MINIMALLY STRONG SUBGRAPH
(k, ℓ)-ARC-CONNECTED DIGRAPHS

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Abstract

Let $D = (V, A)$ be a digraph of order $n$, $S$ a subset of $V$ of size $k$ and $2 \leq k \leq n$. A subdigraph $H$ of $D$ is called an $S$-strong subgraph if $H$ is strong and $S \subseteq V(H)$. Two $S$-strong subgraphs $D_1$ and $D_2$ are said to be arc-disjoint if $A(D_1) \cap A(D_2) = \emptyset$. Let $\lambda_S(D)$ be the maximum number of arc-disjoint $S$-strong digraphs in $D$. The strong subgraph $k$-arc-connectivity is defined as $\lambda_k(D) = \min \{\lambda_S(D) \mid S \subseteq V, |S| = k\}$. A digraph $D = (V, A)$ is called minimally strong subgraph $(k, \ell)$-arc-connected if $\lambda_k(D) \geq \ell$ but for any arc $e \in A$, $\lambda_k(D - e) \leq \ell - 1$. Let $\mathcal{G}(n, k, \ell)$ be the set of all minimally strong subgraph $(k, \ell)$-arc-connected digraphs with order $n$. We define $G(n, k, \ell) = \max\{|A(D)| \mid D \in \mathcal{G}(n, k, \ell)\}$ and $g(n, k, \ell) = \min\{|A(D)| \mid D \in \mathcal{G}(n, k, \ell)\}$.

In this paper, we study the minimally strong subgraph $(k, \ell)$-arc-connected digraphs. We give a characterization of the minimally strong subgraph $(3, n - 2)$-arc-connected digraphs, and then give exact values and bounds for the functions $g(n, k, \ell)$ and $G(n, k, \ell)$.

Keywords: strong subgraph $k$-connectivity, strong subgraph $k$-arc-connectivity, subdigraph packing.

2010 Mathematics Subject Classification: 05C20, 05C35, 05C40, 05C70, 05C75.

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1. Introduction

1.1. Motivation and concepts

The generalized $k$-connectivity $\kappa_k(G)$ of a graph $G = (V,E)$ was introduced by Hager [8] in 1985 ($2 \leq k \leq |V|$). For a graph $G = (V,E)$ and a set $S \subseteq V$ of at least two vertices, an $S$-Steiner tree or, simply, an $S$-tree is a subgraph $T$ of $G$ which is a tree with $S \subseteq V(T)$. Two $S$-trees $T_1$ and $T_2$ are said to be edge-disjoint if $E(T_1) \cap E(T_2) = \emptyset$. Two edge-disjoint $S$-trees $T_1$ and $T_2$ are said to be internally disjoint if $V(T_1) \cap V(T_2) = S$. The generalized local connectivity $\kappa_S(G)$ is the maximum number of internally disjoint $S$-trees in $G$. For an integer $k$ with $2 \leq k \leq n$, the generalized $k$-connectivity is defined as

$$\kappa_k(G) = \min\{\kappa_S(G) \mid S \subseteq V(G), |S| = k\}.$$ 

Observe that $\kappa_2(G) = \kappa(G)$. Li, Mao and Sun [10] introduced the following concept of generalized $k$-edge-connectivity. The generalized local edge-connectivity $\lambda_S(G)$ is the maximum number of edge-disjoint $S$-trees in $G$. For an integer $k$ with $2 \leq k \leq n$, the generalized $k$-edge-connectivity is defined as

$$\lambda_k(G) = \min\{\lambda_S(G) \mid S \subseteq V(G), |S| = k\}.$$ 

Observe that $\lambda_2(G) = \lambda(G)$. Generalized connectivity of graphs has become a well-established area in graph theory, see a recent monograph [9] by Li and Mao on generalized connectivity of undirected graphs.

To extend generalized $k$-connectivity to directed graphs, Sun, Gutin, Yeo and Zhang [14] observed that in the definition of $\kappa_S(G)$, one can replace “an $S$-tree” by “a connected subgraph of $G$ containing $S$.” Therefore, Sun et al. [14] defined strong subgraph $k$-connectivity by replacing “connected” with “strongly connected” (or, simply, “strong”) as follows. Let $D = (V,A)$ be a digraph of order $n$, $S$ a subset of $V$ of size $k$ and $2 \leq k \leq n$. A subdigraph $H$ of $D$ is called an $S$-strong subgraph if $H$ is strong and $S \subseteq V(H)$. Two $S$-strong subgraphs $D_1$ and $D_2$ are said to be arc-disjoint if $A(D_1) \cap A(D_2) = \emptyset$. Two arc-disjoint $S$-strong subgraphs $D_1$ and $D_2$ are said to be internally disjoint if $V(D_1) \cap V(D_2) = S$. Let $\kappa_S(D)$ be the maximum number of internally disjoint $S$-strong subgraphs in $D$. The strong subgraph $k$-connectivity is defined as

$$\kappa_k(D) = \min\{\kappa_S(D) \mid S \subseteq V, |S| = k\}.$$ 

As a natural counterpart of the strong subgraph $k$-connectivity, Sun and Gutin [11] introduced the concept of strong subgraph $k$-arc-connectivity. Let $D = (V(D),A(D))$ be a digraph of order $n$, $S \subseteq V$ a $k$-subset of $V(D)$ and $2 \leq k \leq n$. Let $\lambda_S(D)$ be the maximum number of arc-disjoint $S$-strong subgraphs in $D$. The strong subgraph $k$-arc-connectivity is defined as

