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GAME-PERFECT SEMIORIENTATIONS OF FORESTS

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Abstract

We consider digraph colouring games where two players, Alice and Bob, alternately colour vertices of a given digraph D with a colour from a given colour set in a feasible way. The game ends when such move is not possible any more. Alice wins if every vertex is coloured at the end, otherwise Bob wins. The smallest size of a colour set such that Alice has a winning strategy is the game chromatic number of D. The digraph D is game-perfect if, for every induced subdigraph H of D, the game chromatic number of H equals the size of the largest symmetric clique of H. In the strong game, colouring a vertex is feasible if its colour is different from the colours of its in-neighbours. In the weak game, colouring a vertex is feasible unless it creates a monochromatic directed cycle. There are six variants for each game, which specify the player who begins and whether skipping is allowed

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for some player. For all six variants of both games, we characterise the class of game-perfect semiorientations of forests by a set of forbidden induced subdigraphs and by an explicit structural description.

Keywords: game chromatic number, game-perfect digraph, forest, dichromatic number, game-perfect graph, forbidden induced subdigraph.

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1. Introduction

In this paper, we consider the *strong digraph colouring game* and the *weak digraph colouring game* introduced by Andres [1] and Yang and Zhu [20], respectively.

The strong digraph colouring game was studied in some recent papers [1, 2, 5, 12, 20]. In this game, two players, Alice and Bob, alternately choose a colour c from a given set and colour an uncoloured vertex v of an initially uncoloured, simple and finite digraph D, under the constraint that v does not have any in-neighbour which has been coloured with c. Alice wins if all vertices of D can be coloured finally; otherwise, Bob wins.

Andres [5] considered six variants of the game. Depending on which one we play with, Alice or Bob is the first player, and one of them may be allowed to skip turns. We denote these variants by $g = [X, Y]$. The player $X \in \{A, B\}$ takes the first move and $Y \in \{A, B, -\}$ has the right to skip any number of turns. $A, B, -$ denote Alice, Bob, and none of the players, respectively. The *g-game chromatic number* $\chi_g(D)$ of a digraph D is the smallest $t \in \mathbb{N}$ such that Alice has a winning strategy for the strong digraph colouring game with t colours under the g variant.

The concept of game-perfect digraphs was introduced and first studied by Andres [5]. A *symmetric clique* is a digraph such that between any two different vertices u, v the arcs (u, v) and (v, u) exist. The clique number $\omega(D)$ of a digraph D is the number of vertices of the largest symmetric clique in D. It is clear that $\omega(D) \leq \chi_g(D)$ for any D and g, since all vertices of a symmetric clique should have different colours. For each variant g, a digraph D is *g-perfect* if any induced subdigraph H of D has $\omega(H) = \chi_g(H)$.

A non-game analogue of the game chromatic number is the *dichromatic number* of a digraph introduced by Neumann-Lara [18], which is the smallest number of colours used in a (not necessarily proper) colouring of the vertices of the digraph such that the colour classes do not contain monochromatic directed cycles. A digraph D is *perfect* if, for any induced subdigraph H of D, the dichromatic number and clique number of H are equal. Since the dichromatic number is an obvious lower bound on the game chromatic number, any game-perfect digraph is also a perfect digraph. Using the Strong Perfect Graph Theorem [13], which concerns a characterisation of perfect undirected graphs, Andres and Hochstättler [7] characterised perfect digraphs by a set of forbidden induced subdigraphs, which generalizes the Strong Perfect Graph Theorem. In this paper, we consider similar characterisations with respect to games.

By considering undirected graphs as symmetric digraphs, the dichromatic number generalizes the chromatic number of an undirected graph. In the same way, the digraph colouring game is a generalization of the well-known graph colouring game [15] that was made popular by the works of Bodlaender [10] and Faigle *et al.* [14]. More references on the general topic of graph colouring games can be found in the recent survey by Tuza and Zhu [19].

Game-perfect undirected graphs were introduced by Andres [3]. A characterisation of game-perfect undirected graphs by forbidden induced subgraphs and by explicit structural descriptions was given by Andres [4] for the games $[B, B]$, $[A, B]$, and $[A, -]$ and by Lock [17] and Andres and Lock [8] for the game $[B, -]$.

To deal with digraphs, a *semiorientation* of a graph G is a digraph D on the same vertex set such that every edge vw of G is replaced by either an arc (v, w) or (w, v) or both. Andres [5] proposed the problem of characterising g-perfect digraphs D and partially solved it with respect to the clique number of D. The problem with respect to clique number 1 is trivial and that with respect to clique number 2 was partially solved: Andres characterised g-perfect semiorientations of paths, cycles and complete graphs with clique number 2 for all the six variants.

In this paper, we give further results on the characterisation problem with respect to clique number 2. We characterise game-perfect semiorientations of forests, which have clique number at most 2, by a set of forbidden induced subdigraphs and by an explicit structural description. Since paths are forests, our results include the result on semiorientations of paths given by Andres [5]. The two main results of this paper are stated as follows.

Theorem 1. *For a semiorientation D of a forest, the following are equivalent.*
- (i) *D is $[A, A]$-perfect.*
- (ii) *D does not contain any of the following* 24 *forbidden configurations* (*depicted in Figure* 2) *as an induced subdigraph: the* 6 *in-chairs, the* 6 *in-brooms, the* 2 *in-P_5s, F_4, $F_{3,1}$, $F_{3,2}$, F_8, $F_{\rightarrow}^{(1)}$, $F_{\rightarrow}^{(2)}$, $F_{+}^{(1)}$, $F_{+}^{(2)}$, $F_{+}^{(3)}$, $F_{+}^{(4)}$.*
- (iii) *D is either empty or D has a component of one of the types $E_1, \ldots, E_{12}$* (*depicted in Figure* 6) *and every other component of D is a P_4 or a star.*

Theorem 2. *For a semiorientation D of a forest, the following are equivalent.*
- (i) *D is $[A, B]$-perfect.*
- (i') *D is $[A, -]$-perfect.*
- (ii) *D does not contain any of the following* 7 *configurations as an induced subdigraph: the* 3 *in-P_4s* (*see Figure* 10), *F_4, $F_{3,1}$, $F_{3,2}$, $F_{+}^{(3)}$.*
- (iii) *D is either empty or D has a component of one of the types $E_1^A, \ldots, E_4^A$* (*depicted in Figure* 11) *and every other component of D is a star.*

For the games that Bob begins, i.e., $[B, A]$, $[B, -]$, and $[B, B]$, which are much easier to handle, we give similar characterisations in Theorems 36, 37, and 38, respectively, in Section 5. Thus, we characterise game-perfect directed forests for all six possible variants of the strong digraph colouring game.

Yang and Zhu [20] proposed a different digraph colouring game: the weak digraph colouring game. Both games are identical to the undirected graph colouring game when restricted to undirected graphs. A notion of game-perfectness can be defined also for the weak digraph colouring games (cf. [6]). In Section 6, we give characterisations for weakly game-perfect semiorientations of forests for any variant of the weak digraph colouring game.

Here is an outline of the rest of this paper. Terminologies and notations will be introduced in Section 2. The proofs of Theorem 1 and 2 will be given in Section 3 and 4, respectively. Section 5 is for the three variants of the strong digraph colouring game that Bob begins. We deal with the weak digraph colouring game for all the six variants in Section 6. Some open questions on digraph colouring games will be discussed in Section 7.

2. Preliminaries

2.1. Basic notation and terminology of digraphs

We consider digraphs of the form (V, A), where V is a finite set of *vertices* and $A \subseteq V \times V \setminus \{(v, v) \mid v \in V\}$ is the set of *arcs*. In particular, this means the digraphs we consider have neither loops nor multiple arcs.

An *in-arc* of a vertex v is (u, v) for some vertex u, an *out-arc* of vertex v is (v, w) for some vertex w. A *single arc* is an arc (u, v) such that (v, u) does not exist. If both (u, v) and (v, u) exist, $uv = \{(u, v), (v, u)\}$ is called an *edge*, and u is called a *symmetric neighbour* of v. An arc is either a single arc or an element of an edge.

A *directed component* of a digraph is a component containing at least one single arc. A *symmetric digraph* is a digraph without single arcs. Therefore, a symmetric digraph can be interpreted as and also called an undirected graph by interpreting the two arcs of every edge as an edge in the context of undirected graphs. The *underlying graph* $G(D)$ of a digraph D is the undirected graph obtained by replacing all single arcs (v, u) in D by the edge vu, which makes D symmetric. A *semiorientation* D of an undirected graph G is any digraph D such that $G(D) = G$. There is no common terminology for the concept of semiorientation. For example, semiorientations are also called *biorientations* (by Bang-Jensen and Gutin [9]) and *superorientations* (by Boros and Gurvich [11]).

For short, in the rest of this paper, we will call any semiorientation of a tree, a forest, or a path simply a *tree*, a *forest*, or a *path*, respectively. Also, a connected induced subdigraph of a semiorientation of a tree will be simply called a *subtree*.

The *symmetric part* $S(D)$ of a digraph $D = (V, A)$ is the digraph (V, E), where E is the union of all edges. In other words, $S(D)$ is the maximal symmetric subdigraph of D. Recall that a *symmetric clique* is a symmetric complete digraph. The *clique number* $\omega(D)$ of a digraph D is the number of vertices of the largest symmetric clique in D.

If (u, v) exists, regardless of the existence of (v, u), u is called an *in-neighbour* of v and v is called an *out-neighbour* of u. The *degree* of a vertex v in a digraph D is the degree of v in $G(D)$.

The *distance* between two arcs (u, v) and (u', v') in a semiorientation of a tree D, where $\{u, v\} \neq \{u', v'\}$, is denoted by $\mathrm{dist}((u, v), (u', v'))$ and defined as follows. In the undirected tree $G(D)$, there is a unique path with its starting and ending edges being uv and $u'v'$. This path P has length 2 if uv and $u'v'$ are adjacent; otherwise, it has length at least 3. The distance between (u, v) and (u', v') in D is then defined as $\ell - 2$ where ℓ is the length of P.

We refer to the monography of Bang-Jensen and Gutin [9] for undefined terms or notation in this paper. For example, $d^+(v)$ and $d^-(v)$ denote the out-degree and the in-degree of a vertex v, respectively.

2.2. Terminology of strong digraph colouring games

The following definitions in this Section 2.2 and the definitions in Section 2.3 refer to strong digraph colouring games. Definitions for weak digraph colouring games will be given in Section 6 when they are needed. A *partial colouring* of a digraph is an assignment of colours to some of the vertices. For a strong digraph colouring game g, an uncoloured or a partially coloured digraph D is *k-g-permitted* if Alice has a winning strategy for g played with k colours on D and *k-g-unpermitted* otherwise. An uncoloured digraph D is *g-nice* if $\omega(D) = \chi_g(D)$. Thus, D is g-perfect if and only if every of its induced subdigraphs is g-nice. By definition, a g-nice digraph with clique number k is k-g-permitted.

