MINIMALLY STRONG SUBGRAPH 
(k, ℓ)-ARC-CONNECTED DIGRAPHS

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Abstract

Let D = (V, A) be a digraph of order n, S a subset of V of size k and 2 ≤ k ≤ n. A subdigraph H of D is called an S-strong subgraph if H is strong and S ⊆ V(H). Two S-strong subgraphs D₁ and D₂ are said to be arc-disjoint if A(D₁) ∩ A(D₂) = ∅. Let λₘ(S) be the maximum number of arc-disjoint S-strong digraphs in D. The strong subgraph k-arc-connectivity is defined as λₖ(D) = min{λₘ(S) | S ⊆ V, |S| = k}. A digraph D = (V, A) is called minimally strong subgraph (k, ℓ)-arc-connected if λₖ(D) ≥ ℓ but for any arc e ∈ A, λₖ(D − e) ≤ ℓ − 1. Let Σ(n, k, ℓ) be the set of all minimally strong subgraph (k, ℓ)-arc-connected digraphs with order n. We define G(n, k, ℓ) = max{|A(D)| | D ∈ Σ(n, k, ℓ)} and g(n, k, ℓ) = min{|A(D)| | D ∈ Σ(n, k, ℓ)}.

In this paper, we study the minimally strong subgraph (k, ℓ)-arc-connected digraphs. We give a characterization of the minimally strong subgraph (3, n − 2)-arc-connected digraphs, and then give exact values and bounds for the functions g(n, k, ℓ) and G(n, k, ℓ).

Keywords: strong subgraph k-connectivity, strong subgraph k-arc-connectivity, subgraph packing.

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1. Introduction

1.1. Motivation and concepts

The generalized k-connectivity $\kappa_k(G)$ of a graph $G = (V, E)$ was introduced by Hager [8] in 1985 ($2 \leq k \leq |V|$). For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an $S$-Steiner tree or, simply, an $S$-tree is a subgraph $T$ of $G$ which is a tree with $S \subseteq V(T)$. Two $S$-trees $T_1$ and $T_2$ are said to be edge-disjoint if $E(T_1) \cap E(T_2) = \emptyset$. Two edge-disjoint $S$-trees $T_1$ and $T_2$ are said to be internally disjoint if $V(T_1) \cap V(T_2) = S$. The generalized local connectivity $\kappa_S(G)$ is the maximum number of internally disjoint $S$-trees in $G$. For an integer $k$ with $2 \leq k \leq n$, the generalized $k$-connectivity is defined as

$$\kappa_k(G) = \min\{\kappa_S(G) \mid S \subseteq V(G), |S| = k\}.$$ 

Observe that $\kappa_2(G) = \kappa(G)$. Li, Mao and Sun [9] introduced the following concept of generalized $k$-arc-connectivity. The generalized local edge-connectivity $\lambda_S(G)$ is the maximum number of edge-disjoint $S$-trees in $G$. For an integer $k$ with $2 \leq k \leq n$, the generalized $k$-arc-connectivity is defined as

$$\lambda_k(G) = \min\{\lambda_S(G) \mid S \subseteq V(G), |S| = k\}.$$ 

Observe that $\lambda_2(G) = \lambda(G)$. Generalized connectivity of graphs has become a well-established area in graph theory, see a recent monograph [10] by Li and Mao on generalized connectivity of undirected graphs.

To extend generalized $k$-connectivity to directed graphs, Sun, Gutin, Yeo and Zhang [11] observed that in the definition of $\kappa_S(G)$, one can replace “an $S$-tree” by “a connected subgraph of $G$ containing $S$.” Therefore, Sun et al. [11] defined strong subgraph $k$-connectivity by replacing “connected” with “strongly connected” (or, simply, “strong”) as follows. Let $D = (V, A)$ be a digraph of order $n$, $S$ a subset of $V$ of size $k$ and $2 \leq k \leq n$. A subdigraph $H$ of $D$ is called an $S$-strong subgraph if $H$ is strong and $S \subseteq V(H)$. Two $S$-strong subgraphs $D_1$ and $D_2$ are said to be arc-disjoint if $A(D_1) \cap A(D_2) = \emptyset$. Two arc-disjoint $S$-strong subgraphs $D_1$ and $D_2$ are said to be internally disjoint if $V(D_1) \cap V(D_2) = S$. Let $\kappa_S(D)$ be the maximum number of internally disjoint $S$-strong subgraphs in $D$. The strong subgraph $k$-connectivity is defined as