$$\lambda_k(D) = \min\{\lambda_S(D) \mid S \subseteq V(D), |S| = k\}.$$
The strong subgraph $k$-(arc-)connectivity is not only a natural extension of the concept of generalized $k$-(edge-)connectivity, but also relates to important problems in graph theory. For $k = 2$, $\kappa_2(G) = \kappa(G)$ and $\lambda_2(G) = \lambda(G)$ [11]. Hence, $\kappa_k(D)$ and $\lambda_k(D)$ could be seen as generalizations of connectivity and edge-connectivity of undirected graphs, respectively. For $k = n$, $\kappa_n(D) = \lambda_n(D)$ is the maximum number of arc-disjoint spanning strong subgraphs of $D$. Moreover, we know that $\kappa(S)$ and $\lambda(S)$ denote the number of internally-disjoint and arc-disjoint strong subgraphs containing a given set $S$, respectively. Hence, these parameters are relevant to the subdigraph packing problem, see [2–5,7,13]. For a recent survey on the topic of strong subgraph connectivity, the readers can see [12].

A digraph $D = (V(D), A(D))$ is called minimally strong subgraph $(k, \ell)$-arc-connected if $\lambda_k(D) \geq \ell$ but for any arc $e \in A(D)$, $\lambda_k(D - e) \leq \ell - 1$. Note that $2 \leq k \leq n, 1 \leq \ell \leq n - 1$ by the definition of $\lambda_k(D)$ and Theorem 3. Let $\mathcal{G}(n, k, \ell)$ be the set of all minimally strong subgraph $(k, \ell)$-arc-connected digraphs with order $n$. We define

$$G(n, k, \ell) = \max\{|A(D)| \mid D \in \mathcal{G}(n, k, \ell)\}$$

and

$$g(n, k, \ell) = \min\{|A(D)| \mid D \in \mathcal{G}(n, k, \ell)\}.$$ 

We further define

$$Ex(n, k, \ell) = \{D \mid D \in \mathcal{G}(n, k, \ell), |A(D)| = G(n, k, \ell)\}$$

and

$$ex(n, k, \ell) = \{D \mid D \in \mathcal{G}(n, k, \ell), |A(D)| = g(n, k, \ell)\}.$$ 

In [11], Sun and Gutin first studied the minimally strong subgraph $(k, \ell)$-arc-connected digraphs and gave some characterizations for some special cases (Proposition 7 and Theorem 8). In this paper, we continue to study the minimally strong subgraph $(k, \ell)$-arc-connected digraphs. We first give a characterization of the minimally strong subgraph $(3, n - 2)$-arc-connected digraphs (Theorem 4), then give exact values and bounds for the functions $g(n, k, \ell)$ and $G(n, k, \ell)$ (Theorem 6 and Proposition 10).

1.2. Preliminaries

We will use the following Tillson’s decomposition theorem.

**Theorem 1** [15]. The arcs of $\overrightarrow{K}_n$ can be decomposed into Hamiltonian cycles if and only if $n \neq 4, 6$.

The following proposition will also be used in our argument.
**Proposition 2** [11]. Let $D$ be a digraph of order $n$, and let $k \geq 2$ be an integer. Then

1. $\lambda_{k+1}(D) \leq \lambda_k(D)$ for every $k \leq n - 1$,
2. $\lambda_k(D') \leq \lambda_k(D)$ where $D'$ is a spanning subgraph of $D$,
3. $\kappa_k(D) \leq \lambda_k(D) \leq \min\{\delta^+(D), \delta^-(D)\}$,

Sun and Gutin [11] obtained a sharp lower bound and a sharp upper bound of $\lambda_k(D)$ for $2 \leq k \leq n$.

**Theorem 3.** Let $2 \leq k \leq n$. For a strong digraph $D$ of order $n$, we have

$$1 \leq \lambda_k(D) \leq n - 1.$$ 

Moreover, both bounds are sharp, and the upper bound holds if and only if $D \cong K_n^2$, where $k \notin \{4, 6\}$, or, $k \in \{4, 6\}$ and $k < n$.

### 2. Characterization of the Minimally Strong Subgraph $(3, n - 2)$-Arc-Connected Digraphs

For a digraph $D$, its reverse $D^{rev}$ is a digraph with same vertex set and such that $xy \in A(D^{rev})$ if and only if $yx \in A(D)$.

**Theorem 4.** A digraph $D$ is minimally strong subgraph $(3, n - 2)$-arc-connected if and only if $D$ is a digraph obtained from the complete digraph $K_n^2$ by deleting an arc set $M$ such that $K_n^2[M]$ is a union of vertex-disjoint cycles which cover all but at most one vertex of $K_n^2$.