During a game, colouring a vertex v with colour c is a *Bob-winning move* if v is uncoloured, c is available for v, and colouring v with c makes some out-neighbour of v uncolourable. Two or more Bob-winning moves that exist at the same turn are called *independent* if colouring any single vertex at this turn can eliminate at most one of them.

Therefore, if it is Alice's turn and colouring v with c is a Bob-winning move, Bob can win on his next turn unless Alice colours v or some of its neighbours at this turn. If there exist at least two independent Bob-winning moves, Bob wins the game. These observations will be employed in the proof of Theorem 1.

For any digraph D, the six games are related in the following way (see [5]).

$$\omega(D) \leq \chi(D) \leq \chi_{[A,A]}(D) \leq \left\{ \begin{array}{c} \chi_{[A,-]}(D) \leq \chi_{[A,B]}(D) \\ \chi_{[B,A]}(D) \leq \chi_{[B,-]}(D) \end{array} \right\} \leq \chi_{[B,B]}(D). \tag{1}$$

Consequently, if we denote the set of g-perfect digraphs by $\mathcal{GP}_g$ for each g, and the set of perfect digraphs by $\mathcal{P}$, we have

$$\mathcal{GP}_{[B,B]} \subseteq \left\{ \begin{array}{c} \mathcal{GP}_{[A,B]} \subseteq \mathcal{GP}_{[A,-]} \\ \mathcal{GP}_{[B,-]} \subseteq \mathcal{GP}_{[B,A]} \end{array} \right\} \subseteq \mathcal{GP}_{[A,A]} \subseteq \mathcal{P}. \tag{2}$$

2.3. Threatening out-degree

In this subsection, we will introduce the concept of threatening out-degree. The motivation of it is to simplify the proof of Theorem 1 (i)$\Longrightarrow$(ii).

In a digraph D, a vertex v is *safe* if $d^-(v) < \omega(D)$, otherwise it is *unsafe*. We remark that an unsafe vertex might become uncolourable during the game, whereas a safe vertex can always be coloured. In an uncoloured digraph D, the *threatening out-degree* of a vertex v, denoted by $d^+_{\text{thr}}(v)$, is the number of unsafe out-neighbours of v.

For example, in the subfigure with caption $F_{7,1}$ in Figure 5, vertices are attached with their corresponding threatening out-degrees. Safe and unsafe vertices are represented by unfilled and filled vertices, respectively. In $F_{7,1}$, v is safe since v has exactly one in-neighbour and $\omega(F_{7,1}) = 2$. Also, $d^+_{\text{thr}}(v) = 2$ since v has exactly two unsafe out-neighbours. Since the leaf adjacent to u is safe, u has two neighbours but only one unsafe out-neighbour, which implies $d^+_{\text{thr}}(u) = 1$.

Intuitively, the threatening out-degree of a vertex v measures the threat of colouring v to Alice at the beginning of the game. Therefore, at the beginning of any variant where Alice takes the first move, Alice may prefer to skip or colour vertices with threatening out-degree 0. This intuition will be rigorously presented in the following lemma.

Lemma 3. *Let $Y \in \{A, B, -\}$. Let D be a digraph with $\omega(D) \leq 2$. For the $[A, Y]$-game played with $\omega(D)$ colours on D, if Alice colours a vertex v with $d^+_{\text{thr}}(v) \geq 1$ in her first move of the digraph colouring game, then Bob will win the game. Equivalently, in any of Alice's winning strategies for this game, Alice's first move is either colouring a vertex v with $d^+_{\text{thr}}(v) = 0$ or skipping.*

Proof. If Alice colours a vertex v with $d^+_{\text{thr}}(v) \geq 1$ in her first move, an unsafe out-neighbour of v, denoted by u, will have no available colours in the game played with one colour. For the game with two colours, let again u be an unsafe out-neighbour of v, the vertex Alice has coloured. The vertex u exists, since

$d^+_{\mathrm{thr}}(v) \geq 1$. Since u is unsafe and $\omega(D) = 2$, the vertex u has at least two in-neighbours. Therefore, Bob can colour an in-neighbour w of u with $w \neq v$ with the other colour so that u has no available colours. ■

The above lemma will be employed in the proof of Theorem 1 (i)$\Longrightarrow$(ii).

2.4. Notation concerning structures

By P_n, C_n, K_n, and $(n-1)$-*star* we denote the undirected path, cycle, complete graph and star of n vertices, respectively. The smallest star is the 0-star, which consists of one vertex. An *out-leaf arc* is a single arc (u, v), so that u is the unique neighbour of v, i.e., $d^-(v) = 1$ and $d^+(v) = 0$. A *k-in-star* is a digraph consisting of $k+1$ vertices and k single arcs which point towards a unique central vertex.

Let v be a vertex in a semiorientation D of a tree. For an integer $k \geq 2$ we define the following types of subdigraphs of D.

- A *pending star at v* is an undirected k-star such that v is a leaf of the star.
- A *P_k at v* is an undirected P_k such that v is a leaf of the P_k.
- A *broken P_k at v* is an undirected P_k such that v is an internal vertex of the P_k.

Moreover, for vertex v and an integer $k \geq 0$, we define, as depicted in Figure 1.

- A *star* (*k-star*) *at v* is an undirected star (undirected k-star) such that v is the center of the star. (If $k \in \{0, 1\}$, then an arbitrary vertex of the star can be considered as center.)
- A *2-gadget at v* is a star or P_3 at v.
- A *P-gadget at v* is a star or pending star at v.
- A *4-gadget at v* is a star, pending star, P_4 or broken P_4 at v.
- A *3-gadget at v* is a 3-star, P_4 or broken P_4 at v.

Note that since k can be zero, a star can be a single vertex, i.e., just v exists. Therefore, a 2-gadget, a P-gadget or a 4-gadget can also be a single vertex.

Let T be a tree and v be one of its vertices. A *v-branch* of T is the subtree induced by v and all vertices of a component of $T - v$. For the v-branch H of T containing a vertex $w \neq v$, we define the *truncated v-branch* containing w, denoted by H_w, as $H_w = H - v$.

2.5. Explanation of the figures

In the figures of this paper, single arcs are depicted by arrows and edges are depicted by lines. Configurations that might be repeated an arbitrary number of times are indicated by multiple dots. Stars, 2-gadgets, P-gadgets, 4-gadgets and 3-gadgets at a vertex v are depicted by the triangles given in Figure 1.

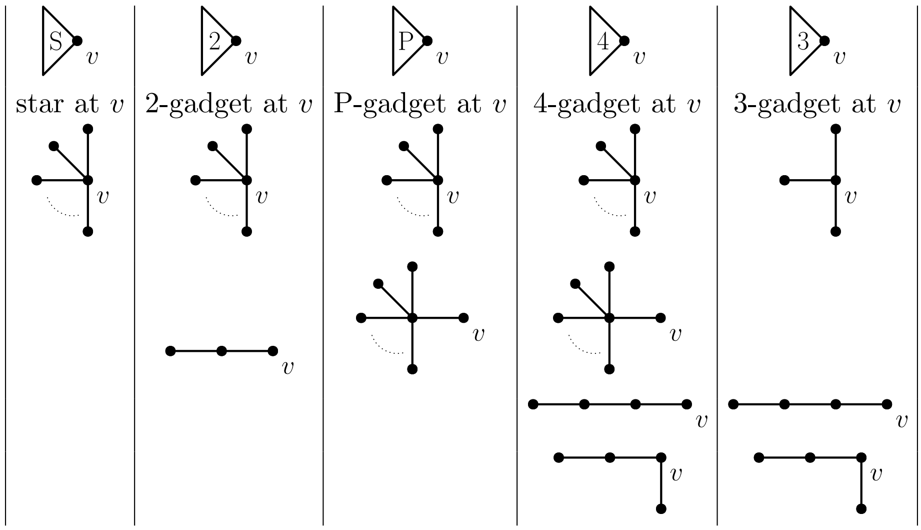

Figure 1. The gadgets and the types of graphs they represent.

3. $[A, A]$-Perfect Forests: Proof of Theorem 1

Our method to prove Theorem 1 is inspired by the methods developed in [4], which were also used by Lock [17] and Andres and Lock [8]. We start with an outline of the proof.

Outline of the proof of Theorem 1. In Section 3.1 we will define 24 digraphs, which we call *forbidden digraphs*. In Section 3.2 we will prove that Bob has a winning strategy for the game $[A, A]$ on each of the 24 forbidden digraphs when the number of colours equals its clique number. This means that the forbidden digraphs are not $[A, A]$-perfect, thus (i)$\Longrightarrow$(ii) is proved by contraposition. We will define the 12 classes $E_1, \ldots, E_{12}$ of digraphs, which we call *permitted types*, in Section 3.3. Section 3.4 contains a structural characterisation of forests that do not contain any of the forbidden digraphs as an induced subdigraph. We will first remark that any component of such a forest is a P_4 or a star, except for at most one special component. Then, by a number of case distinctions, we will show that if such a special component exists, then the special component must be of one of the permitted types, which proves the implication (ii)$\Longrightarrow$(iii). Finally, in Section 3.5 we will prove that, for any permitted type, every digraph D belonging to this type is $[A, A]$-nice, by describing an explicit winning strategy of Alice. Also, we will prove that any subdigraph of D belongs to some permitted type. The two results together imply that the digraphs of each permitted type are $[A, A]$-perfect, establishing the implication (iii)$\Longrightarrow$(i).

3.1. Forbidden configurations

In Theorem 1, $[A, A]$-perfect forests are characterised by the thirteen forbidden types of induced subdigraphs shown in Figure 2. These thirteen types totally consist of twenty-four forbidden configurations. All the configurations of the types in-P_5, in-chair and in-broom are depicted in Figures 3, 4 and 5, respectively.