$$\kappa_k(D) = \min\{\kappa_S(D) \mid S \subseteq V, |S| = k\}.$$ 

As a natural counterpart of the strong subgraph $k$-connectivity, Sun and Gutin [11] introduced the concept of strong subgraph $k$-arc-connectivity. Let $D = (V(D), A(D))$ be a digraph of order $n$, $S \subseteq V$ a $k$-subset of $V(D)$ and $2 \leq k \leq n$. Let $\lambda_S(D)$ be the maximum number of arc-disjoint $S$-strong subgraphs in $D$. The strong subgraph $k$-arc-connectivity is defined as

$$\lambda_k(D) = \min\{\lambda_S(D) \mid S \subseteq V(D), |S| = k\}.$$
The strong subgraph \( k \)-arc-connectivity is not only a natural extension of the concept of generalized \( k \)-(edge-)connectivity, but also relates to important problems in graph theory. For \( k = 2 \), \( \kappa_2(G) = \kappa(G) \) [14] and \( \lambda_2(G) = \lambda(G) \) [11]. Hence, \( \kappa_k(D) \) and \( \lambda_k(D) \) could be seen as generalizations of connectivity and edge-connectivity of undirected graphs, respectively. For \( k = n \), \( \kappa_n(D) = \lambda_n(D) \) is the maximum number of arc-disjoint spanning strong subgraphs of \( D \).

Moreover, we know that \( \kappa_S(D) \) and \( \lambda_S(D) \) denote the number of internally-disjoint and arc-disjoint strong subgraphs containing a given set \( S \), respectively. Hence, these parameters are relevant to the subdigraph packing problem, see [2–5,7,13]. For a recent survey on the topic of strong subgraph connectivity, the readers can see [12].

A digraph \( D = (V(D), A(D)) \) is called \textit{minimally strong subgraph \((k, \ell)\)-arc-connected} if \( \lambda_k(D) \geq \ell \) but for any arc \( e \in A(D) \), \( \lambda_k(D - e) \leq \ell - 1 \). Note that \( 2 \leq k \leq n, 1 \leq \ell \leq n - 1 \) by the definition of \( \lambda_k(D) \) and Theorem 3.

Let \( \mathcal{G}(n,k,\ell) \) be the set of all minimally strong subgraph \((k, \ell)\)-arc-connected digraphs with order \( n \). We define

\[
G(n,k,\ell) = \max\{|A(D)| \mid D \in \mathcal{G}(n,k,\ell)\}
\]

and

\[
g(n,k,\ell) = \min\{|A(D)| \mid D \in \mathcal{G}(n,k,\ell)\}.
\]

We further define

\[
Ex(n,k,\ell) = \{D \mid D \in \mathcal{G}(n,k,\ell), |A(D)| = G(n,k,\ell)\}
\]

and

\[
ex(n,k,\ell) = \{D \mid D \in \mathcal{G}(n,k,\ell), |A(D)| = g(n,k,\ell)\}.
\]

In [11], Sun and Gutin first studied the minimally strong subgraph \((k, \ell)\)-arc-connected digraphs and gave some characterizations for some special cases (Proposition 7 and Theorem 8). In this paper, we continue to study the minimally strong subgraph \((k, \ell)\)-arc-connected digraphs. We first give a characterization of the minimally strong subgraph \((3, n - 2)\)-arc-connected digraphs (Theorem 4), then give exact values and bounds for the functions \( g(n,k,\ell) \) and \( G(n,k,\ell) \) (Theorem 6 and Proposition 10).

1.2. Preliminaries

We will use the following Tillson’s decomposition theorem.

**Theorem 1** [15]. The arcs of \( \overset{\rightarrow}{K}_n \) can be decomposed into Hamiltonian cycles if and only if \( n \neq 4, 6 \).

The following proposition will also be used in our argument.
Proposition 2 [11]. Let $D$ be a digraph of order $n$, and let $k \geq 2$ be an integer. Then

1. $\lambda_{k+1}(D) \leq \lambda_k(D)$ for every $k \leq n - 1$,
2. $\lambda_k(D') \leq \lambda_k(D)$ where $D'$ is a spanning subgraph of $D$,
3. $\kappa_k(D) \leq \lambda_k(D) \leq \min\{\delta^+(D), \delta^-(D)\}$,

Sun and Gutin [11] obtained a sharp lower bound and a sharp upper bound of $\lambda_k(D)$ for $2 \leq k \leq n$.

Theorem 3. Let $2 \leq k \leq n$. For a strong digraph $D$ of order $n$, we have

$$1 \leq \lambda_k(D) \leq n - 1.$$ 

Moreover, both bounds are sharp, and the upper bound holds if and only if $D \cong \overrightarrow{K}_n$, where $k \not\in \{4, 6\}$, or, $k \in \{4, 6\}$ and $k < n$.