**Proof.** Let $D$ be a digraph obtained from the complete digraph $K_n^2$ by deleting an arc set $M$ such that $K_n^2[M]$ is a union of vertex-disjoint cycles which cover all but at most one vertex of $K_n^2$. To prove the theorem it suffices to show that (a) $D$ is minimally strong subgraph $(3, n - 2)$-arc-connected, that is, $\lambda_3(D) \geq n - 2$ but for any arc $e \in A(D)$, $\lambda_3(D - e) \leq n - 3$, and (b) if a digraph $H$ is minimally strong subgraph $(3, n - 2)$-arc-connected then it must be constructed from $K_n^2$ as the digraph $D$ above. Thus, the remainder of the proof has two parts.

**Part (a).** We just consider the case that $K_n^2[M]$ is a union of vertex-disjoint cycles which cover all vertices of $K_n^2$, since the argument for the other case is similar. For any $e \in A(K_n^2) \setminus M$, observe that $e$ must be adjacent to at least one element of $M$, so $\lambda_3(D - e) \leq \min\{\delta^+(D - e), \delta^-(D - e)\} = n - 3$ by (3). Hence, it suffices to show that $\lambda_3(D) = n - 2$ in the following. So we will show that for $S = \{x, y, z\} \subseteq V(D)$, there are at least $n - 2$ arc-disjoint $S$-strong subgraphs in $D$. 
Case 1. \( x, y, z \) belong to the same cycle, say \( C = u_1 u_2 \cdots u_t u_1 \), of \( K_n^r[M] \).

Subcase 1.1. \( S \) induces a path of length two in \( C \). Without loss of generality, assume that \( x = u_1, y = u_2, z = u_3 \).

For the case that \( t = 3 \), we construct the following \( n - 2 \) arc-disjoint \( S \)-strong subgraphs: let \( D_1 \) be the cycle \( yzx \); for any \( u \in V(D) \setminus S \), let \( D_u \) be the subdigraph of \( D \) with vertex set \( S \cup \{ u \} \) and arc set \( \{ xu, ux, yu, uy, zu, uz \} \).

For the case that \( t = 4 \), we construct the following \( n - 2 \) arc-disjoint \( S \)-strong subgraphs: let \( D_1 \) be the cycle \( zyxz \); let \( D_2 \) be the subdigraph of \( D \) with vertex set \( V(C) \) and arc set \( \{ xu, zx, yu, u_y, uz \} \); for any \( u \in V(D) \setminus V(C) \), let \( D_u \) be the subdigraph of \( D \) with vertex set \( S \cup \{ u \} \) and arc set \( \{ xu, ux, yu, uy, zu, uz \} \).

For the case that \( t \geq 5 \), we construct the following \( n - 2 \) arc-disjoint \( S \)-strong subgraphs: let \( D_1 \) be the cycle \( yz \); let \( D_2 \) be the cycle \( xxu_zu_y \); let \( D_3 \) be the subdigraph of \( D \) with vertex set \( S \cup \{ u \} \) and arc set \( \{ xu_3, u_3x, u_3y, u_4y, u_4 \} \); for any \( u \in V(D) \setminus (\{ u \} \cup S) \), let \( D_u \) be the subdigraph of \( D \) with vertex set \( S \cup \{ u \} \) and arc set \( \{ xu, ux, yu, uy, zu, uz \} \).

Subcase 1.2. Exactly two elements of \( S \) are adjacent. Without loss of generality, assume that \( x = u_1, y = u_2 \). We know \( t \geq 5 \) in this case.

If \( t = 5 \), then \( z = u_4 \). We construct the following \( n - 2 \) arc-disjoint \( S \)-strong subgraphs: let \( D_1 \) be the cycle \( yzxz \); let \( D_2 \) be the cycle \( zxxu_zu_y \); let \( D_3 \) be the subdigraph of \( D \) with vertex set \( V(C) \) and arc set \( \{ xu_3, u_3x, u_3y, yu, u_z, z_u, z_u \} \); for any \( u \in V(D) \setminus V(C) \), let \( D_u \) be the subdigraph of \( D \) with vertex set \( S \cup \{ u \} \) and arc set \( \{ xu, ux, yu, uy, zu, uz \} \).

We now consider the case that \( t \geq 6 \). If \( z = u_4 \), then we construct the following \( n - 2 \) arc-disjoint \( S \)-strong subgraphs: let \( D_1 \) be the cycle \( yzxz \); let \( D_2 \) be the cycle \( zxxu_zu_y \); let \( D_3 \) be the subdigraph of \( D \) with vertex set \( \{ x, y, u_3, z, u_4 \} \) and arc set \( \{ xu_3, u_3x, u_3y, yu, u_z, z_u, z_u \} \); let \( D_4 \) be the subdigraph of \( D \) with vertex set \( \{ x, y, z, u_5, u_5 \} \) and arc set \( \{ xu_5, u_5x, u_5y, yu_5, zu_5, u_5u_5 \} \); for any \( u \in V(D) \setminus \{ x, y, u_3, z, u_5, u_5 \} \), let \( D_u \) be the subdigraph of \( D \) with vertex set \( S \cup \{ u \} \) and arc set \( \{ xu, ux, yu, uy, zu, uz \} \).