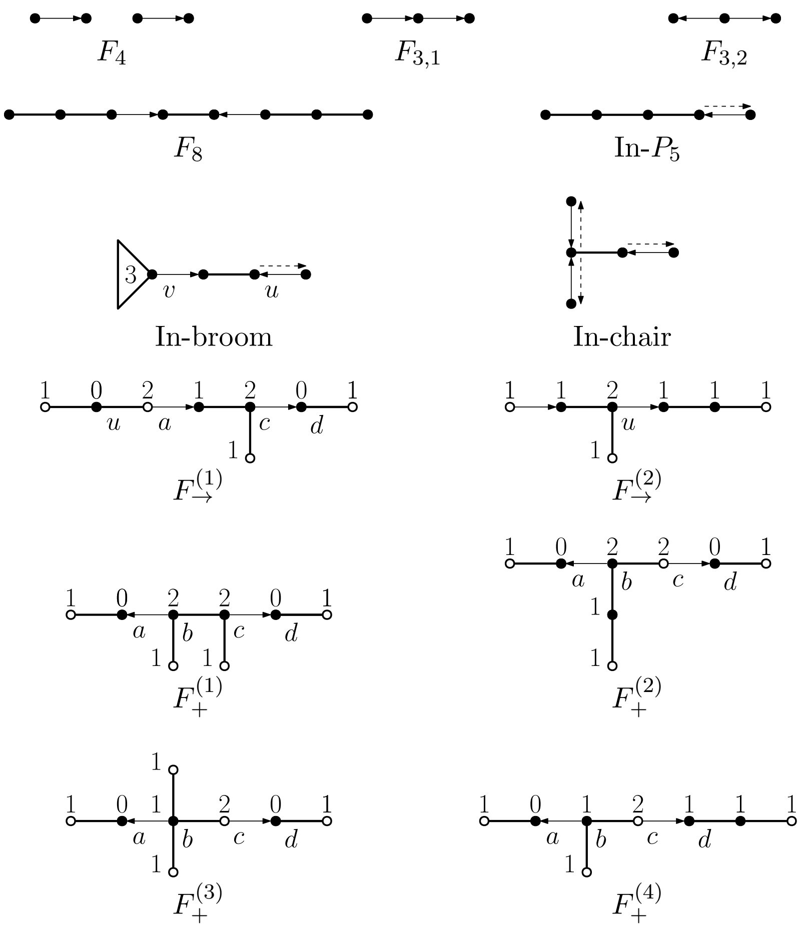

Figure 2. The thirteen types containing twenty-four forbidden configurations for $[A, A]$-perfect digraphs. In the last six depictions, unfilled circles denote safe vertices, filled circles unsafe vertices, and the numbers are the threatening out-degree of each vertex.

3.2. Proof of Theorem 1 (i)$\Longrightarrow$(ii)

It is sufficient to show that every digraph F in the list of forbidden types depicted in Figure 2 is $[A, A]$-*forbidden*, i.e., Bob has a winning strategy for the $[A, A]$-colouring game played on F with $\omega(F)$ colours. We will show them one by one. The case of forests of paths has been already discussed in [5].

Figure 3. The two in-P_5s.

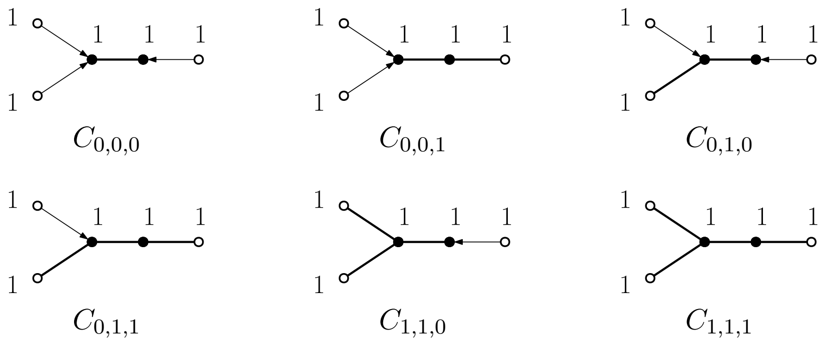

Figure 4. The six in-chairs. In the depictions, unfilled circles denote safe vertices, filled circles unsafe vertices, and the numbers are the threatening out-degree of each vertex.

Proposition 4 [5]**.** *The F_4 and the paths $F_{3,1}$, $F_{3,2}$, F_8 and the two in-P_5s are $[A, A]$-forbidden.*

Note that all in-chairs, all in-brooms, $F_{\rightarrow}^{(i)}$ $(1 \leq i \leq 2)$ and $F_{+}^{(i)}$ $(1 \leq i \leq 4)$ have clique number 2. Therefore, in the proofs of the following propositions, we will describe winning strategies of Bob for the $[A, A]$-colouring game with two colours played on these digraphs.

Proposition 5. *The six in-chairs and the six in-brooms are $[A, A]$-forbidden.*

Proof. For all the six in-chairs, let v be the vertex with degree 3 and u be the leaf that is not adjacent to v. For all the six in-brooms, by v and u we denote the corresponding vertices depicted in Figure 5.

Note that every vertex in any in-chair, respectively in any in-broom has threatening out-degree at least 1 (see Figures 4 and 5). Therefore, by Lemma 3, Alice's first move in any of her winning strategies on an in-chair or in-broom is skipping.

Then, in case of an in-broom that contains a P_4 or a broken P_4 at v, Bob can win by colouring v in his first move to generate two independent Bob-winning moves.

Otherwise, i.e., in case of an in-chair or an in-broom that contains a 3-star at v, Bob may colour a leaf adjacent to v with colour 1. Since colouring anyone of the remaining one or two leaves adjacent to v with colour 2 is a Bob-winning move, Alice must colour v with colour 2 in her second turn or, in case of an in-chair, colour the unique uncoloured leaf adjacent to v with colour 1. Then,

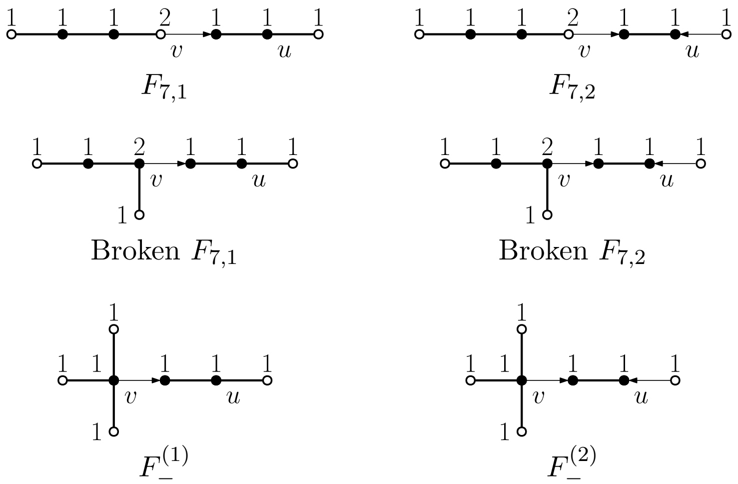

Figure 5. The six in-brooms. In the depictions, unfilled circles denote safe vertices, filled circles unsafe vertices, and the numbers are the threatening out-degree of each vertex.

in both cases, Bob will win after he colours u with 1. Therefore, Alice has no winning strategy. ■

Proposition 6. *$F_{\rightarrow}^{(1)}$ and $F_{\rightarrow}^{(2)}$ are $[A, A]$-forbidden.*

Proof. Suppose Alice has a winning strategy for $F_{\rightarrow}^{(1)}$. Since only u and d have threatening out-degree 0, in her first move, by Lemma 3, she colours u, d or skips. If she colours u, Bob may colour c with the same colour to generate two independent Bob-winning moves. For the remaining two choices of her first move, Bob may colour a to generate two independent Bob-winning moves.

Consider $F_{\rightarrow}^{(2)}$. Since all the vertices have non-zero threatening out-degree, by Lemma 3, Alice's first move in any of her winning strategies is skipping. Then, Bob can win by colouring u in his first move to generate two independent Bob-winning moves. ■

Proposition 7. *$F_{+}^{(1)}$, $F_{+}^{(2)}$, $F_{+}^{(3)}$, $F_{+}^{(4)}$ are $[A, A]$-forbidden.*

Proof. We have $d_{\text{thr}}^{+}(a) = 0$ for all the 4 digraphs and $d_{\text{thr}}^{+}(d) = 0$ for $F_{+}^{(1)}$, $F_{+}^{(2)}$, $F_{+}^{(3)}$ and all other vertices have nonzero threatening out-degree. Therefore, in Alice's first move of any of her winning strategies for $F_{+}^{(1)}$, $F_{+}^{(2)}$, $F_{+}^{(3)}$, she colours a, d or skips. Her first move of her winning strategy for $F_{+}^{(4)}$ is either colouring a or skipping.

For all the 4 digraphs, if Alice colours a or skips in her first move, Bob may colour c to generate two independent Bob-winning moves.

If Alice colours d in her first move on the digraphs $F_+^{(1)}$ and $F_+^{(2)}$, Bob may colour b to generate two independent Bob-winning moves.

If Alice colours d in her first move on $F_+^{(3)}$, Bob colours c. To avoid the two threats of b given by the leaf neighbours of b, Alice must colour b. Then Bob colours the leaf adjacent to a with the other colour and wins. ■

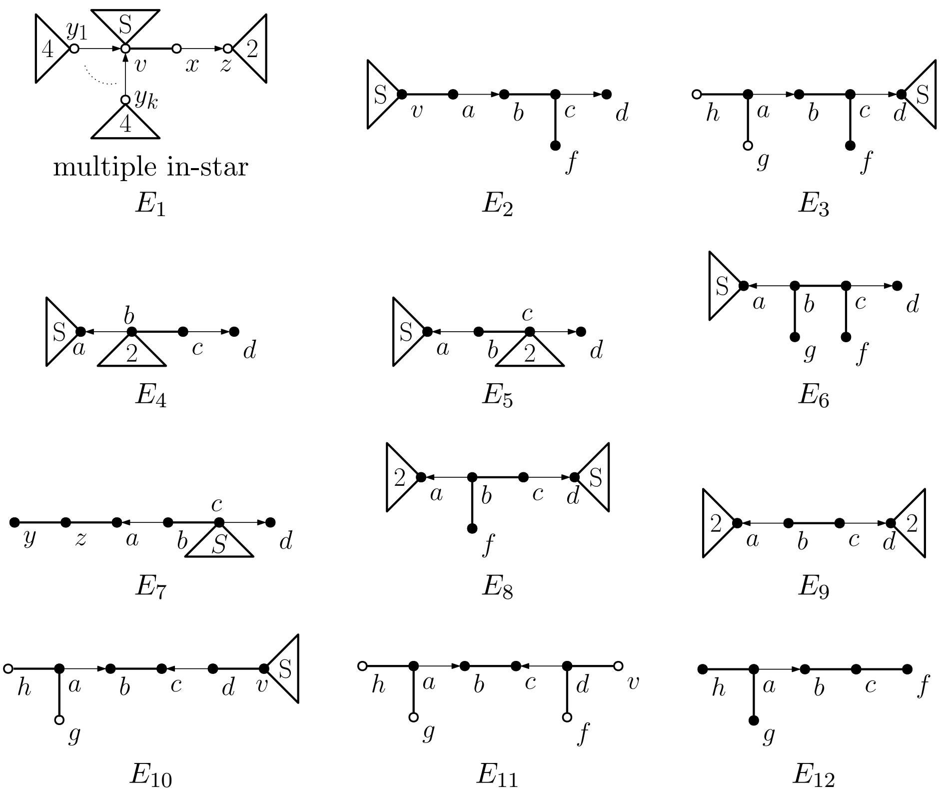

Figure 6. The permitted types. In E_1, every vertex could be optional, under the constraint that E_1 is an induced connected subdigraph of the configuration depicted above.