2. Characterization of the Minimally Strong Subgraph $(3, n - 2)$-Arc-Connected Digraphs

For a digraph $D$, its reverse $D^{\text{rev}}$ is a digraph with same vertex set and such that $xy \in A(D^{\text{rev}})$ if and only if $yx \in A(D)$.

Theorem 4. A digraph $D$ is minimally strong subgraph $(3, n - 2)$-arc-connected if and only if $D$ is a digraph obtained from the complete digraph $\overrightarrow{K}_n$ by deleting an arc set $M$ such that $\overrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all but at most one vertex of $\overrightarrow{K}_n$.

Proof. Let $D$ be a digraph obtained from the complete digraph $\overrightarrow{K}_n$ by deleting an arc set $M$ such that $\overrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all but at most one vertex of $\overrightarrow{K}_n$. To prove the theorem it suffices to show that (a) $D$ is minimally strong subgraph $(3, n - 2)$-arc-connected, that is, $\lambda_3(D) \geq n - 2$ but for any arc $e \in A(D)$, $\lambda_3(D - e) \leq n - 3$, and (b) if a digraph $H$ is minimally strong subgraph $(3, n - 2)$-arc-connected then it must be constructed from $\overrightarrow{K}_n$ as the digraph $D$ above. Thus, the remainder of the proof has two parts.

Part (a). We just consider the case that $\overrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all vertices of $\overrightarrow{K}_n$, since the argument for the other case is similar. For any $e \in A(\overrightarrow{K}_n) \setminus M$, observe that $e$ must be adjacent to at least one element of $M$, so $\lambda_3(D - e) \leq \min\{\delta^+(D - e), \delta^-(D - e)\} = n - 3$ by (3). Hence, it suffices to show that $\lambda_3(D) = n - 2$ in the following. So we will show that for $S = \{x, y, z\} \subseteq V(D)$, there are at least $n - 2$ arc-disjoint $S$-strong subgraphs in $D$. 
Case 1. \(x, y, z\) belong to the same cycle, say \(C = u_1u_2 \cdots u_{t}u_{1}\), of \(\overrightarrow{K}_n[M]\).

Subcase 1.1. \(S\) induces a path of length two in \(C\). Without loss of generality, assume that \(x = u_1, y = u_2, z = u_3\).

For the case that \(t = 3\), we construct the following \(n - 2\) arc-disjoint \(S\)-strong subgraphs: let \(D_1\) be the cycle \(zyxz\); for any \(u \in V(D) \setminus S\), let \(D_u\) be the subdigraph of \(D\) with vertex set \(S \cup \{u\}\) and arc set \(\{xz, ux, xu, uy, zu, uz\}\).

For the case that \(t = 4\), we construct the following \(n - 2\) arc-disjoint \(S\)-strong subgraphs: let \(D_1\) be the cycle \(zyxz\); let \(D_2\) be the subdigraph of \(D\) with vertex set \(V(C)\) and arc set \(\{xz, xu, uy, u_{t-1}, u_{t-2}\}\); for any \(u \in V(D) \setminus V(C)\), let \(D_u\) be the subdigraph of \(D\) with vertex set \(S \cup \{u\}\) and arc set \(\{xz, ux, xu, uy, zu, uz\}\).

For the case that \(t = 5\), we construct the following \(n - 2\) arc-disjoint \(S\)-strong subgraphs: let \(D_1\) be the cycle \(zyxz\); let \(D_2\) be the cycle \(zxu_3u_4z\); let \(D_3\) be the subdigraph of \(D\) with vertex set \(S \cup \{u_4, u_5\}\) and arc set \(\{xu_3, u_3x, u_3y, uy, u_{t-1}, u_{t-2}, u_{t-3}, u_{t-4}\}\); for any \(u \in V(D) \setminus (\{u_4, u_5\} \cup S)\), let \(D_u\) be the subdigraph of \(D\) with vertex set \(S \cup \{u\}\) and arc set \(\{xz, ux, xu, uy, zu, uz\}\).

Subcase 1.2. Exactly two elements of \(S\) are adjacent. Without loss of generality, assume that \(x = u_1, y = u_2\). We know \(t \geq 5\) in this case.

If \(t = 5\), then \(z = u_4\). We construct the following \(n - 2\) arc-disjoint \(S\)-strong subgraphs: let \(D_1\) be the cycle \(zyxz\); let \(D_2\) be the cycle \(zxu_3u_4z\); let \(D_3\) be the subdigraph of \(D\) with vertex set \(V(C)\) and arc set \(\{xu_3, u_3x, u_3y, uy, u_{t-1}, u_{t-2}, u_{t-3}, u_{t-4}\}\); for any \(u \in V(D) \setminus V(C)\), let \(D_u\) be the subdigraph of \(D\) with vertex set \(S \cup \{u\}\) and arc set \(\{xz, ux, xu, uy, zu, uz\}\).