If \( z = u_{t-1} \), then we construct the following \( n - 2 \) arc-disjoint \( S \)-strong subgraphs: let \( D_1 \) be the cycle \( yzxz \); let \( D_2 \) be the cycle \( zxxu_zu_y \); let \( D_3 \) be the subdigraph of \( D \) with vertex set \( \{ x, y, u_3, u_{t-2}, z, u_t \} \) and arc set \( \{ xu_3, u_3x, u_3y, yu_t, uz, zu_{t-2}, u_{t-2}u_3 \} \); let \( D_4 \) be the subdigraph of \( D \) with vertex set \( \{ x, y, u_3, u_{t-2}, z, u_t \} \) and arc set \( \{ xu_{t-2}, u_{t-2}x, u_{t-2}y, yu_{t-2}, u_{t-2}u_t, u_tu_{t-2}, u_tu_3, u_3u_t, u_3z, zu_3 \} \); for any \( u \in V(D) \setminus \{ x, y, u_3, u_{t-2}, z, u_t \} \), let \( D_u \) be the subdigraph of \( D \) with vertex set \( S \cup \{ u \} \) and arc set \( \{ xu, ux, yu, uy, zu, uz \} \).

If \( z \notin \{ u_4, u_{t-1} \} \), say \( z = u_5 \), then we construct the following \( n - 2 \) arc-disjoint \( S \)-strong subgraphs: let \( D_1 \) be the cycle \( yzxz \); let \( D_2 \) be the cycle \( zxxu_zu_y \); let \( D_3 \) be the subdigraph of \( D \) with vertex set \( \{ x, y, u_3, u_4, z, u_{t-1} \} \) and arc set \( \{ xu_3, u_3x, u_3y, yu_{t-1}, u_{t-1}z, zu_4, u_4u_3 \} \); let \( D_4 \) be the subdigraph of \( D \) with ver-
tex set \( \{x, y, z, u_{t-1}, u_t\} \) and arc set \( \{xu_{t-1}, u_{t-1}x, u_tz, zu_t, yu_t, u_tu_{t-1}, u_{t-1}y\} \); let \( D_5 \) be the subdigraph of \( D \) with vertex set \( \{x, y, u_3, z, u_{t-1}\} \) and arc set \( \{xu_4, u_4x, u_4y, yu_4, u_4u_{t-1}, u_{t-1}u_4, u_{t-1}u_3, u_3u_{t-1}, u_3z, zu_3\} \); for any \( u \in V(D) \setminus \{x, y, u_3, z, u_{t-1}, u_t\} \), let \( D_u \) be the subdigraph of \( D \) with vertex set \( S \cup \{u\} \) and arc set \( \{xu, xu, yu, uy, zu, zu\} \).

Subcase 1.3. Any two elements of \( S \) are nonadjacent. Without loss of generality, assume that \( x = u_1 \). We know \( t \geq 6 \) in this case.

If \( t = 6 \), then we can assume that \( y = u_3, z = u_5 \). We construct the following \( n - 2 \) arc-disjoint \( S \)-strong subgraphs: let \( D_1 \) be the cycle \( yzyz \); let \( D_2 = D_1^{rev} \); let \( D_3 \) be the subdigraph of \( D \) with vertex set \( S \cup \{u_2, u_4\} \) and arc set \( \{xu_6, u_6y, yu_2, u_2x, u_2z, zu_2\} \); let \( D_4 \) be the subdigraph of \( D \) with vertex set \( S \cup \{u_4, u_1\} \) and arc set \( \{zu_4, u_4y, yu_4, u_4z, u_4x, xu_4\} \); let \( D_5 \) be the subdigraph of \( D \) with vertex set \( S \cup \{u_6, u_1\} \) and arc set \( \{zu_6, u_6y, u_6z, u_6x, xu_6, u_6y, yu_6, zu_6, u_6z\} \); for any \( u \in V(D) \setminus (S \cup \{u_2, u_4, u_6, u_1\}) \), let \( D_u \) be the subdigraph of \( D \) with vertex set \( S \cup \{u\} \) and arc set \( \{xu, xu, yu, uy, zu, zu\} \).

We now consider the case that exactly one pair of elements, say \( x \) and \( y \), of \( S \) does not have a common neighbor in the cycle \( C \). Without loss of generality, assume that \( y = u_3, z = u_5 \) (observe that \( x \) and \( y \) have a common neighbor \( u_2\), and \( z \) have a common neighbor \( u_4 \), but \( z \) and \( z \) do not have a common neighbor in the cycle \( C \)). We construct the following \( n - 2 \) arc-disjoint \( S \)-strong subgraphs: let \( D_1 \) be the cycle \( zyxy \); let \( D_2 = D_1^{rev} \); let \( D_3 \) be a subdigraph of \( D \) with vertex set \( S \cup \{u_2, u_4\} \) and arc set \( \{xu_6, u_6y, yu_2, u_2x, u_2z, zu_2\} \); let \( D_4 \) be a subdigraph of \( D \) with vertex set \( S \cup \{u_4, u_1\} \) and arc set \( \{zu_4, u_4y, yu_4, u_4z, u_4x, xu_4\} \); let \( D_5 \) be the subdigraph of \( D \) with vertex set \( S \cup \{u_6, u_1\} \) and arc set \( \{zu_6, u_6y, u_6z, u_6x, xu_6, u_6y, yu_6, zu_6, u_6z\} \); for any \( u \in V(D) \setminus (S \cup \{u_2, u_4, u_6, u_1\}) \), let \( D_u \) be the subdigraph of \( D \) with vertex set \( S \cup \{u\} \) and arc set \( \{xu, xu, yu, uy, zu, zu\} \).