3.3. Permitted structures

The main permitted type of digraphs for $[A, A]$-perfect digraphs is type E_1. We say a digraph is of type E_1 if it is a connected induced subdigraph of a *multiple in-star*, which is a digraph built from an edge vx by adding a (non-symmetric) out-neighbour z to x, and by adding a 2-gadget at z, some leaf edges incident to v and some (non-symmetric) in-neighbours $y_1, \ldots, y_k$ to v with a 4-gadget at each of them. Note that, by definition, in a digraph of type E_1 the vertices v, x or z need not exist and the number k of vertices y_i may be zero. Furthermore, recall that the 2-, 4- and star-gadgets may be a single vertex.

The other types $E_2, \dots, E_{12}$ are more special types not fitting to the definition of a multiple in-star. The permitted types of $[A, A]$-perfect digraphs are depicted in Figure 6. In this figure, unfilled circles indicate optional vertices, whereas filled circles indicate the vertices compulsory for a digraph to be of the type considered.

3.4. Proof of Theorem 1 (ii)$\Longrightarrow$(iii)

3.4.1. Preliminary lemmas

The proofs of the next two lemmas are obvious.

Lemma 8. *If in a tree that contains no induced F_4 there is a single arc $\vec{e}_1$ and an arc $\vec{e}_2$ with dist$(\vec{e}_1, \vec{e}_2) \geq 2$, then $\vec{e}_2$ is part of an edge.*

Lemma 9. *In any tree that does not contain $F_{3,2}$ every vertex is incident with at most one single out-arc.*

Lemma 10. *An undirected tree T that does neither contain P_5 nor the chair is either the P_4 or a star.*

Proof. Since P_5 is not contained in T, the diameter of T is at most 3. Since no in-chair is contained in T, the tree T is a path when its diameter is 3. Therefore, T is either a P_4 or a star. ∎

Lemma 11 (Out-Arc-4-Gadget Lemma)**.** *Let (v, w) be a single arc in a tree for which* (ii) *holds. Assume that the truncated w-branch H_v containing v does not contain a single arc. Then H_v is a* 4*-gadget at v.*

Proof. Since (ii) is true, H_v does not contain a P_5 nor a chair. By Lemma 10, H_v is a star or P_4, thus H_v is either a star, pending star, P_4 or broken P_4 at v. ∎

Lemma 12 (In-Arc-2-Gadget Lemma)**.** *Let (v, w) be a single arc in a tree for which* (ii) *holds. Assume that the truncated v-branch H_w containing w does not contain a single arc. Then H_w is a* 2*-gadget at w.*

Proof. As in the proof of Lemma 11, since (ii) is true and by Lemma 10, H_w is a star, pending star, P_4 or broken P_4 at w. Since the in-P_5 is forbidden, H_w is not a P_4 at w. Since the in-chair is forbidden, H_w is neither a broken P_4 nor a pending k-star at w with $k \geq 3$. If it is a pending 2-star at w, it is a P_3 at w. Thus, H_w is either a star or a P_3 at w. ∎

3.4.2. Proof of Theorem 1 (ii)$\Longrightarrow$(iii): Case analysis

***Proof of Theorem* 1 (ii)$\Longrightarrow$(iii).** Let D be a semiorientation of a forest that does not contain any of the 24 forbidden configurations from (ii) as induced subdigraph. Since F_4 is forbidden in D, at most one component of D contains a single arc. By Lemma 10, every other component is a star or a P_4. If D contains no single arc, the proof is complete. Otherwise let T be the component of D containing a single arc.

By a case distinction we prove that T is of one of the types $E_1, \ldots, E_{12}$.

Lemma 13. *Either T contains a vertex with at least two single in-arcs or T has at most two single arcs, and if there are two, then they have distance* 1.

Proof. Since F_4 is forbidden in T, any pair of single arcs has distance 1 or is adjacent. Since $F_{3,1}$ and $F_{3,2}$ are forbidden in T, two adjacent single arcs are two in-arcs of the same vertex. In a tree it is not possible to have three single arcs which are pairwise at distance 1. □

By Lemma 13, we consider the five cases shown in Figure 7.

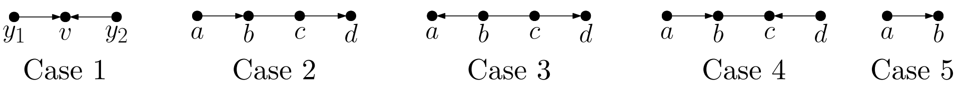

Figure 7. The five cases: T contains a vertex with at least two single in-arcs in Case 1, but not in Cases 2–5. In Cases 2–4, T contains exactly two single arcs; in Case 5, T contains exactly one single arc.

We now describe our approach for showing that T, in each case, is of one of the types E_i. In each case, we first employ the preliminary lemmas to restrict the possible configurations of T; then, we eliminate some of these configurations by using the assumption (ii) that T does not contain any forbidden type. After that we point out that all remaining configurations have some structure E_i.

Case 1. The tree T has a vertex v incident with at least two single in-arcs. We aim to show that T, in this case, is of type E_1. Assume v has k single in-arcs, say $(y_1, v), \ldots, (y_k, v)$, and $k \geq 2$. For each one of them, say (y, v), the truncated v-branch H_y containing y does not contain any single arc, since otherwise the existence of such a single arc (a, b) would imply that T contains $F_{3,1}$ or $F_{3,2}$ (induced by the vertices a, b, v) or F_4 (induced by the vertices a, b, v, y_i for some i with $y_i \neq y$). Thus, H_y is a 4-gadget by Lemma 11. Observe the following.

- v has no single out-arc, since otherwise the existence of such an out-arc, say (v, z), would imply that T contains $F_{3,1}$ (induced by the vertices y_1, v, z).
- No symmetric neighbour x of v is incident to a single in-arc or another edge than vx, since otherwise the existence of such an edge, say xz, or such a single in-arc, say (z, x), would imply that T contains an in-chair (induced by the vertices y_1, y_2, v, x, z).
- No symmetric neighbour x of v is incident to more than one single out-arc, since otherwise, by Lemma 9, T would contain an induced $F_{3,2}$.

- There is at most one symmetric neighbour of v incident to a single out-arc, since otherwise, by Lemma 8, T would contain an induced F_4.

If v has a symmetric neighbour, say x, which has an out-neighbour, say z, then the truncated x-branch H_z containing z does not contain a single arc, since otherwise the existence of such a single arc, say (a, b), would imply T contains F_4 (induced by the vertices a, b, v, y_1). Thus, by Lemma 12, H_z is a 2-gadget. Therefore, in Case 1, T is of type E_1.

In the following Cases 2–5 we explicitly exclude Case 1, i.e., we assume that there is no vertex in T incident with two single in-arcs. By Lemma 13, then T contains at most two single arcs.

Case 2. The tree T has single arcs (a, b) and (c, d) and an edge bc. We aim to prove that, in this case, T is of type E_1, E_2 or E_3. First observe that, by Lemmas 11 and 12, the truncated b-branch H_a containing a is a 4-gadget and the truncated c-branch H_d containing d is a 2-gadget. Moreover, consider T', the component containing a of $T - (c, d)$. Since (ii) is true for T, it is true for T', and so by Lemma 12, the truncated a-branch H'_b containing b of T' is a 2-gadget, i.e., it is either a star at b or a P_3 at b. Observe that if H'_b is a star at b, then T is of type E_1. In the following we assume H'_b is a P_3 at b, i.e., c has a symmetric neighbour f other than b.

We make the following observations.

- H_a does not contain a 3-gadget, since otherwise the 3-gadget together with b, c and f would induce an in-broom. So H_a is either a pending star at a or a k-star at a for some $k \leq 2$.
- If H_a is a pending star at a, then d is a leaf, since otherwise the existence of a symmetric neighbour g of d would imply that T contains $F_{\rightarrow}^{(1)}$ (induced by the vertices b, c, d, f, g and a P_3 of H_a containing a). Therefore, T is of type E_2.
- H_d is a star at d, since otherwise it would be a P_3 at d, which would imply that T contains $F_{\rightarrow}^{(2)}$ (induced by H_d and the vertices a, b, c, f).
- If H_a is a k-star at a, then as said $k \leq 2$ and T is of type E_3.

Thus, in Case 2 we only get the structures E_1, E_2 and E_3.

Case 3. The tree T has single arcs (b, a) and (c, d) and an edge bc. We aim to prove that, in this case, T is of type E_4, E_5, E_6, E_7, E_8 or E_9. By Lemma 12, the truncated b-branch H_a containing a and the truncated c-branch H_d containing d are both 2-gadgets. We denote by T' the component of $T - (c, d)$ containing a and by T'' the component of $T - (b, a)$ containing c. By Lemma 11, the truncated a-branch containing b in T' and the truncated d-branch containing c in T'' are both 4-gadgets.

Observe that, by definition of a 4-gadget, this is equivalent to say that in T there are a 2-gadget H_b at b and a 2-gadget H_c at c, and that either one of them is reduced to a single vertex or both are reduced to a single pending edge. If both are reduced to a single vertex, then T is of the permitted type E_9; we assume in the following it is not the case.

Moreover, if H_a is a star at a and H_d reduced to a single vertex, then T is of one of the permitted types E_4, E_5 or E_6. By symmetry, we obtain the same permitted types if H_a is a single vertex and H_d is a star. We are left to consider the cases that neither H_a nor H_d is trivial or one of them is a P_3 at its vertex.

- If neither H_a nor H_d is trivial, then, since $F_+^{(1)}$, $F_+^{(2)}$ and $F_+^{(3)}$ are forbidden, H_b or H_c has to be trivial and the non-trivial one, say H_b, has to be reduced to a single edge. Since $F_+^{(4)}$ is forbidden, H_d cannot be a P_3 at d and so T is of the permitted type E_8.
- If otherwise H_a or H_d is a P_3 at its vertex, say H_a is a P_3 at a and H_d is trivial, then, since T has no induced in-broom, we conclude the following.
  - H_b or H_c has to be trivial (otherwise H_c, H_b and H_a would induce a broken $F_{7,1}$).
  - H_b cannot be a P_3 at b (otherwise c, H_b and H_a would induce again a broken $F_{7,1}$).
  - H_c cannot be a P_3 at c (otherwise H_c, b, and H_a would induce $F_{7,1}$).
  - If H_b is a k-star at b, then $k \leq 1$ (otherwise the digraph induced by c, H_b and H_a would contain an induced $F_-^{(1)}$).

  Thus, by what we already stated about H_b and H_c, either H_b is trivial and H_c is a star or H_b is reduced to an edge and H_c is trivial. In the former, T is of the permitted type E_7. In the latter, T is of the permitted type E_8.

Thus, in Case 3 we only get the structures E_4, E_5, E_6, E_7, E_8 and E_9.

Case 4. The tree T has single arcs (a, b) and (d, c) and an edge bc. We aim to prove that, in this case, T is of type E_{10} or E_{11}. First we remark that b and c do not have neighbours outside the set $\{a, b, c, d\}$. If b has another symmetric neighbour than c, say f, then the vertices a, b, c, d, f induce an in-chair, which contradicts (ii). The same is true if we change the roles of b and c.