We now consider the case that \(t \geq 6\). If \(z = u_4\), then we construct the following \(n - 2\) arc-disjoint \(S\)-strong subgraphs: let \(D_1\) be the cycle \(zyxz\); let \(D_2\) be the cycle \(zxu_3u_4z\); let \(D_3\) be the subdigraph of \(D\) with vertex set \(\{x, y, u_3, z, u_t\}\) and arc set \(\{xu_3, u_3x, u_3y, uy, u_{t-1}, u_{t-2}, u_{t-3}, u_{t-4}\}\); let \(D_4\) be the subdigraph of \(D\) with vertex set \(\{x, y, u_3, u_4, z, u_{t-1}, u_{t-2}, u_{t-3}, u_{t-4}\}\); for any \(u \in V(D) \setminus \{x, y, u_3, z, u_{t-1}, u_{t-2}\}\), let \(D_u\) be the subdigraph of \(D\) with vertex set \(S \cup \{u\}\) and arc set \(\{xz, ux, xu, uy, zu, uz\}\).

If \(z = u_{t-1}\), then we construct the following \(n - 2\) arc-disjoint \(S\)-strong subgraphs: let \(D_1\) be the cycle \(zyxz\); let \(D_2\) be the cycle \(zxu_3u_4z\); let \(D_3\) be the subdigraph of \(D\) with vertex set \(\{x, y, u_3, u_{t-2}, z, u_t\}\) and arc set \(\{xu_3, u_3x, u_3y, uy, u_{t-1}, u_{t-2}, u_{t-3}, u_{t-4}\}\); let \(D_4\) be the subdigraph of \(D\) with vertex set \(\{x, y, u_3, u_{t-2}, z, u_t\}\) and arc set \(\{xu_{t-2}, u_{t-2}x, u_{t-2}y, uy_{t-1}, u_{t-2}u_t, u_{t-2}u_{t-3}, u_{t-2}u_{t-4}, u_{t-2}u_3, u_{t-2}u_4, z, u_{t-3}, u_{t-4}\}\); for any \(u \in V(D) \setminus \{x, y, u_3, u_{t-2}, z, u_t\}\), let \(D_u\) be the subdigraph of \(D\) with vertex set \(S \cup \{u\}\) and arc set \(\{xz, ux, xu, uy, zu, uz\}\).

If \(z \not\in \{u_4, u_{t-1}\}\), say \(z = u_5\), then we construct the following \(n - 2\) arc-disjoint \(S\)-strong subgraphs: let \(D_1\) be the cycle \(zyxz\); let \(D_2\) be the cycle \(zxu_3u_4z\); let \(D_3\) be the subdigraph of \(D\) with vertex set \(\{x, y, u_3, u_{t-2}, z, u_t\}\) and arc set \(\{xu_3, u_3x, u_3y, uy_{t-1}, u_{t-2}z, u_{t-3}, u_{t-4}\}\); let \(D_4\) be the subdigraph of \(D\) with ver-
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tex set \{x, y, z, u_{t-1}, u_t\} and arc set \{xu_{t-1}, u_{t-1}x, u_tz, zu_t, yu_t, u_tu_{t-1}, u_{t-1}y\}; let \(D_5\) be the subdigraph of \(D\) with vertex set \(\{x, y, u_3, z, u_{t-1}\}\) and arc set \(\{xu_4, u_4x, u_4y, yu_4, u_4u_{t-1}, u_{t-1}u_4, u_{t-1}u_3, u_3u_{t-1}, u_3z, zu_4\}\); for any \(u \in V(D) \setminus \{x, y, u_3, z, u_{t-1}, u_t\}\), let \(D_u\) be the subdigraph of \(D\) with vertex set \(S \cup \{u\}\) and arc set \(\{xu, ux, yu, uy, zu, uz\}\).

Subcase 1.3. Any two elements of \(S\) are nonadjacent. Without loss of generality, assume that \(x = u_1\). We know \(t \geq 6\) in this case.

If \(t = 6\), then we can assume that \(y = u_3, z = u_5\). We construct the following \(n - 2\) arc-disjoint \(S\)-strong subgraphs: let \(D_1\) be the cycle \(zyxz\); let \(D_2 = D_1^{xy}\); let \(D_3\) be the subdigraph of \(D\) with vertex set \(S \cup \{u_2, u_4\}\) and arc set \(\{xu_6, u_6y, yu_2, u_2x, u_2z, zu_2\}\); let \(D_4\) be the subdigraph of \(D\) with vertex set \(S \cup \{u_4, u_t\}\) and arc set \(\{zu_4, u_4y, yu_6, u_6z, zu_4, xu_4\}\); for any \(u \in V(D) \setminus \{x, u_2, u_4, u_t, u_4\}\), let \(D_u\) be the subdigraph of \(D\) with vertex set \(S \cup \{u\}\) and arc set \(\{xu, ux, yu, uy, zu, uz\}\).