We consider the remaining case that any pair of elements of \( S \) does not have a common neighbor in the cycle \( C \). Without loss of generality, assume that \( y = u_4, z = u_7 \) (we know \( x \) and \( y \) do not have a common neighbor \( u_2\), and \( z \) do not have a common neighbor, \( z \) and \( x \) do not have a common neighbor in the cycle \( C \)). We
construct the following \( n-2 \) arc-disjoint \( S \)-strong subgraphs: let \( D_1 \) be the cycle \( zyxxz \); let \( D_2 = D_1^{\text{ev}} \); let \( D_3 \) be a subdigraph of \( D \) with vertex set \( S \cup \{u_2, u_1\} \) and arc set \( \{xu, u_1u_2, u_2x, u_2y, yu_2, u_2z, zu_2\} \); let \( D_4 \) be a subdigraph of \( D \) with vertex set \( S \cup \{u_3, u_2\} \) and arc set \( \{u_3u_1, u_1y, yu_3, u_3x, xu_3, uz_3\} \); let \( D_5 \) be a subdigraph of \( D \) with vertex set \( S \cup \{u_3, u_2, u_5\} \) and arc set \( \{u_5y, yu_1, u_1u_5, u_5x, xu_5, uz_5\} \); let \( D_6 \) be a subdigraph of \( D \) with vertex set \( S \cup \{u_6, u_5\} \) and arc set \( \{u_6u_1, u_1z, zu_6, xu_6, u_6y, u_6y\} \); let \( D_7 \) be a subdigraph of \( D \) with vertex set \( S \cup \{u_8, u_1\} \) and arc set \( \{u_8z, zu_1, u_1u_8, xu_8, u_8y, u_8y\} \); for any \( u \in V(D) \setminus (S \cup \{u_2, u_3, u_5, u_6, u_8, u_1\}) \), let \( D_u \) be the subdigraph of \( D \) with vertex set \( S \cup \{u\} \) and arc set \( \{xu, xu, yu, uy, zu, uz\} \).

**Case 2.** Exactly two elements of \( S \) belong to the same cycle, say \( C_1 = u_1 u_2 \cdots u_{t} u_1, \) of \( K_n[M] \), and the remaining element belongs to the other cycle \( C_2 = v_1 v_2 \cdots v_h v_1 \). Without loss of generality, assume that \( x, y \in V(C_1), z = v_1 \).

**Subcase 2.1.** \( x \) and \( y \) are adjacent. Without loss of generality, assume that \( x = u_1, y = u_2 \). We just consider the case that \( t \geq 4 \) and \( h \geq 3 \), since the arguments for the other cases are similar and simpler. We construct the following \( n-2 \) arc-disjoint \( S \)-strong subgraphs: let \( D_1 \) be the cycle \( zyxxz \); let \( D_2 = D_1^{\text{ev}} \); let \( D_3 \) be a subdigraph of \( D \) with vertex set \( S \cup \{u_4\} \) and arc set \( \{xu, u_1y, yu_2, u_2x, u_2z, zu_2\} \); let \( D_4 \) be a subdigraph of \( D \) with vertex set \( S \cup \{u_3, u_4\} \) and arc set \( \{yu_4, u_4u_5, u_5y, u_3x, xu_3, uz_3\} \); let \( D_5 \) be a subdigraph of \( D \) with vertex set \( S \cup \{v_2, v_h\} \) and arc set \( \{zv_h, v_h v_2, v_2 z, v_2 x, v_2 y, v_2 y\} \); for any \( u \in V(D) \setminus (S \cup \{u_3, u_4, u_2, v_h\}) \), let \( D_u \) be the subdigraph of \( D \) with vertex set \( S \cup \{u\} \) and arc set \( \{xu, xu, yu, uy, zu, uz\} \).

**Subcase 2.2.** \( x \) and \( y \) are nonadjacent. Without loss of generality, assume that \( x = u_1 \).

We first consider the case that \( t = 4 \), and observe that \( y = u_3 \). Furthermore, assume that \( h \geq 3 \) since the argument for the remaining case is similar and simpler. We construct the following \( n-2 \) arc-disjoint \( S \)-strong subgraphs: let \( D_1 \) be the cycle \( zyxxz \); let \( D_2 = D_1^{\text{ev}} \); let \( D_3 \) be a subdigraph of \( D \) with vertex set \( S \cup \{u_2, u_1\} \) and arc set \( \{xu, u_1 y, yu_2, u_2 x, u_2 z, zu_2\} \); let \( D_4 \) be a subdigraph of \( D \) with vertex set \( S \cup \{v_2, u_1\} \) and arc set \( \{xv_2, v_2 x, yv_2, yv_2, u_1 z, u_1 v_2, v_2 u_1\} \); let \( D_5 \) be a subdigraph of \( D \) with vertex set \( S \cup \{v_2, v_h\} \) and arc set \( \{xv_h, v_h x, yv_h, v_h y, v_h v_2, v_2 z, zv_h\} \); for any \( u \in V(D) \setminus (S \cup \{u_2, u_1, v_2, v_h\}) \), let \( D_u \) be the subdigraph of \( D \) with vertex set \( S \cup \{u\} \) and arc set \( \{xu, xu, yu, uy, zu, uz\} \).

