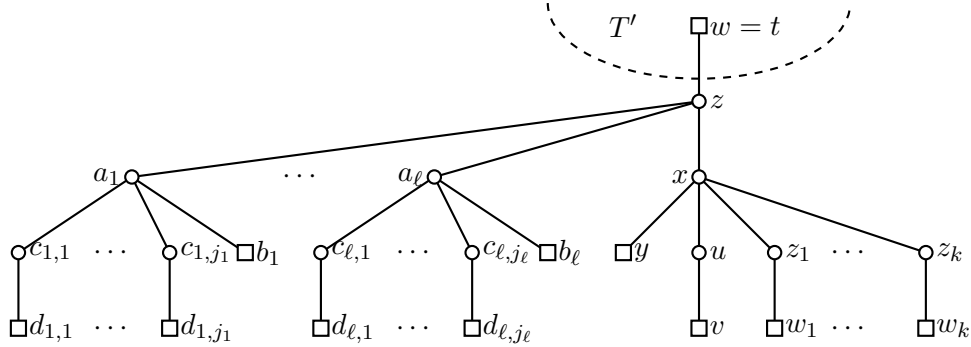
Figure 10. The case when $z \notin R$, $\deg(x) \geq 3$ and $z = y$.

a down-neighbor of a_i . In addition every a_i , $1 \leq i \leq \ell$, can have $c_{i,1}, \dots, c_{i,j_i}$ down-neighbors which must be in Q (they cannot be in R by Lemma 5 and are not in P as $d(c_{i,s}, b_i) = 2$ for $1 \leq s \leq j_i$). Furthermore, every $c_{i,s}$, $1 \leq i \leq \ell$ and $1 \leq s \leq j_i$, has exactly one down-neighbor $d_{i,s}$ by the choice of v and by Lemma 2 and $d_{i,s}$ must be in P by Lemma 1.

If $w = t$ we obtain a tree T' from T by deleting a subtree rooted at z , see Figure 11. If there exist two $\rho(T')$ -sets P_1 and P_2 with $|P| - 2 - k - \ell - j_1 - \dots - j_\ell$ elements, then for

$$A = \{v, y, w_1, \dots, w_k, b_1, \dots, b_\ell, d_{1,1}, \dots, d_{1,j_1}, \dots, d_{\ell,1}, \dots, d_{\ell,j_\ell}\}$$

sets $P_1 \cup A$ and $P_2 \cup A$ are both $\rho(T)$ -sets which is a contradiction with $\rho(T)$ -set being unique. So T' has a unique $\rho(T')$ -set. By the induction hypothesis T' can be built from K_1 by a sequence of operations $O_1 - O_5$. If we add the operation O_1 for z , x and y and operation O_2 once for u and v , k times for z_i and w_i , $1 \leq i \leq k$, ℓ times for a_i and b_i , $1 \leq i \leq \ell$, and $j_1 + \dots + j_\ell$ times for $c_{i,q}$ and $d_{i,q}$, $1 \leq i \leq \ell$ and $1 \leq q \leq j_i$, at the end of this sequence, then we obtain T from K_1 by a sequence of operations $O_1 - O_5$.

Figure 11. The case when $z \notin R$, $\deg(x) \geq 3$, $z \neq y$ and $w = t$.

If $w \neq t$ (so t is a down-neighbor of z) we obtain a tree T' from T by

deleting vertices u and v , see Figure 12. Notice that no $\rho(T')$ -set contains x since $(P' - \{x\}) \cup \{y, t, w_1, \dots, w_k\}$ would be a packing of cardinality greater than a $\rho(T')$ -set. If there exist two $\rho(T')$ -sets P_1 and P_2 with $|P_i| = \rho(T')$ elements, then $P_1 \cup \{v\}$ and $P_2 \cup \{v\}$ are both $\rho(T)$ -sets which is a contradiction with $\rho(T)$ -set being unique. So T' has a unique $\rho(T')$ -set. By the induction hypothesis T' can be built from K_1 by a sequence of operations $O_1 - O_5$. If we add the operation O_2 for u and v at the end of this sequence, then we obtain T from K_1 by a sequence of operations $O_1 - O_5$.