In the following we assume that \(t \geq 7\). We consider the case that exactly one pair of elements, say \(x\) and \(z\), of \(S\) does not have a common neighbor in the cycle \(C\). Without loss of generality, assume that \(y = u_3, z = u_5\) (observe that \(x\) and \(y\) have a common neighbor \(u_2\), and \(z\) and \(x\) do not have a common neighbor \(u_4\), but \(z\) and \(x\) do not have a common neighbor in the cycle \(C\)). We construct the following \(n - 2\) arc-disjoint \(S\)-strong subgraphs: let \(D_1\) be the cycle \(zyxz\); let \(D_2 = D_1^{xy}\); let \(D_3\) be the subdigraph of \(D\) with vertex set \(S \cup \{u_2, u_4\}\) and arc set \(\{xu_6, u_6y, yu_2, u_2x, u_2z, zu_2\}\); let \(D_4\) be the subdigraph of \(D\) with vertex set \(S \cup \{u_4, u_t\}\) and arc set \(\{zu_4, u_4y, yu_t, u_tz, u_4x, xu_4\}\); for any \(u \in V(D) \setminus \{S \cup \{u_2, u_4, u_t, u_4\}\}\), let \(D_u\) be the subdigraph of \(D\) with vertex set \(S \cup \{u\}\) and arc set \(\{xu, ux, yu, uy, zu, uz\}\).

We now consider the case that exactly one pair of elements, say \(x\) and \(y\), of \(S\) has a common neighbor in the cycle \(C\). Without loss of generality, assume that \(y = u_3, z = u_6\) (we know \(x\) and \(y\) have a common neighbor \(u_2\), \(z\) and \(x\) do not have a common neighbor in the cycle \(C\)). We construct the following \(n - 2\) arc-disjoint \(S\)-strong subgraphs: let \(D_1\) be the cycle \(zyxz\); let \(D_2 = D_1^{xy}\); let \(D_3\) be a subdigraph of \(D\) with vertex set \(S \cup \{u_2, u_4\}\) and arc set \(\{xu_6, u_6y, yu_2, u_2x, u_2z, zu_2\}\); let \(D_4\) be a subdigraph of \(D\) with vertex set \(S \cup \{u_4, u_t\}\) and arc set \(\{zu_4, u_4y, yu_6, u_6z, zu_4, xu_4\}\); for any \(u \in V(D) \setminus \{S \cup \{u_2, u_4, u_t, u_4\}\}\), let \(D_u\) be the subdigraph of \(D\) with vertex set \(S \cup \{u\}\) and arc set \(\{xu, ux, yu, uy, zu, uz\}\).

We consider the remaining case that any pair of elements of \(S\) does not have a common neighbor in the cycle \(C\). Without loss of generality, assume that \(y = u_4, z = u_7\) (we know \(x\) and \(y\) do not a common neighbor \(u_2\), \(y\) and \(z\) do not have a common neighbor \(u_4\), \(z\) and \(x\) do not have a common neighbor in the cycle \(C\)). We
construct the following $n - 2$ arc-disjoint $S$-strong subgraphs: let $D_1$ be the cycle $zyxz$; let $D_2 = D_1^{\text{ev}}$; let $D_3$ be a subdigraph of $D$ with vertex set $S \cup \{u_2, u_1\}$ and arc set $\{xu, u_1u_2, u_2x, u_2y, yu, u_2z, zu\}$; let $D_4$ be a subdigraph of $D$ with vertex set $S \cup \{u_3, u_2\}$ and arc set $\{u_3u, u_1y, yu_3, u_3x, xu_3, u_3z, zu_3\}$; let $D_5$ be a subdigraph of $D$ with vertex set $S \cup \{u_5, u_3\}$ and arc set $\{u_5y, yu_1, u_1u_5, u_5x, xu_5, u_5z, zu_5\}$; let $D_6$ be a subdigraph of $D$ with vertex set $S \cup \{u_6, u_1\}$ and arc set $\{u_6u, u_1z, zu, xu_6, xu_1, u_6y, u_6y\}$; let $D_7$ be a subdigraph of $D$ with vertex set $S \cup \{u_8, u_1\}$ and arc set $\{u_8z, zu, u_1u_8, xu_8, u_8y, u_8y\}$; for any $u \in V(D) \setminus (S \cup \{u_2, u_3, u_5, u_6, u_8, u_1\})$, let $D_u$ be the subdigraph of $D$ with vertex set $S \cup \{u\}$ and arc set $\{xu, xu, yu_1, xu, zu, zu\}$.