Now we assume that \( t \geq 5 \). We first consider the case that \( x \) and \( y \) have exactly one common neighbor in the cycle \( C_1 \). With a similar argument to that of the case that \( t \geq 7 \) and exactly one pair of elements, say \( x \) and \( y \), of \( S \) has a common neighbor in the cycle \( C \) in Subcase 1.3, we can construct \( n-2 \) arc-disjoint \( S \)-strong subgraphs.

We next consider the case that \( x \) and \( y \) do not have a common neighbor in
the cycle $C_1$. If $h \geq 3$, then with a similar argument to that of the case that $t \geq 7$
and any pair of elements of $S$ does not have a common neighbor in the cycle $C$ in
Subcase 1.3, we can construct $n - 2$ arc-disjoint strong subgraphs containing $S$.
Otherwise, we have $h = 2$. Without loss of generality, assume that $y = u_4$. We
construct the following $n - 2$ arc-disjoint $S$-strong subgraphs: let $D_1$ be the cycle
$xuyz$; let $D_2 = D_1^{xy}$; let $D_3$ be a subdigraph of $D$ with vertex set $S \cup \{u_2, u_t\}$
and arc set $\{xu_4, u_2u_3, u_2x, u_2y, yu_2, u_2z, zu_2\}$; let $D_4$ be a subdigraph of $D$ with
vertex set $S \cup \{u_3, u_t\}$ and arc set $\{u_3u_4, u_3y, u_3u_2, xu_3, xz_3, zu_3\}$; let $D_5$ be a
subdigraph of $D$ with vertex set $S \cup \{u_5, u_t\}$ and arc set $\{u_5x, yu_1, u_1u_5, u_5x, xu_5,$
$u_5z, zu_5\}$; let $D_6$ be a subdigraph of $D$ with vertex set $S \cup \{u_t, v_h\}$ and arc set
$\{xv_h, v_hx, v_hu_t, u_tv_h, v_hy, yv_h, u_tv_1, v_1u_t\}$; for any $u \in V(D) \setminus (S \cup \{u_2, u_3, u_t, u_5,$
$v_h\})$, let $D_u$ be the subdigraph of $D$ with vertex set $S \cup \{u\}$ and arc set $\{xu, xu,$
$yu, uy, zu, uz\}$.

Case 3. The elements of $S$ belong to distinct cycles, say $x \in V(C_1)$, $y \in V(C_2)$,
$z \in V(C_3)$, of $\overrightarrow{K}_n[M]$.

Subcase 3.1. $|V(C_i)| \geq 3$ for all $1 \leq i \leq 3$. With a similar argument to the case
that $t \geq 7$ and exactly one pair of elements, say $x$ and $y$, of $S$ has a common
neighbor in the cycle $C$ in Subcase 1.3, we can construct $n - 2$ arc-disjoint $S$-strong
subgraphs.

Subcase 3.2. $|V(C_{i_0})| = 2$ for some $1 \leq i_0 \leq 3$. With a similar argument to
the case that $x, y$ do not have a common neighbor in the cycle $C_1$ and $h = 2$
in last paragraph of Subcase 2.2, we can construct $n - 2$ arc-disjoint $S$-strong
subgraphs.

Subcase 3.3. $|V(C_{i_0})| = |V(C_{j_0})| = 2$ for some $1 \leq i_0, j_0 \leq 3$. Without loss
of generality, we assume that $i_0 = 2, j_0 = 3$ and furthermore, $u_1x, xu_2 \in E(C_1)$,
u_3y, yu_3 \in E(C_2), u_4z, zu_4 \in E(C_3)$. We construct the following $n - 2$ arc-disjoint
$S$-strong subgraphs: let $D_1$ be the cycle $xyzz$; let $D_2 = D_1^{xy}$; let $D_3$ be a
subdigraph of $D$ with vertex set $S \cup \{u_1, u_2\}$ and arc set $\{u_1u_2, u_2x, xu_1, u_2y, yu_2,$
u_2z, zu_2\}$; let $D_4$ be the cycle $xu_4yu_1zu_3x$; let $D_5 = D_4^{yz}$; for any $u \in V(D) \setminus (S \cup \{u_1, u_2, u_3, u_4\})$, let $D_u$ be a subdigraph of $D$ with vertex set $S \cup \{u\}$ and arc set $\{xu, xu, yu, uy, zu, uz\}$.

Subcase 3.4. $|V(C_i)| = 2$ for all $1 \leq i \leq 3$. This case is easy and we omit the
details.

Part (b). Let $H$ be minimally strong subgraph $(3, n - 2)$-arc-connected. By
Theorem 3, we have that $H \not\cong \overrightarrow{K}_n$, that is, $H$ can be obtained from a complete
digraph $\overrightarrow{K}_n$ by deleting a nonempty arc set $M$. To end our argument, we need
the following claim. Let us start from a simple yet useful observation, which follows from (3).
Proposition 5. No pair of arcs in $M$ has a common head or tail.

Thus, $\overrightarrow{K}_n[M]$ must be a union of vertex-disjoint cycles or paths, otherwise, there are two arcs of $M$ such that they have a common head or tail, a contradiction with Proposition 5.

Claim 1. $\overrightarrow{K}_n[M]$ does not contain a path of order at least two.