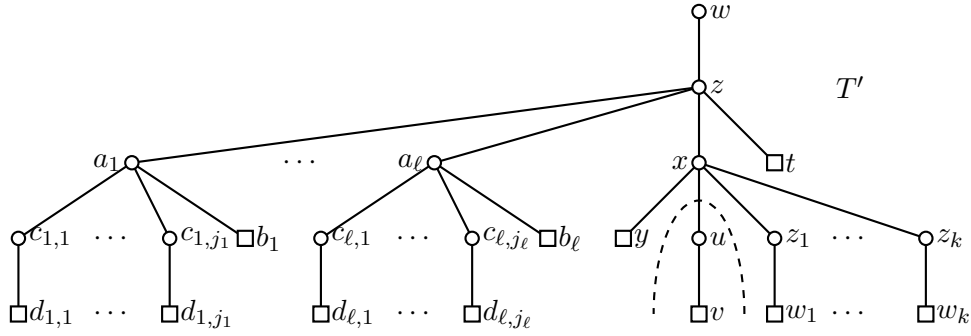


Figure 12. The case when $z \notin R$, $\deg(x) \geq 3$, $z \neq y$ and $w \neq t$.

Case 2. $z \in R$. In this case there exists $y \in P$ that is a down-neighbor of x otherwise also $(P - \{v\}) \cup \{u\}$ is a $\rho(T)$ -set which is not possible. Denote by z_1, \dots, z_k down-neighbors of x (if they exist) different from y and u . Clearly, every z_i , $1 \leq i \leq k$, is in Q as they cannot be in R by Lemma 5 nor in P because $d(z_i, y) = 2$. By Lemma 1 every z_i , $1 \leq i \leq k$, has a down-neighbor w_i which is in P . By the choice of v and by Lemma 2 w_i is the unique down-neighbor of z_i . Let a_1, \dots, a_ℓ be down-neighbors of z if they exist (see Figure 13 for the case that they do not exist). Clearly, every a_i , $1 \leq i \leq \ell$, is in Q because a_i cannot be in R by Lemma 5 and the choice of v and not in P as neighbors of z . Every a_i , $1 \leq i \leq \ell$, has exactly one down-neighbor t_i which is in P . All the other down-neighbors of a_i are in Q because they are at distance two from t_i and therefore not in P and not in R by the choice of v and by Lemma 5. We denote them by $b_{a_i, j}$, $1 \leq i \leq \ell$, $1 \leq j \leq m_i$. Every $b_{a_i, j}$, $1 \leq i \leq \ell$, $1 \leq j \leq m_i$, has exactly one down-neighbor $c_{a_i, j} \in P$. By the choice of v and by Lemma 2, $c_{a_i, j}$ is the unique down-neighbor of $b_{a_i, j}$. Notice that by Lemma 3 for t_i and a_i we have $m_i \geq 1$ for every $1 \leq i \leq \ell$. See Figures 14 and 15 for this constellation. We will use the following notation

$$A = \{v, y, w_1, \dots, w_k, t_1, \dots, t_\ell, c_{a_1, 1}, \dots, c_{a_1, m_1}, \dots, c_{a_\ell, 1}, \dots, c_{a_\ell, m_\ell}\}.$$

Subcase 2.1. $\deg(z) = 2$. Notice that in this case vertices a_i , $1 \leq i \leq \ell$, do not exist whenever $z \neq r$ and if $z = r$, then $\ell = 1$. We obtain a tree T' from T

by deleting a subtree rooted at z , see Figure 13. Suppose T' has two $\rho(T')$ -sets P_1 and P_2 . In that case $P_1 \cup \{v, y, w_1, \dots, w_k\}$ and $P_2 \cup \{v, y, w_1, \dots, w_k\}$ are both $\rho(T)$ -sets which is a contradiction with the uniqueness of $\rho(T)$ -set. Meaning that T' has a unique $\rho(T')$ -set. By the induction hypothesis T' can be built from K_1 by a sequence of operations $O_1 - O_5$. If we add the operation O_4 for vertices z, y, x, u, v at the end of this sequence and then continue with k times adding operation O_2 for z_i and w_i , $1 \leq i \leq k$, then we obtain T from K_1 by a sequence of operations $O_1 - O_5$.

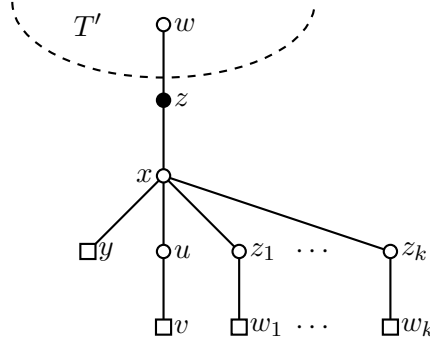


Figure 13. The case when $z \in R$ and $\deg(z) = 2$.

Subcase 2.2. $\deg(z) \geq 3$ and z does not have any neighbors in R . Clearly, $w \in Q$, since z does not have any neighbors in R . So w has a neighbor $s_1 \in P$ see Figure 14. By Lemma 3 there exists a neighbor $s_2 \in Q$ of w and a neighbor $s_3 \in P$ of s_2 .