By Lemma 11 the truncated b-branch H_a containing a and the truncated c-branch H_d containing d are both 4-gadgets. Neither of them is a P_4 or broken P_4 or k-star at their vertex for some $k \geq 3$, since otherwise it would imply that T contains an in-broom ($F_{7,2}$, a broken $F_{7,2}$ or $F_-^{(2)}$, respectively), contradicting (ii). Thus they are either k-stars, for $k \leq 2$, or pending stars at their vertices.

- If one of them, say H_d, is a pending star, then H_a is not also a pending star, since otherwise T would contain F_8 (induced by the vertices b, c and a P_3

of each of H_a and H_d containing a and d, respectively). Thus T is of the permitted type E_{10}.

- If both of them are k-stars, $k \leq 2$, then T is of the permitted type E_{11}.

Thus, in Case 4 we only get the structures E_{10} and E_{11}.

Case 5. The tree T has (edges and) only one single arc (a, b). We aim to prove that, in this case, T is of type E_1 or E_{12}. By Lemma 11, the truncated b-branch H_a containing a is a 4-gadget, and, by Lemma 12, the truncated a-branch H_b containing b is a 2-gadget. We distinguish two cases.

- If H_b is a star at b, then T is of type E_1.
- If H_b is a P_3 at b, then H_a is neither a P_4 nor a broken P_4 nor a k-star at a for some $k \geq 3$, since otherwise, by a similar argument as in Case 4, it would imply that T contains an in-broom ($F_{7,1}$, broken $F_{7,1}$ or $F_-^{(1)}$, respectively). Thus in this case, either H_a is a 2-star at a and T is of permitted type E_{12}, or H_a is a 1-star or a pending star at a and T is of the permitted type E_1.

Thus, in Case 5 we only get the structures E_1 and E_{12}.

This completes the proof of (ii)$\Longrightarrow$(iii). ∎

3.5. Proof of Theorem 1 (iii)$\Longrightarrow$(i)

We start with an outline of the proof.

***Proof of Theorem* 1 (iii)$\Longrightarrow$(i).** First, we prove that every digraph for which (iii) is true is $[A, A]$-nice (Lemma 14). Then we will prove that (iii) is also true for every one of its induced subdigraphs (Lemma 15). This implies that D is $[A, A]$-perfect. ∎

Lemma 14. *For any i, the disjoint union of a digraph of type E_i and any number of stars and P_4s is $[A, A]$-nice.*

Lemma 15. *If D is a digraph for which* (iii) *is true and D' is a non-empty induced subdigraph of D, then* (iii) *is true for D'.*

In the following we will prove Lemma 14 and Lemma 15. For the proof of Lemma 14, we begin with a folklore observation.

Observation 16. *Every star, the P_4, every star with its center having an additional in-arc and the P_4^- (depicted in Figure* 8*) are $[B, A]$-nice and thus $[A, A]$-nice.*

Lemma 17 (Arc Deletion Rule)**.** *Let $X \in \{A, B\}$. For any tree T with clique number* 2 *and an out-leaf arc (u, v), T is $[X, A]$-nice if and only if $T - v$ is $[X, A]$-nice.*

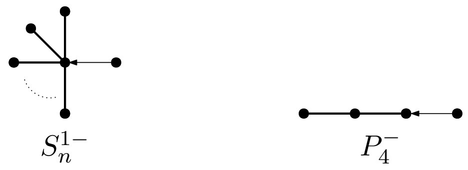

Figure 8. A star S_n^{1-} with its center having an additional in-arc, and the P_4^-.

Proof. ($\Leftarrow$): Suppose Alice has a winning strategy for the $[X, A]$-colouring game on $T - v$. During the game on T, if Bob never colours the leaf v, Alice may use her strategy for $T - v$; she will win because v has only one in-neighbour, so v can always be coloured in the game with two colours. If Bob colours v at some turn, Alice skips her next turn. Because v has no out-neighbours, the colouring of v will not affect the subsequent colouring of any vertices in $T - v$. So, after the skip, Alice can resume her winning strategy for $T - v$.

($\Rightarrow$): Suppose Alice has a winning strategy for the $[X, A]$-colouring game on T. During the game on $T - v$, Alice may use her strategy for T. This strategy fails only if at some Alice's turn, she chooses to colour the leaf v in her strategy for T but now v is deleted. In this case, she may skip her turn and resume the strategy starting from her next turn. Because in the game on T, the colouring of v does not affect any subsequent colouring of the other vertices, Alice can successfully resume her strategy and win the game. ■

Recall that for game g, an uncoloured or partially coloured digraph G is *k-g-permitted* if Alice can win g with k colours on G, and *k-g-unpermitted* otherwise.

Lemma 18 (P_5-Lemma)**.** *The partially coloured path in Figure* 9(a), *where the vertex a is coloured, is* 2-$[B, A]$*-permitted and so* 2-$[A, A]$*-permitted.*

Proof. We present a winning stategy for Alice in game $[B, A]$. If Bob colours b, then Alice colours d with the same colour. If Bob colours d, then Alice colours c with the other colour. If Bob colours c, then Alice colours f with the same colour, and vice versa. After that Alice wins. ■

Lemma 19. *The digraphs in Figures* 9(a), 9(b), 9(c) *and* 9(d) *and the partially coloured digraph in Figure* 9(d) *with a coloured are* 2-$[B, A]$*-permitted and so* 2-$[A, A]$*-permitted.*

In particular, Lemma 19 implies that the digraphs in Figures 9(a), 9(b), 9(c) and 9(d) are $[A, A]$-nice.

Proof. We denote the cases that the game digraph is defined in Figure 9(a), Figure 9(b), Figure 9(c) or Figure 9(d) (no matter whether a is coloured or uncoloured at the beginning) simply as cases (a), (b), (c) or (d), respectively. We consider all possible first moves of Bob.

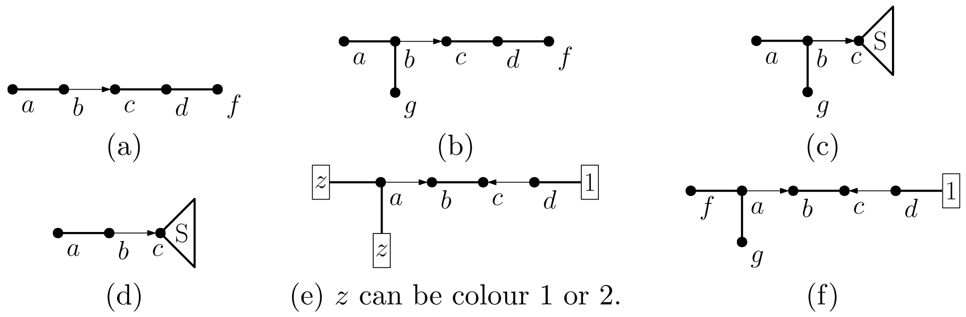

Figure 9. Some 2-$[A,A]$-permitted types. The variable or number inside a rectangle is the colour that has been put on the vertex.

If Bob colours a, then Alice colours g with the same colour, and vice versa, in case g exists. If, in case g does not exist, Bob colours a, then Alice skips. After that, in the cases (a) or (b), Alice may use her winning strategy in the P_5-Lemma (Lemma 18); in the cases (c) or (d), she may ensure that c is coloured after her next move, so that the remaining vertices must be coloured finally.

If Bob colours b, then Alice colours d (in the cases (a) or (b)) with the same colour, respectively, c (in the cases (c) or (d)), or vice versa. Then, Alice wins.

If, in the cases (a) or (b), Bob colours c, then Alice colours f with the same colour, and vice versa. If, in the cases (c) or (d), Bob colours a leaf in the star gadget, then Alice colours c with the other colour. In the cases (b) or (c), the generated uncoloured P_3 induced by a, b, g is $[B,A]$-nice. Thus, Alice wins. ∎

Lemma 20. *The partially coloured subtree in Figure* 9(e) *is* 2-$[B,A]$*-permitted.*

Proof. Alice may respond to Bob's first move as follows.

If Bob colours a, then she may colour c with the same colour, and vice versa, so that the remaining uncoloured vertices b and d must have available colours. If Bob colours b, then she may colour c with the other colour. Thus, the remaining uncoloured vertices a and d must have available colours. If Bob colours d with 2, then she may colour b with 2 so that the remaining uncoloured vertices a and c must have available colours. ∎

Lemma 21. *The partially coloured subtree in Figure* 9(f) *is* 2-$[B,A]$*-permitted.*

Proof. Alice may respond to Bob's first move as follows.

If Bob colours f, then she may colour g with the same colour, and vice versa, to generate the 2-$[B,A]$-permitted subtree in Figure 9(e). If Bob colours a, then she may colour c with the same colour, and vice versa, so that the remaining uncoloured vertices b and d must have available colours. If Bob colours b, then she may colour c with the other colour. Thus, the remaining uncoloured vertex

d must have an available colour and a, f, g induce a P_3, which is $[B, A]$-nice. If Bob colours d with 2, then she may colour b with 2, so that a similar situation arises as in the previous case. ■

Proposition 22. *Every digraph of type E_1 is $[A, A]$-nice.*

Proof. Let H be a digraph of type E_1. If $\omega(H) = 1$, then, since H is connected, H is an r-in-star for some non-negative r. Note that all the r leaves are safe. Therefore, the winning strategy of Alice for the game on this in-star with one colour is to colour the sink in her first move.

Now we may assume $\omega(H) = 2$. In the following the vertex names refer to Figure 6. We first consider the case that the vertex v exists. Then Alice may colour the vertex v in her first move to generate some uncoloured or partially coloured subtrees. Observe that any generated subtree must be a P_4, a star, a star with the central vertex coloured or one of the subtrees in Figure 9(a) or 9(d) with vertex a coloured. By Observation 16, Lemma 18 and Lemma 19, each such subtree is 2-$[B, A]$-permitted. Then, Alice will eventually win the game if she employs the following strategy in the rest of the game. Suppose Bob acted on some subtree T in his last move. If T is not fully coloured yet, she acts on T according to her winning strategy for the game $[B, A]$ on T with two colours; if T is fully coloured, she passes her turn.

Second, consider the case that v does not exist. Since H is connected, H is either the P_4, a star, a star whose center has an additional in-arc (when the 2-gadget is a star), or the P_4^- (when the 2-gadget is a P_3) which is depicted in Figure 8. All of them are $[A, A]$-nice by Observation 16. ■

Proposition 23. *Every digraph of type E_2 or E_7 is $[A, A]$-nice.*

Proof. With the Arc Deletion Rule (Lemma 17), we may consider the games on E_2 and E_7 with their out-leaf arcs deleted. In her first move, Alice may colour v (for the game on E_2) or c (for the game on E_7) to generate the path in Figure 9(a) with a coloured. This subtree is 2-$[B, A]$-permitted by the P_5-Lemma (Lemma 18). ■

Proposition 24. *Every digraph of type E_3 or E_8 is $[A, A]$-nice.*

Proof. In her first move, Alice may colour d to generate a star with the central vertex coloured and either a P_4^- (see Figure 8) or the subtree in Figure 9(a), 9(b) or 9(c). By the same strategy as in the proof of Lemma 18 or by Lemma 19 Alice wins, respectively. ■

Proposition 25. *Every digraph of type E_4, E_5 or E_6 is $[A, A]$-nice.*

Proof. With the Arc Deletion Rule (Lemma 17), we may consider the game on a digraph of type E_4, E_5 or E_6 with its out-leaf arc (c, d) deleted, which is a digraph of type E_1. By Proposition 22, Alice wins. ∎

Proposition 26. *Every digraph of type* E_9 *is* $[A, A]$*-nice.*

Proof. Alice may skip her first move and respond to Bob's first move as follows. By the structural symmetry of this digraph, it is sufficient to consider the cases when Bob plays on the left half of the digraph.