Proof. Suppose that $\overrightarrow{K}_n[M]$ contains a path of order at least two. Let $M' \supseteq M$ be a set of arcs obtained from $M$ by adding some arcs from $\overrightarrow{K}_n - M$ such that the digraph $\overrightarrow{K}_n[M']$ contains no path of order at least two. For example, if $\overrightarrow{K}_n[M]$ contains a path $u_1, \ldots, u_\ell$ with $\ell \geq 2$, then add the arc $u_\ell u_1$ to $M'$. Note that $\overrightarrow{K}_n[M']$ is a supergraph of $\overrightarrow{K}_n[M]$ and is a union of vertex-disjoint cycles which cover all but at most one vertex of $\overrightarrow{K}_n$. By Part (a), we have that $\lambda_3(\overrightarrow{K}_n - M') = n - 2$, so $H$ is not minimally strong subgraph $(3, n - 2)$-arc-connected, a contradiction.

It follows from Claim 1 and its proof that $\overrightarrow{K}_n[M]$ must be a union of vertex-disjoint cycles which cover all but at most one vertex of $\overrightarrow{K}_n$, which completes the proof of Part (b).

3. Results for $g(n, k, \ell)$, $G(n, k, \ell)$, ex$(n, k, \ell)$ and Ex$(n, k, \ell)$

The following result concerns the precise value for $g(n, k, \ell)$.

Theorem 6. For any triple $(n, k, \ell)$ with $2 \leq k \leq n, 1 \leq \ell \leq n - 1$ such that $(n, k, \ell) \notin \{(4, 4, 3), (6, 6, 5)\}$, we have

$$g(n, k, \ell) = n\ell.$$

Proof. By Theorem 3 and the definition of $g(n, k, \ell)$, we have $(n, k, \ell) \notin \{(4, 4, 3), (6, 6, 5)\}$.

For all digraphs $D$ and $k \geq 2$, we have $\lambda_k(D) \leq \delta^+(D)$ and $\lambda_k(D) \leq \delta^-(D)$ by (3). Hence for each $D$ with $\lambda_k(D) = \ell$, we have that $\delta^+(D), \delta^-(D) \geq \ell$, so $|A(D)| \geq n\ell$ and then $g(n, k, \ell) \geq n\ell$.

We first consider the case that $n \notin \{4, 6\}$. Let $D \cong \overrightarrow{K}_n$. By Theorem 1, $D$ can be decomposed into $n - 1$ Hamiltonian cycles $H_i$ ($1 \leq i \leq n - 1$). Let $D_\ell$ be the spanning subdigraph of $D$ with arc sets $A(D_\ell) = \bigcup_{1 \leq i \leq \ell} A(H_i)$. Clearly, we have $\lambda_k(D_\ell) \geq \ell$ for $2 \leq k \leq n, 1 \leq \ell \leq n - 1$. Furthermore, by (3), we have $\lambda_k(D_\ell) \leq \ell$ since the in-degree and out-degree of each vertex in $D_\ell$ are both $\ell$. Hence, $\lambda_k(D_\ell) = \ell$ for $2 \leq k \leq n, 1 \leq \ell \leq n - 1$. For any $e \in A(D_\ell)$, we have $\delta^+(D_\ell - e) = \delta^-(D_\ell - e) = \ell - 1$, so $\lambda_k(D_\ell - e) \leq \ell - 1$ by (3). Thus,
$D_\ell$ is minimally strong subgraph $(k, \ell)$-arc-connected. As $|A(D_\ell)| = n\ell$, we have $g(n, k, \ell) \leq n\ell$. From the lower bound that $g(n, k, \ell) \geq n\ell$, we have $g(n, k, \ell) = n\ell$ for the case that $n \not\in \{4, 6\}$.

Now we assume that $n \in \{4, 6\}$. We just consider the case that $n = 6$, since the remaining case is similar and simpler. Let $D$ be a digraph with vertex set $V(D) = \{u_i \mid 1 \leq i \leq 6\}$ such that $D$ is a union of four arc-disjoint cycles $C_i$, where $C_1 : u_1u_2u_3u_4u_5u_6u_1$, $C_2 = C_1^{\text{rev}}$, $C_3 : u_1u_3u_5u_2u_4u_6u_1$ and $C_4 = C_3^{\text{rev}}$.

Let $D_\ell$ $(1 \leq \ell \leq 4)$ be the spanning subdigraph of $D$ with arc sets $A(D_\ell) = \bigcup_{1 \leq i \leq \ell} A(C_i)$. Let $D_5 = \overrightarrow{K_6}$. Clearly, we have $\lambda_k(D_\ell) \geq \ell$ for $2 \leq k \leq 5, 1 \leq \ell \leq 5$. Furthermore, by (3), we have $\lambda_k(D_\ell) \leq \ell$ since the in-degree and out-degree of each vertex in $D_\ell$ are both $\ell$. Hence, $\lambda_k(D_\ell) = \ell$ for $2 \leq k \leq 5, 1 \leq \ell \leq 5$. For any $e \in A(D_\ell)$, we have $\delta^+(D_\ell - e) = \delta^-(D_\ell - e) = \ell - 1$, so $\lambda_k(D_\ell - e) \leq \ell - 1$ by (3). Thus, $D_\ell$ is minimally strong subgraph $(k, \ell)$-arc-connected. As $|A(D_\ell)| = n\ell$, we have $g(n, k, \ell) \leq n\ell$. Hence, $g(n, k, \ell) = n\ell$ holds for this case by the lower bound that $g(n, k, \ell) \geq n\ell$. For the case that $k = n = 6$, we have $1 \leq \ell \leq 4$, with a similar argument, we can also deduce that $g(n, k, \ell) = n\ell$.