**Case 2.** Exactly two elements of $S$ belong to the same cycle, say $C_1 = u_1u_2 \cdots u_1u_1$, of $\overrightarrow{K}_n[M]$, and the remaining element belongs to the other cycle $C_2 = v_1v_2 \cdots v_nv_1$. Without loss of generality, assume that $x, y \in V(C_1), z = v_1$.

**Subcase 2.1.** $x$ and $y$ are adjacent. Without loss of generality, assume that $x = u_1, y = u_2$. We just consider the case that $t \geq 4$ and $h \geq 3$, since the arguments for the other cases are similar and simpler. We construct the following $n - 2$ arc-disjoint $S$-strong subgraphs: let $D_1$ be the cycle $zyxz$; let $D_2 = D_1^{\text{ev}}$; let $D_3$ be a subdigraph of $D$ with vertex set $S \cup \{u_1\}$ and arc set $\{xu_1, u_1y, yu, u_1z, zuu\}$; let $D_4$ be a subdigraph of $D$ with vertex set $S \cup \{u_3, u_1\}$ and arc set $\{yu_1, u_1u_3, u_3y, u_3x, xu_3, u_3z, zu_3\}$; let $D_5$ be a subdigraph of $D$ with vertex set $S \cup \{v_2, v_1\}$ and arc set $\{zv_1, v_1v_2, v_2z, xv_2, v_2y, v_2y\}$; for any $u \in V(D) \setminus (S \cup \{u_3, u_1, v_2, v_1\})$, let $D_u$ be the subdigraph of $D$ with vertex set $S \cup \{u\}$ and arc set $\{xu, xu, yu, zu, zu\}$.

**Subcase 2.2.** $x$ and $y$ are nonadjacent. Without loss of generality, assume that $x = u_1$.

We first consider the case that $t = 4$, and observe that $y = u_3$ now. Furthermore, assume that $h \geq 3$ since the argument for the remaining case is similar and simpler. We construct the following $n - 2$ arc-disjoint $S$-strong subgraphs: let $D_1$ be the cycle $zyxz$; let $D_2 = D_1^{\text{ev}}$; let $D_3$ be a subdigraph of $D$ with vertex set $S \cup \{u_1\}$ and arc set $\{xu, u_1y, yu, u_1z, zuu\}$; let $D_4$ be a subdigraph of $D$ with vertex set $S \cup \{v_2, u_1\}$ and arc set $\{xv_2, v_2x, yv_2, v_2y, v_2u_1, v_2u_1\}$; let $D_5$ be a subdigraph of $D$ with vertex set $S \cup \{v_2, v_1\}$ and arc set $\{xv_1, v_1x, yv_1, v_1y, v_1v_2, v_2z, zuv_1\}$; for any $u \in V(D) \setminus (S \cup \{u_2, u_1, v_2, v_1\})$, let $D_u$ be the subdigraph of $D$ with vertex set $S \cup \{u\}$ and arc set $\{xu, xu, yu, zu, zu\}$.

Now we assume that $t \geq 5$. We first consider the case that $x$ and $y$ have exactly one common neighbor in the cycle $C_1$. With a similar argument to that of the case that $t \geq 7$ and exactly one pair of elements, say $x$ and $y$, of $S$ has a common neighbor in the cycle $C$ in Subcase 1.3, we can construct $n - 2$ arc-disjoint $S$-strong subgraphs.

We next consider the case that $x$ and $y$ do not have a common neighbor in
the cycle $C_1$. If $h \geq 3$, then with a similar argument to that of the case that $t \geq 7$
and any pair of elements of $S$ does not have a common neighbor in the cycle $C$ in
Subcase 1.3, we can construct $n - 2$ arc-disjoint strong subgraphs containing $S$.
Otherwise, we have $h = 2$. Without loss of generality, assume that $y = u_4$. We
construct the following $n - 2$ arc-disjoint $S$-strong subgraphs: let $D_1$ be the cycle
$xyzz$; let $D_2 = D_1^{ev}$; let $D_3$ be a subdigraph of $D$ with vertex set $S \cup \{u_2, u_t\}$
and arc set $\{ xu_4, u_4u_2, u_2x, u_2y, yu_2, u_2z, zu_2 \}$; let $D_4$ be a subdigraph of $D$
with vertex set $S \cup \{u_3, u_4\}$ and arc set $\{ u_3u_1, u_1y, yu_3, u_3x, xu_3, u_3z, zu_3 \}$; let $D_5$
be a subdigraph of $D$ with vertex set $S \cup \{u_5, u_t\}$ and arc set $\{ u_5y, yu_5, u_1u_5, u_5x, xu_5,$
$u_5z, zu_5 \}$; let $D_6$ be a subdigraph of $D$ with vertex set $S \cup \{u_t, v_h\}$ and arc set
$\{ xv_h, v_hx, v_hi, u_1v_h, v_hy, yv_h, u_1v_1, v_1u_1 \}$; for any $u \in V(D) \setminus (S \cup \{u_2, u_3, u_t, u_5,$
v_h\})$, let $D_u$ be the subdigraph of $D$ with vertex set $S \cup \{u\}$ and arc set $\{ xu, ux,$
yu, uy, zu, uz \}.