We obtain a tree T' from T by deleting a subtree rooted at z . If there exist two different $\rho(T')$ -sets P_1 and P_2 with $|P| - 2 - k - \ell - m_1 - \dots - m_\ell$ elements, then sets $P_1 \cup A$ and $P_2 \cup A$ are both $\rho(T)$ -sets which is a contradiction with $\rho(T)$ -set being unique. So T' has a unique $\rho(T')$ -set. By the induction hypothesis T' can be built from K_1 by a sequence of operations $O_1 - O_5$. If we add the operation O_4 for z, x, y, u, v , ℓ times operation O_3 for $a_i, t_i, b_{a_i,1}, c_{a_i,1}$, $1 \leq i \leq \ell$, $m_1 + \dots + m_\ell - \ell$ times operation O_2 for $b_{a_i,r}, c_{a_i,r}$, $1 \leq i \leq \ell$, $2 \leq r \leq m_i$, and k times operation O_2 for z_i, w_i , $1 \leq i \leq k$, at the end of this sequence, then we obtain T from K_1 by a sequence of operations $O_1 - O_5$.

Subcase 2.3. $\deg(z) \geq 3$ and z does have a neighbor in R . By Lemma 5 and the choice of v , vertex w is in the set R , see Figure 15. Tree T' is obtained from T by deleting a subtree rooted at z . By Lemma 5 z has at least one down-neighbor different from x , which means that $\ell \geq 1$. Suppose T' has two different $\rho(T')$ -sets P_1 and P_2 . In that case $P_1 \cup A$ and $P_2 \cup A$ are both $\rho(T)$ -sets which is a contradiction with the uniqueness of $\rho(T)$ -set. So T' has a unique $\rho(T')$ -set. By the induction hypothesis T' can be built from K_1 by a sequence of operations

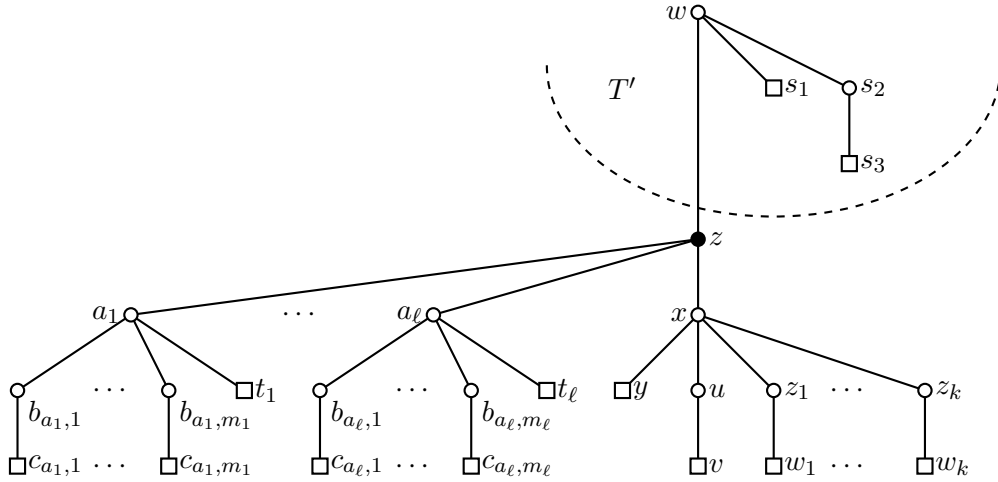


Figure 14. The case when $z \in R$, $\deg(z) \geq 3$ and z does not have any neighbors in R .

$O_1 - O_5$. If we add the operation O_5 for $z, x, y, u, v, a_1, t_1, b_{a_1,1}, c_{a_1,1}$, $\ell - 1$ times operation O_3 for $a_i, t_i, b_{a_i,1}, c_{a_i,1}$, $2 \leq i \leq \ell$, $m_1 + \dots + m_\ell - \ell$ times operation O_2 for $b_{a_i,r}, c_{a_i,r}$, $1 \leq i \leq \ell$, $2 \leq r \leq m_i$, and k times operation O_2 for z_i, w_i , $1 \leq i \leq k$, at the end of this sequence, then we obtain T from K_1 by a sequence of operations $O_1 - O_5$.