If Bob colours b, we distinguish two cases. When the 2-gadget at a is P_3, Alice may colour a symmetric neighbour of a with the same colour so that they totally generate a partially coloured P_2, a partially coloured star with the central vertex a having an available colour, and the partially coloured subtree in Figure 9(a) or 9(d) with a coloured. When the 2-gadget at a is a star, she colours a, so that they totally generate a partially coloured star with a coloured central vertex a, and the partially coloured subtree in Figure 9(a) or 9(d) with a coloured.

In the following we consider the case that Bob colours a vertex in the 2-gadget at a.

When the gadget is $P_3 = yza$, if Bob colours y (respectively, a), then she may colour a (respectively, y) with the same colour to generate a partially coloured P_3 with the central vertex z having an available colour, and the uncoloured subtree in Figure 9(a) or 9(d). If Bob colours z, then she may colour a so that they totally generate a partially coloured P_2 and the uncoloured subtree in Figure 9(a) or 9(d).

When the gadget is a star, if Bob colours a leaf adjacent to a (respectively, a), then she may colour a (respectively, a leaf adjacent to a) so that they totally generate a partially coloured star with the central vertex a coloured, and the uncoloured subtree in Figure 9(a) or 9(d).

In any case, by Lemma 19, Alice wins. ∎

Proposition 27. *Every digraph of type* E_{10} *is* $[A, A]$*-nice.*

Proof. We only discuss the case that both optional vertices g and h exist, the strategies for the other cases are very similar.

Alice may colour v with 1 to generate a star with the central vertex coloured and the partially coloured subtree in Figure 9(f). By Lemma 21, the latter is 2-$[B, A]$-permitted. ∎

Proposition 28. *Every digraph of type* E_{11} *is* $[A, A]$*-nice.*

Proof. Again, we only discuss the case that all four optional vertices exist, the strategies for the other cases are very similar.

Alice may skip her first move and respond to Bob's first move as follows. By the structural symmetry of this digraph, we may consider the cases when Bob plays on the left half of the digraph.

If Bob colours f, then Alice may colour v with the same colour to generate a partially coloured subtree, denoted by T. After that, the subtree induced by all the uncoloured vertices (a, b, c, d, h, g) of T is the same as that induced by all the uncoloured vertices (a, b, c, d, h, g) of the partially coloured subtree in Figure 9(f). Moreover, any two vertices with the same label in the two induced subtrees have the same set of available colours. Therefore, the games on T and the partially coloured subtree in Figure 9(f) are equivalent. Consequently, T is also 2-$[B, A]$-permitted.

If Bob colours d, then Alice may colour b with the same colour so that they totally generate a star with the central vertex d coloured, an uncoloured P_3 and a subtree in which c must have an available colour.

If Bob colours c with 1, then Alice may colour b with 2 so that they totally generate two uncoloured P_3 and a completely coloured P_2. ∎

Proposition 29. *The digraph of type E_{12} is $[A, A]$-nice.*

Proof. This was proven in Lemma 19 (case (b)). ∎

***Proof of Lemma* 14.** Alice has the following winning strategy with 2 colours for the game $[A, A]$ played on the disjoint union of a digraph D_0 of type E_i and stars $S_1, \ldots, S_p$ and P_4s $P_1, \ldots, P_q$, where $p, q \geq 0$.

By Observation 16, she has a winning strategy for the game $[B, A]$ on each of $S_1, \ldots, S_p, P_1, \ldots, P_q$. By Propositions 22–29, she has a winning strategy for D_0. Alice combines these strategies in the following way.

In her first move she acts according to her winning stategy for D_0 (this act might be a skip if required by her strategy). After that, whenever Bob plays on one of the components $D_0, S_1, \ldots, S_p, P_1, \ldots, P_q$, Alice acts according to her winning strategy for this component on this component, unless the component is fully coloured. In case such a component is fully coloured Alice misses her turn.

Since the colouring of a component does not affect the colouring of any other component, Alice will win finally. ∎

***Proof of Lemma* 15.** In Table 1, for each digraph H of type E_i and each vertex u, we list the types of the components of $H - u$. In the left column of the tables, we give the name of the vertex u or, for inner vertices of the gadgets which are not shown in Figure 6, the name of the gadget containing u (S means star-gadget, 2 means 2-gadget, 4 means 4-gadget). In the right column of the tables, S denotes a star of arbitrary size, K_1 an isolated vertex (which is also a star) and K_2 the 1-star, and A/B is either an A or a B. $A^{\cup}$ means a (maybe empty) disjoint union of some non-negative number of graphs A. In particular $A^{\cup}$ might be optional.

Type E_1	
u	comp. of $H-u$
4	$S/K_1^{\cup}, E_1$
y_i	$S/[K_2, K_1]/K_1^{\cup}, E_1$
v	$S^{\cup}, P_4^{\cup}, K_1^{\cup}, E_1$
S	E_1
x	E_1, S
z	$E_1, K_2/K_1^{\cup}$
2	E_1, K_1

Type E_2	
u	comp. of $H-u$
S	E_2
v	$K_1^{\cup}, E_3$
a	S, E_1
b	S, E_1
c	E_1, K_1, K_1
d	E_1
f	E_1

Type E_3	
u	comp. of $H-u$
a	K_1, K_1, E_1
b	S, E_1
c	E_1, K_1, S
d	$E_1/E_{12}, K_1^{\cup}$
S	E_3
f	E_1
g	E_3 by def.
h	E_3 by def.

Type E_4	
u	comp. of $H-u$
S	E_4
a	$K_1 \cdots K_1, E_1$
b	$S, K_2/K_1^{\cup}, E_1$
c	E_1, K_1
d	E_1
2	K_1, E_4

Type E_5	
u	comp. of $H-u$
S	E_5
a	$K_1 \cdots K_1, E_1$
b	S, E_1
c	$E_1, K_2/K_1^{\cup}, K_1$
d	E_1
2	E_5, K_1

Type E_6	
u	comp. of $H-u$
S	E_6
a	$K_1^{\cup}, E_1$
b	S, K_1, E_1
c	E_1, K_1, K_1
d	E_1
f	E_4
g	E_5

Type E_7	
u	comp. of $H-u$
y	E_5
z	K_1, E_5
a	K_2, E_1
b	S, E_1
c	E_1, K_1, K_1
d	E_1
f	E_9

Type E_8	
u	comp. of $H-u$
2	E_8, K_1
a	$E_1, K_2/K_1^{\cup}$
b	E_1, K_1, S
c	S, E_1
d	$K_1^{\cup}, E_1/E_{12}$
f	E_9
S	E_8

Type E_9	
u	comp. of $H-u$
2	K_1, E_9
a	$K_2/K_1^{\cup}, E_1$
b	S, E_1
(use symmetry)	

Type E_{10}	
u	comp. of $H-u$
a	K_1, K_1, E_1
b	S, E_1
c	E_1, S
d	E_1, S
v	$E_9, K_1^{\cup}$
S	E_{10}
h	E_{10} by def.
g	E_{10} by def.

Type E_{11}	
u	comp. of $H-u$
a	K_1, K_1, E_1
b	S, E_1
f	E_{11} by def.
g	E_{11} by def.
(use symmetry)	

Type E_{12}	
u	comp. of $H-u$
a	K_1, K_1, S
b	S, K_2
c	E_1, K_1
f	E_1
g	E_1
h	E_1

Table 1. Listing the types of the components of $H - u$ for any digraph H from each type E_i and any vertex u from H. Not all components must appear, some are optional.

Since all these types are contained in some E_j and at most one of the components is neither a star nor a P_4, the table proves that (iii) is true for every digraph obtained from a digraph of type E_i by deleting a vertex.

Observe that every induced subdigraph of the P_4 or a star is the P_4 or a star. Thus, by induction, (iii) is true for every induced subdigraph of a digraph of type E_i, and, actually, for every induced subdigraph of a digraph for which (iii) is true. ■

This completes the proof of (iii)$\Longrightarrow$(i), thus the whole proof of Theorem 1 is complete.

4. $[A, -]$-Perfect Forests: Proof of Theorem 2

Similar to the proof technique in Theorem 1, we will prove the four implications (i)$\Longrightarrow$(i')$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(i) of Theorem 2, separately.

In Figure 11 we display the permitted types for the game $[A, -]$. Note that in a *reduced multiple in-star* the vertices v, x and z exist, whereas in a general digraph of type E_1^A, which is a connected induced subdigraph of a reduced multiple in-star, the vertices v, x or z need not exist. In Figure 10 we display the additional forbidden types.

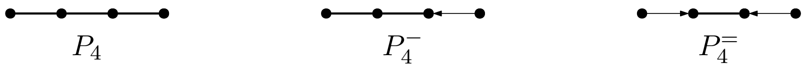

Figure 10. The 3 in-P_4s.

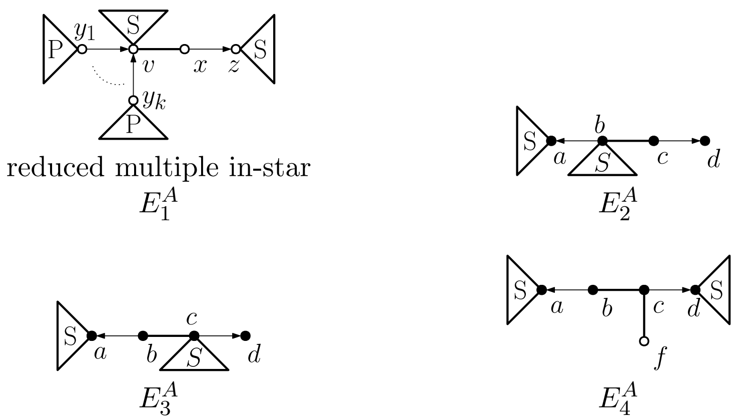

Figure 11. The permitted types for $[A, -]$-perfect digraphs. In E_1^A, every vertex could be optional, under the constraint that E_1^A is an induced connected subdigraph of the configuration depicted above.