Case 3. The elements of $S$ belong to distinct cycles, say $x \in V(C_1)$, $y \in V(C_2)$,
z $\in V(C_3)$, of $K^{-}[M]$.

Subcase 3.1. $|V(C_i)| \geq 3$ for all $1 \leq i \leq 3$. With a similar argument to the
case that $t \geq 7$ and exactly one pair of elements, say $x$ and $y$, of $S$ has a common
neighbor in the cycle $C$ in Subcase 1.3, we can construct $n - 2$ arc-disjoint $S$-strong
subgraphs.

Subcase 3.2. $|V(C_{i_0})| = 2$ for some $1 \leq i_0 \leq 3$. With a similar argument to
the case that $x, y$ do not have a common neighbor in the cycle $C_1$ and $h = 2$
in last paragraph of Subcase 2.2, we can construct $n - 2$ arc-disjoint $S$-strong
subgraphs.

Subcase 3.3. $|V(C_{i_0})| = |V(C_{j_0})| = 2$ for some $1 \leq i_0, j_0 \leq 3$. Without loss
of generality, we assume that $i_0 = 2, j_0 = 3$ and furthermore, $u_1x, xu_2 \in E(C_1)$,
$u_3y, yu_3 \in E(C_2), u_4z, zu_4 \in E(C_3)$. We construct the following $n - 2$ arc-disjoint
$S$-strong subgraphs: let $D_1$ be the cycle $xyzz$; let $D_2 = D_1^{ev}$; let $D_3$ be a subdigraph of $D$
with vertex set $S \cup \{u_1, u_2\}$ and arc set $\{ u_1u_2, u_2x, xu_1, u_2y, yu_2,$
u_2z, zu_2 \}$; let $D_4$ be the cycle $xu_4yu_4zux_3x$; let $D_5 = D_4^{ev}$; for any $u \in V(D) \setminus (S \cup \{u_1,$
u_2, u_3, u_4\})$, let $D_u$ be a subdigraph of $D$ with vertex set $S \cup \{u\}$ and arc set $\{ xu, ux,$
yu, uy, zu, uz \}.

Subcase 3.4. $|V(C_i)| = 2$ for all $1 \leq i \leq 3$. This case is easy and we omit the
details.

Part (b). Let $H$ be minimally strong subgraph $(3, n - 2)$-arc-connected. By
Theorem 3, we have that $H \not\sim K^{-}[n]$, that is, $H$ can be obtained from a complete
digraph $K^{-}$ by deleting a nonempty arc set $M$. To end our argument, we need
the following claim. Let us start from a simple yet useful observation, which follows from (3).
Proposition 5. No pair of arcs in $M$ has a common head or tail.

Thus, $\overrightarrow{K}_n[M]$ must be a union of vertex-disjoint cycles or paths, otherwise, there are two arcs of $M$ such that they have a common head or tail, a contradiction with Proposition 5.

Claim 1. $\overrightarrow{K}_n[M]$ does not contain a path of order at least two.

Proof. Suppose that $\overrightarrow{K}_n[M]$ contains a path of order at least two. Let $M' \supseteq M$ be a set of arcs obtained from $M$ by adding some arcs from $\overrightarrow{K}_n - M$ such that the digraph $\overrightarrow{K}_n[M']$ contains no path of order at least two. For example, if $\overrightarrow{K}_n[M]$ contains a path $u_1, \ldots, u_\ell$ with $\ell \geq 2$, then add the arc $u_\ell u_1$ to $M'$. Note that $\overrightarrow{K}_n[M']$ is a supergraph of $\overrightarrow{K}_n[M]$ and is a union of vertex-disjoint cycles which cover all but at most one vertex of $\overrightarrow{K}_n$. By Part (a), we have that $\lambda_3(\overrightarrow{K}_n - M') = n - 2$, so $H$ is not minimally strong subgraph $(3, n - 2)$-arc-connected, a contradiction. \hfill \Box

It follows from Claim 1 and its proof that $\overrightarrow{K}_n[M]$ must be a union of vertex-disjoint cycles which cover all but at most one vertex of $\overrightarrow{K}_n$, which completes the proof of Part (b).