A digraph $D$ is minimally strong if $D$ is strong but $D - e$ is not for every arc $e$ of $D$. Sun and Gutin [11] gave the following characterizations.

**Proposition 7** [11]. The following assertions hold.

(i) A digraph $D$ is minimally strong subgraph $(k, 1)$-arc-connected if and only if $D$ is minimally strong digraph.

(ii) Let $2 \leq k \leq n$. If $k \not\in \{4, 6\}$, or, $k \in \{4, 6\}$ and $k < n$, then a digraph $D$ is minimally strong subgraph $(k, n - 1)$-arc-connected if and only if $D \cong \overrightarrow{K_n}$.

**Theorem 8** [11]. A digraph $D$ is minimally strong subgraph $(2, n - 2)$-arc-connected if and only if $D$ is a digraph obtained from the complete digraph $\overrightarrow{K_n}$ by deleting an arc set $M$ such that $\overrightarrow{K_n}[M]$ is a union of vertex-disjoint cycles which cover all but at most one vertex of $\overrightarrow{K_n}$.

To prove upper bounds on the number of arcs in a minimally strong subgraph $(k, \ell)$-arc-connected digraph, we will use the following result, see e.g. Corollary 5.3.6 of [1].

**Theorem 9.** Every strong digraph $D$ on $n$ vertices has a strong spanning subgraph $H$ with at most $2n - 2$ arcs and equality holds only if $H$ is a symmetric digraph whose underlying undirected graph is a tree.

**Proposition 10.** We have (i) $G(n, n, \ell) \leq 2\ell(n - 1)$; (ii) For every $k$ $(2 \leq k \leq n)$, $G(n, k, 1) = 2(n - 1)$ and $Ex(n, k, 1)$ consists of symmetric digraphs whose underlying undirected graphs are trees; (iii) $G(n, k, n - 2) = (n-1)^2$ for $k \in \{2, 3\}$.

**Proof.** (i) Let $D = (V, A)$ be a minimally strong subgraph $(n, \ell)$-arc-connected
digraph, and let $D_1, \ldots, D_\ell$ be arc-disjoint strong spanning subgraphs of $D$. Since $D$ is minimally strong subgraph $(n, \ell)$-arc-connected and $D_1, \ldots, D_\ell$ are pairwise arc-disjoint, $|A| = \sum_{i=1}^{\ell} |A(D_i)|$. Thus, by Theorem 9, $|A| \leq 2\ell(n-1)$.

(ii) In the proof of Proposition 7, Sun and Gutin [11] showed that a digraph $D$ is strong if and only if $\lambda_k(D) \geq 1$. Now let $\lambda_k(D) \geq 1$ and a digraph $D$ has a minimal number of arcs. By Theorem 9, we have that $|A(D)| \leq 2(n-1)$, and if $D \in Ex(n, k, 1)$ then $|A(D)| = 2(n-1)$ and $D$ is a symmetric digraph whose underlying undirected graph is a tree.

Part (iii) follows directly from Theorems 4 and 8.

By Theorems 4 and 8, we can get the following result on $ex(n, k, \ell)$ and $Ex(n, k, \ell)$.

Proposition 11. The following assertions hold.

(i) For $k \in \{2, 3\}$, $Ex(n, k, n-2) = \{\overrightarrow{K}_n - M\}$ where $M$ is an arc set such that $\overrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all but exactly one vertex of $\overrightarrow{K}_n$.

(ii) For $k \in \{2, 3\}$, $ex(n, k, n-2) = \{\overrightarrow{K}_n - M\}$ where $M$ is an arc set such that $\overrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all vertices of $\overrightarrow{K}_n$.

4. Discussion

In this paper, we give the characterization of minimally strong subgraph $(3, n-2)$-arc-connected digraphs. We determine the precise values for $g(n, k, \ell)$ completely and the precise values for $G(n, k, n-2)$ for $k \in \{2, 3\}$. So it would be interesting to determine $G(n, k, n-2)$ for every value of $k \geq 2$, as obtaining characterizations of all $(k, n-2)$-arc-connected digraphs for $2 \leq k \leq n$ seems a very difficult problem. It would also be interesting to find a sharp upper bound for $G(n, k, \ell)$ for all $k \geq 2$ and $\ell \geq 2$.

Acknowledgements

We would like to thank two anonymous referees for helpful comments and suggestions which indeed help us greatly to improve the quality of our paper. Yuefang Sun was supported by Zhejiang Provincial Natural Science Foundation (No. LY20A010013) and National Natural Science Foundation of China (No. 11401389). Zemin Jin was supported by National Natural Science Foundation of China (No. 11571320) and Zhejiang Provincial Natural Science Foundation (No. LY19A010018).

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doi:10.1007/978-1-84800-998-1


Received 14 September 2018
Revised 14 January 2020
Accepted 14 January 2020