***Proof of Theorem* 2 (i)$\Longrightarrow$(i').** For any digraph D, we know by (2) that every $[A, B]$-perfect digraph is $[A, -]$-perfect. ■

***Proof of Theorem* 2 (i')$\Longrightarrow$(ii).** Let D be $[A, -]$-perfect. By (2), D is also $[A, A]$-perfect. By Theorem 1, D does not contain F_4, $F_{3,1}$, $F_{3,2}$, or $F_+^{(3)}$ as induced subdigraph. It remains to show that Bob has a winning strategy with 2 colours for the game $[A, -]$ on any of the three in-P_4s. This is trivial: Alice is forced to colour a vertex in her first move. Then Bob can colour a vertex at distance 2 with the other colour and wins. ■

***Proof of Theorem* 2 (ii)$\Longrightarrow$(iii).** Note that F_4, $F_{3,1}$, $F_{3,2}$, and $F_+^{(3)}$ are all included in the twenty-four forbidden types of Theorem 1(ii). Moreover, it can be easily checked that any of the remaining seventeen forbidden types given in Theorem 1(ii) contains at least one of the three in-P_4s. Therefore, (ii) implies Theorem 1(ii). Since P_4 is forbidden by (ii), all components of D are undirected stars, except maybe one which is of type E_i for some $i \in \{1, 4, 5, 8, 9\}$ (the other E_is are excluded since they contain an in-P_4).

Since Theorem 1(ii) is true for D, so are Lemma 11 and 12. Moreover, since the in-P_4s are forbidden, we can state stronger versions of these lemmas.

Lemma 30 (Out-Arc-P-Gadget Lemma)**.** *Let (v, w) be a single arc in a tree for which* (ii) *holds. Assume that the truncated w-branch H_v containing v does not contain any single arc. Then H_v is a P-gadget at v.*

Lemma 31 (In-Arc-Star-Gadget Lemma)**.** *Let (v, w) be a single arc in a tree for which* (ii) *holds. Assume that the truncated v-branch H_w containing w does not contain any single arc. Then H_w is a star at w.*

By those lemmas, if T is of type E_1, then it is of type E_1^A, and if it is of type E_8 or E_9, then it is of type E_4^A.

Finally, if T is of type E_4 (respectively, E_5), then observe that the 2-gadget at b (respectively, c) cannot be a P_3, since it would form a P_4 with the edge bc. Thus this 2-gadget is a star, which implies that T is of type E_2^A (respectively, of type E_3^A).

Summarizing, T is of type $E_1^A, \ldots, E_4^A$. ■

***Proof of Lemma* 30.** Since H_v is undirected and P_4 is forbidden by (ii), H_v has diameter at most 2. Thus H_v is a star. ■

***Proof of Lemma* 31.** Since H_w is undirected and P_4 is forbidden by (ii), H_w has diameter at most 2. Thus H_w is a star. Since, by (ii), the v-branch containing w does not contain any in-P_4, H_w does not have a P_3 at w. In particular, H_w does not have a pending star at w. ■

***Proof of Theorem* 2 (iii)$\Longrightarrow$(i).** For the proof it is sufficient to first remark in Observation 32 that the set of permitted digraphs given in (iii) is closed under taking induced subdigraphs and then to show in Propositions 33 and 34 that every digraph of type $E_1^A, \ldots, E_4^A$ is $[A,B]$-nice.

Observation 32. *Let H be a digraph of type E_i^A $(1 \le i \le 4)$ and u be one of its vertices. If $H-u$ is non-empty, one of the components of $H-u$ is of type $E_1^A, \ldots, E_4^A$, then every other component of $H-u$ is a star.*

Type E_1^A	
u	comp$(H-u)$
P	$K_1^\cup, E_1^A$
y_i	$K_2/K_1^\cup, E_1^A$
v	$S^\cup, K_1^\cup, E_1^A$
S	E_1^A
x	E_1^A, S
z	$E_1^A, K_1^\cup$
S	E_1^A

Type E_2^A	
u	comp$(H-u)$
S	E_2^A
a	$K_1^\cup, E_1^A$
b	$S, K_1^\cup, E_1^A$
c	E_1^A, K_1
d	E_1^A
S	E_2^A

Type E_3^A	
u	comp$(H-u)$
S	E_3^A
a	$K_1^\cup, E_1^A$
b	$S, K_1^\cup, E_1^A$
c	$E_1^A, K_1^\cup, K_1$
d	E_1^A
S	E_3^A

Type E_4^A	
u	comp$(H-u)$
S	E_4^A
a	$E_1^A, K_1^\cup$
b	E_1^A, K_1, S
c	S, E_1^A
d	$K_1^\cup, E_1^A$
S	E_4^A
f	E_4^A by def.

Table 2. The proof of Observation 32: Listing the types of the components of $H-u$ for any digraph H from each type E_i^A and any vertex u from H. Not all components must appear, some are optional.

Proof. The proof is given in Table 2. The method is similar to the proof of Lemma 15. In Table 2, P denotes a P-gadget. For the other notation we refer to the proof of Lemma 15. □

Proposition 33. *E_1^A is $[A,B]$-nice.*

Proof. Let T be a digraph of type E_1^A. If $\omega(T)=1$, then T is an in-star, on which Alice wins with one colour if she colours the sink in her first move. Therefore we may assume $\omega(T)=2$.

We call U the set of unsafe vertices of T that are different from v. Observe that every component of $T-v$ (if v does not exist, $T-v=T$) contains at most one vertex of U, and that this vertex is not an out-neighbour of v. The strategy for Alice is as follows. If v exists, she colours it first; otherwise she colours an arbitrary vertex of U (in the case there is none, Alice wins trivially since all vertices are safe). Then, each time Bob colours a vertex w, Alice colours the vertex of U that is in the same component of $T-v$ as w, if it exists and is uncoloured; otherwise she colours any other uncoloured vertex of U. When every vertex of U is coloured, all uncoloured vertices are safe, therefore Alice wins. □

Proposition 34. *E_2^A, E_3^A and E_4^A are $[A,B]$-nice.*

Proof. Let T be a digraph of type E_2^A, E_3^A or E_4^A. The following is a winning strategy of Alice for the game $[A, B]$ with 2 colours on T. In her first move she colours a. After that, vertex b (or c) is the only possible uncolored unsafe vertex in E_2^A (or E_3^A, respectively). In E_4^A, only vertices c or d may be unsafe, and both of them are unsafe if and only if f exists and the star gadget at d is nontrivial. If there is at most one uncolored unsafe vertex v after Alice's first move, Alice must win if she ensures v has been colored after her second move. Therefore, only the case that both c and d are unsafe in E_4^A is left.

In the remaining case

- if Bob colours c in his first move, then Alice colours d with the other colour, and vice versa, so that all remaining uncolored vertices are safe,
- if he colours b, then she colours f with the same color, and vice versa, so that d is the only uncolored unsafe vertex,
- if he colours none of b, c, d, f in his first move, then she colours d, so that c is the only uncolored unsafe vertex.

In the last two cases, Alice must win if she ensures the only uncolored unsafe vertex has been colored after her third move. □

This completes the proof of Theorem 2 (iii)$\Longrightarrow$(i). ■

5. BOB BEGINS: PROOF OF THEOREMS 36, 37, AND 38

Game-perfect forests for the games where Bob begins can be characterised trivially because of the following observation.

Observation 35 [5]. *A game-perfect digraph D with regard to a game $[B, Y]$ where Bob begins is an undirected graph, i.e., D does not contain any single arc.*

Proof. The digraph consisting of two vertices a, b and a single arc (a, b) is $[B, Y]$-forbidden: it has clique number 1 and a winning strategy for Bob with one colour is to colour a in his first move. ■

Observation 35 leads to the following characterisations of game-perfect forests with regard to the games where Bob begins.

Theorem 36. *For a semiorientation D of a forest, the following are equivalent.*

(i) *D is $[B, A]$-perfect.*

(ii) *D does neither contain any single arc nor the chair nor P_5 as an induced subdigraph.*

(iii) *Every component of D is a P_4 or a star.*

Theorem 37. *For a semiorientation D of a forest, the following are equivalent.*
(i) *D is $[B, -]$-perfect.*
(ii) *D does neither contain any single arc nor the chair nor P_5 nor $P_4 \cup K_1$ as an induced subdigraph.*
(iii) *Either D is the P_4 or every component of D is a star.*

Theorem 38. *For a semiorientation D of a forest, the following are equivalent.*
(i) *D is $[B, B]$-perfect.*
(ii) *D does neither contain any single arc nor P_4 as an induced subdigraph.*
(iii) *Every component of D is a star.*

***Proof of Theorem* 36.** (i)$\Longrightarrow$(ii) Let D be $[B, A]$-perfect. By Observation 35, D does not contain single arcs. By (2), D is $[A, A]$-perfect, thus, by Theorem 1, D does neither contain P_5 nor the chair as an induced subdigraph.

(ii)$\Longrightarrow$(iii) Let T be a component of D. Since single arcs are forbidden by (ii), T is an undirected tree. Since P_5 and the chair are forbidden by (ii), by Lemma 10 the component T is a star or a P_4.

(iii)$\Longrightarrow$(i) Since every proper subgraph of the P_4 or a star is a forest of stars, (iii) also holds for every subdigraph of D whenever (iii) holds for D. Therefore it is sufficient to describe a winning strategy for Alice with 2 colours on D: using skipping moves Alice can force Bob to start playing on each component of D and then reply in this component to make it safe. ■

***Proof of Theorem* 37.** (i)$\Longrightarrow$(ii) Let D be $[B, -]$-perfect. By (2), D is $[B, A]$-perfect, thus, by Theorem 36, D does neither contain a single arc nor an induced chair nor an induced P_5. The graph $P_4 \cup K_1$ is also $[B, -]$-forbidden: Bob wins if he colours the isolated vertex in his first move and thus forces Alice to begin colouring the P_4.

(ii)$\Longrightarrow$(iii) By Theorem 36, the components of D are P_4s or stars. If there is a P_4-component, it is the unique component since $P_4 \cup K_1$ is forbidden by (ii).

(iii)$\Longrightarrow$(i) By the same argument as in the proof of Theorem 36(iii)$\Longrightarrow$(ii) it is sufficient to show that Alice has a winning strategy on D. On the P_4 or a forest of stars, Alice wins in the game $[B, -]$ with two colours, obviously. ■

***Proof of Theorem* 38.** (i)$\Longrightarrow$(ii) Let D be $[B, B]$-perfect. Then, by Observation 35, D does not contain single arcs. By (2), D is $[A, -]$-perfect, thus, by Theorem 2, D does not contain P_4 as an induced subdigraph.

(ii)$\Longrightarrow$(iii) Let T be a component of D. Since single arcs are forbidden by (ii), T is an undirected tree. Since P_4 is forbidden by (ii), the diameter of T is at most 2, thus T is a star.