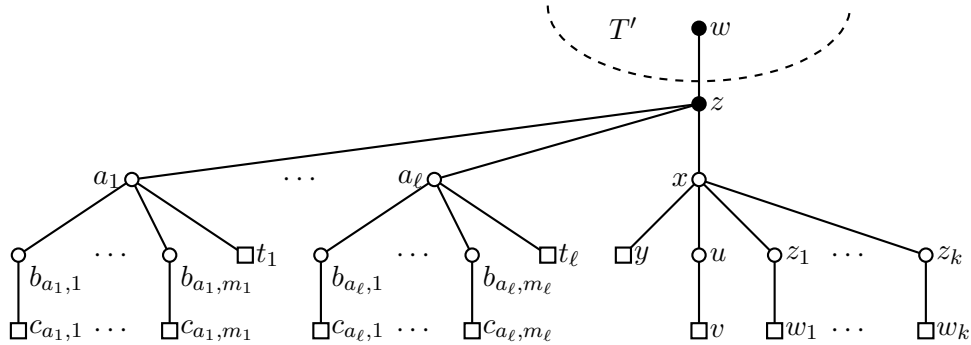


Figure 15. The case when $z \in R$, $\deg(z) \geq 3$ and z does have a neighbor in R . ■

Acknowledgment

Both authors are also with Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia, and their work was partially supported by Slovenian research agency under the grants P1-0297 and J1-9109.

We are also thankful to the anonymous referee for a careful reading of the manuscript which helped us to improve the final version, in particular Theorem 6.

REFERENCES

- [1] N. Biggs, *Perfect codes in graphs*, J. Combin. Theory Ser. B **15** (1973) 289–296.
doi:10.1016/0095-8956(73)90042-7
- [2] B. Brešar, K. Kuenzel and D. F. Rall, *Graphs with a unique maximum open packing* (2019).
arXiv1901.09859
- [3] M.P. Dobson, E. Hinrichsen and V. Leoni, *Generalized limited packings of some graphs with a limited number of P_4 -partners*, Theoret. Comput. Sci. **579** (2015) 1–8.
doi:10.1016/j.tcs.2014.11.014
- [4] R.D. Dutton, *Global domination and packing numbers*, Ars Combin. **101** (2011) 489–501.
- [5] A. Gagarin and V. Zverovich, *The probabilistic approach to limited packings in graphs*, Discrete Appl. Math. **184** (2015) 146–153.
doi:10.1016/j.dam.2014.11.017
- [6] R. Gallant, G. Gunther, B. Hartnell and D.F. Rall, *Limited packings in graphs*, Discrete Appl. Math. **158** (2010) 1357–1364.
doi:10.1016/j.dam.2009.04.014
- [7] G. Gunther, B.L. Hartnell, L.R. Markus and D.F. Rall, *Graphs with unique minimum dominating sets*, Congr. Numer. **101** (1994) 55–63.
- [8] T.W. Haynes and M.A. Henning, *Trees with unique minimum total dominating sets*, Discuss. Math. Graph Theory **22** (2002) 233–246.
doi:10.7151/dmgt.1172
- [9] M.A. Henning, C. Löwenstein and D. Rautenbach, *Dominating sets, packings, and the maximum degree*, Discrete Math. **311** (2011) 2031–2036.
doi:10.1016/j.disc.2011.04.030
- [10] G. Hopkins and W. Staton, *Graphs with unique maximum independent sets*, Discrete Math. **57** (1985) 245–251.
doi:10.1016/0012-365X(85)90177-3
- [11] M.-J. Jou and G.J. Chang, *The number of maximum independent sets in graphs*, Taiwanese J. Math. **4** (2000) 685–695.
doi:10.11650/twjm/1500407302
- [12] K. Junosza-Szaniawski and P. Rzażewski, *On the number of 2-packings in a connected graph*, Discrete Math. **312** (2012) 3444–3450.
doi:10.1016/j.disc.2012.02.005
- [13] A.P. Kazemi, B. Pahlavsay and R.J. Stones, *Cartesian product graphs and k -tuple total domination*, Filomat **32** (2018) 6713–6731.
doi:10.2298/FIL1819713K

- [14] S. Klavžar, I. Peterin and I.G. Yero, *Graphs that are simultaneously efficient open domination and efficient closed domination graphs*, Discrete Appl. Math. **217** (2017) 613–621.
doi:10.1016/j.dam.2016.09.027
- [15] A. Meir and J.W. Moon, *Relations between packing and covering numbers of a tree*, Pacific J. Math. **61** (1975) 225–233.
doi:10.2140/pjm.1975.61.225
- [16] D.A. Mojdeh, B. Samadi, A. Khodkar and H.R. Golmohammadi, *On the packing numbers in graphs*, Australas. J. Combin. **71** (2018) 468–475.
- [17] I. Sahul Hamid and S. Saravanakumar, *Packing parameters in graphs*, Discuss. Math. Graph Theory **35** (2015) 5–16.
doi:10.7151/dmgt.1775

Received 26 March 2019
Revised 20 January 2020
Accepted 20 January 2020