(iii)$\Longrightarrow$(i) On a forest of stars, Alice wins obviously. ■

6. Weakly Game-Perfect Forests

Yang and Zhu [20] introduced the following digraph colouring game, which we call *weak digraph colouring game*, whereas the digraph colouring game considered so far is also called *strong digraph colouring game.* Two players, Alice and Bob alternately colour vertices of a given digraph D with colours of a given colour set C, obeying the rule that creating any monochromatic cycle is forbidden. When no more moves are possible, the game ends. Alice wins if every vertex is coloured at the end, otherwise, Bob wins. The smallest cardinality $|C|$ of the colour set such that Alice has a winning strategy is called the *weak game chromatic number* $\chi_{wg}(D)$.

As for the strong game we may also consider six variants wg of the weak digraph colouring game, where $wg = w[X,Y]$ with $g = [X,Y]$ and $X \in \{A,B\}$ and $Y \in \{A,B,-\}$ has the same meaning concerning the player X who begins and the player Y who is allowed to skip as in the strong digraph colouring game.

A notion of game-perfectness for the weak game was introduced in [6]. For any g, a digraph D is *weakly g-perfect* (or *weakly game-perfect with respect to the game g*) if, for any induced subdigraph H of D, $\chi_{wg}(H) = \omega(H)$.

Observation 39. *The inclusions given in* (2) *for the classes of strongly game-perfect digraphs also hold for the classes of weakly game-perfect digraphs.*

Guo and Surmacs [16] call the weak game chromatic number also *game dichromatic number* as it is nearer to the notion of dichromatic number than the strong number. Their definition is supported by the following two results.

Theorem 40 [20]**.** *For* (*any g and*) *any orientation D of a graph G,*

$$\chi_{wg}(D) \leq \left\lfloor \frac{\mathrm{col}_g(G)}{2} \right\rfloor,$$

where $\mathrm{col}_g(G)$ *denotes the game colouring number introduced by Zhu* [21].

Theorem 41 [6]**.** *For any g, a digraph D is weakly g-perfect if and only if*

(i) *the symmetric part $S(D)$ of D is a g-perfect graph and*

(ii) *D does not contain any induced directed n-cycle with $n \geq 3$.*

Since semiorientations of forests do not contain induced directed cycles of length greater than 2, Theorem 41 immediately implies the following.

Corollary 42. *For any g, a semiorientation D of a forest is weakly g-perfect if and only if $S(D)$ is g-perfect.*

Corollary 42 enables us to characterise weakly game-perfect forests. For the proofs of the following characterisations (Theorem 44, 46, respectively, 47), recall from the definitions at the beginning that P_4 and stars always denote undirected graphs, whereas a *forest* denotes a digraph (a semiorientation of an undirected forest). In the proofs we frequently use the fact that the strong game and the weak game are equivalent when played on undirected graphs.

Observation 43. *For any undirected graph (=symmetric digraph) G we have $\chi_g(G) = \chi_{wg}(G)$.*

Proof. In both colouring games on a graph G, the vertices of any edge which is a directed 2-cycle, must be coloured differently. Thus, the players have to respect that the colouring is proper, which means that both games are equivalent to Bodlaender's graph colouring game when played on a symmetric digraph. ∎

Theorem 44. *For a semiorientation D of a forest, the following are equivalent.*

(i) *D is weakly $[B, A]$-perfect.*

(i') *D is weakly $[A, A]$-perfect.*

(ii) *D does neither contain P_5 nor the chair as an induced subdigraph.*

(iii) *Every component of $S(D)$ is a star or a P_4.*

Proof. The implication (i)$\Longrightarrow$(i') follows directly from Observation 39.

Let D be weakly $[A, A]$-perfect. By Corollary 42, $S(D)$ is $[A, A]$-perfect. By Theorem 1, $S(D)$ does neither contain any induced (undirected) P_5 nor any induced (undirected) chair, which implies (ii). Thus (i') implies (ii).

Let D be a digraph that does neither contain P_5 nor the chair as an induced subdigraph. Since D is a forest, every induced P_5, respectively, chair in $S(D)$, is an induced subdigraph of D, too. Therefore $S(D)$ does neither contain an induced P_5 nor an induced chair. This means that every component of $S(D)$ has diameter at most 3, and if it has diameter 3, it is a P_4. Thus (ii) implies (iii).

Assume (iii) holds. By Observation 16, $S(D)$ is $[B, A]$-perfect. By Corollary 42, D is weakly $[B, A]$-perfect. Thus (iii) implies (i). ∎

For the next theorem, we define the following. A P_4^0 is a digraph on 5 vertices consisting of an undirected P_4 and an additional vertex v_0 and at most one single arc which, in case it exists, connects v_0 and some vertex of the P_4. There are exactly five pairwise nonisomorphic digraphs that are a P_4^0 (see Figure 12).

Lemma 45. *For any digraph D of type P_4^0 we have $\chi_{w[B,-]} > 2$.*

Proof. For any digraph D of type P_4^0, there exists an edge in D such that its two ending vertices cannot have the same colour. Therefore, Bob must win the weak $[B, -]$-game on D with 1 colour. Now we give a winning strategy for Bob

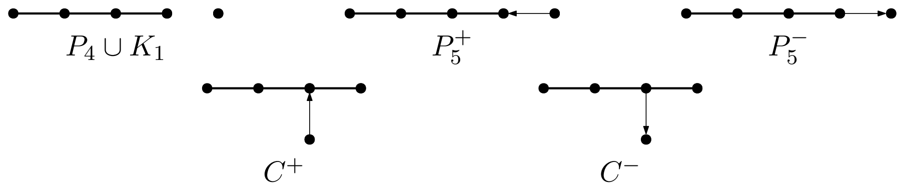

Figure 12. The five digraphs of type P_4^0.

in the weak $[B, -]$-game with 2 colours played on a P_4^0. In his first move, Bob colours v_0. This move does not affect the colouring of any other vertex since v_0 is not contained in a directed cycle. Now Alice is forced to start colouring the P_4, say vertex u. Bob wins by colouring a vertex at distance 2 in $S(P_4^0)$ from u with the other colour. ■

Theorem 46. *For a semiorientation D of a forest, the following are equivalent.*

(i) *D is weakly $[B, -]$-perfect.*

(ii) *D does neither contain P_5 nor the chair nor any of the five P_4^0s as an induced subdigraph.*

(iii) *Either D is the P_4 or every component of $S(D)$ is a star.*

Proof. Let D be weakly $[B, -]$-perfect. By Corollary 42, $S(D)$ is $[B, -]$-perfect. By Theorem 37, $S(D)$ does neither contain any induced P_5 nor any induced chair, which implies that D does neither contain an induced P_5 nor an induced chair. By Lemma 45, D does not contain any induced P_4^0. Thus (i) implies (ii).

Now, let D be such that (ii) holds. Since D is a forest, every induced P_5 or chair of $S(D)$ is induced in D, too. With (ii) this implies that $S(D)$ does neither contain an induced P_5 nor an induced chair. By Theorem 44, every component of $S(D)$ is a star or a P_4. As well, since D does not contain any induced P_4^0, $S(D)$ does not contain an induced $P_4 \cup K_1$. If $S(D)$ contains a P_4-component H, then, since $P_4 \cup K_1$ is forbidden in $S(D)$, $S(D)$ must be connected and consist only of a P_4. In this case, since D has the same vertex set as $S(D)$, the digraph D is a forest on four vertices that has the P_4 as a subdigraph, therefore, $D = S(D) = P_4$. Thus (ii) implies (iii).

Finally, let D be such that (iii) holds. Since every proper subdigraph H of D is a digraph with each component of $S(H)$ being a star, we are left to prove that D is $w[B, -]$-nice. If D is the P_4, then Alice wins with 2 colours, since Bob is forced to start colouring the P_4. Otherwise, $S(D)$ is a forest of stars. Then Alice has the following winning strategy with $\omega(D)$ colours. Whenever Bob starts colouring a star of $S(D)$, Alice colours the center of this star if possible. If this is not possible, she colours the center of any other star or any vertex of a star the center has already been coloured. If such move is not possible, every vertex is coloured. Alice will win by following this strategy on $S(D)$. Thus $S(D)$ is

$[B,-]$-perfect. Therefore, by Corollary 42, the digraph D is $[B,-]$-perfect. Thus (iii) implies (i). ∎

Theorem 47. *For a semiorientation D of a forest, the following are equivalent.*

(i) *D is weakly $[B,B]$-perfect.*

(i') *D is weakly $[A,B]$-perfect.*

(i") *D is weakly $[A,-]$-perfect.*

(ii) *D does not contain P_4 as an induced subdigraph.*

(iii) *Every component of $S(D)$ is a star.*

Proof. The implications (i)$\Longrightarrow$(i')$\Longrightarrow$(i") follow directly from Observation 39.

Let D be weakly $[A,-]$-perfect. By Corollary 42, $S(D)$ is $[A,-]$-perfect. By Theorem 2, $S(D)$ does not contain an induced P_4, which implies that D does not contain an induced P_4. This proves the implication (i")$\Longrightarrow$(ii).

Now, let D be such that (ii) holds. Since D is a forest, every induced P_4 of $S(D)$ is induced in D, too. With (ii) this implies that $S(D)$ does not contain an induced P_4. Thus every component of $S(D)$ has diameter at most 2, i.e., it is a star. Thus (ii) implies (iii).

Finally, let D be such that (iii) holds. Then, by Theorem 38, $S(D)$ is $[B,B]$-perfect. Thus, by Corollary 42, D is $[B,B]$-perfect. Thus (iii) implies (i). ∎

7. Final Remarks and Open Questions

A *cactus* is a graph with the property that any two different of its cycles intersect in at most one vertex. In particular, an undirected forest is a cactus without any cycles. Combining the ideas from this paper with the characterisation of strongly game-perfect semiorientations of cycles in [5], it might be possible to easily solve the following problem.

Problem 48. *Characterise game-perfect semiorientations of cactuses for any of the* 12 *game variants.*

Moreover, the following more general problem, which partially already was proposed in [5], could be the next step towards a characterisation of all game-perfect digraphs for each of the 12 game variants.

Problem 49. *Characterise game-perfect digraphs with clique number* 2 *for any of the* 12 *game variants.*

Problem 50. *Characterise game-perfect digraphs for any of the* 12 *game variants.*

The solution of Problem 50 is known only for the variants $[B,B]$ ([4]) and $[B,-]$ ([8, 17]) of the strong digraph colouring game and for the variants $[B,B]$, $[A,B]$, $[A,-]$ ([4, 6]) and $[B,-]$ ([6, 8, 17]) of the weak digraph colouring game, whereas for the other six, quite more interesting game variants it is still open.

Our results support the following seemingly intuitive conjecture, but which, to our knowledge, still has not been proven.

Conjecture 51. *For any g, if D is strongly g-perfect, then D is weakly g-perfect.*

Or, more generally.

Conjecture 52. *For any g, $\chi_{wg}(D) \leq \chi_g(D)$.*

Let $GP[X,Y]$ be the class of $[X,Y]$-perfect digraphs. Our results support the following, which is true for undirected graphs.

Conjecture 53. $GP[A,B] = GP[A,-]$.

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