3. Results for $g(n, k, \ell)$, $G(n, k, \ell)$, $ex(n, k, \ell)$ and $Ex(n, k, \ell)$

The following result concerns the precise value for $g(n, k, \ell)$.

Theorem 6. For any triple $(n, k, \ell)$ with $2 \leq k \leq n, 1 \leq \ell \leq n - 1$ such that $(n, k, \ell) \notin \{(4, 4, 3), (6, 6, 5)\}$, we have

$$g(n, k, \ell) = n\ell.$$  

Proof. By Theorem 3 and the definition of $g(n, k, \ell)$, we have $(n, k, \ell) \notin \{(4, 4, 3), (6, 6, 5)\}$.

For all digraphs $D$ and $k \geq 2$, we have $\lambda_k(D) \leq \delta^+(D)$ and $\lambda_k(D) \leq \delta^-(D)$ by (3). Hence for each $D$ with $\lambda_k(D) = \ell$, we have that $\delta^+(D), \delta^-(D) \geq \ell$, so $|A(D)| \geq n\ell$ and then $g(n, k, \ell) \geq n\ell$.

We first consider the case that $n \notin \{4, 6\}$. Let $D \cong \overrightarrow{K}_n$. By Theorem 1, $D$ can be decomposed into $n - 1$ Hamiltonian cycles $H_i$ ($1 \leq i \leq n - 1$). Let $D_\ell$ be the spanning subdigraph of $D$ with arc sets $A(D_\ell) = \bigcup_{1 \leq i \leq \ell} A(H_i)$. Clearly, we have $\lambda_k(D_\ell) \geq \ell$ for $2 \leq k \leq n, 1 \leq \ell \leq n - 1$. Furthermore, by (3), we have $\lambda_k(D_\ell) \leq \ell$ since the in-degree and out-degree of each vertex in $D_\ell$ are both $\ell$. Hence, $\lambda_k(D_\ell) = \ell$ for $2 \leq k \leq n, 1 \leq \ell \leq n - 1$. For any $e \in A(D_\ell)$, we have $\delta^+(D_\ell - e) = \delta^-(D_\ell - e) = \ell - 1$, so $\lambda_k(D_\ell - e) \leq \ell - 1$ by (3). Thus,
$D_\ell$ is minimally strong subgraph $(k, \ell)$-arc-connected. As $|A(D_\ell)| = n\ell$, we have $g(n, k, \ell) \leq n\ell$. From the lower bound that $g(n, k, \ell) \geq n\ell$, we have $g(n, k, \ell) = n\ell$ for the case that $n \notin \{4, 6\}$.

Now we assume that $n \in \{4, 6\}$. We just consider the case that $n = 6$, since the remaining case is similar and simpler. Let $D$ be a digraph with vertex set $V(D) = \{u_i \mid 1 \leq i \leq 6\}$ such that $D$ is a union of four arc-disjoint cycles $C_i$, where $C_1 : u_1u_2u_3u_4u_5u_6u_1$, $C_2 = C_1^{\text{rev}}$, $C_3 : u_1u_3u_5u_2u_4u_6u_1$ and $C_4 = C_3^{\text{rev}}$.

Let $D_\ell$ $(1 \leq \ell \leq 4)$ be the spanning subdigraph of $D$ with arc sets $A(D_\ell) = \bigcup_{1 \leq i \leq \ell} A(C_i)$. Let $D_5 = \overrightarrow{K}_6$. Clearly, we have $\lambda_k(D_\ell) \geq \ell$ for $2 \leq k \leq 5, 1 \leq \ell \leq 5$. Furthermore, by (3), we have $\lambda_k(D_\ell) \leq \ell$ since the in-degree and out-degree of each vertex in $D_\ell$ are both $\ell$. Hence, $\lambda_k(D_\ell) = \ell$ for $2 \leq k \leq 5, 1 \leq \ell \leq 5$. For any $e \in A(D_\ell)$, we have $\delta^+(D_\ell - e) = \delta^-(D_\ell - e) = \ell - 1$, so $\lambda_k(D_\ell - e) \leq \ell - 1$ by (3). Thus, $D_\ell$ is minimally strong subgraph $(k, \ell)$-arc-connected. As $|A(D_\ell)| = n\ell$, we have $g(n, k, \ell) \leq n\ell$. Hence, $g(n, k, \ell) = n\ell$ holds for this case by the lower bound that $g(n, k, \ell) \geq n\ell$. For the case that $k = n = 6$, we have $1 \leq \ell \leq 4$, with a similar argument, we can also deduce that $g(n, k, \ell) = n\ell$.

A digraph $D$ is minimally strong if $D$ is strong but $D - e$ is not for every arc $e$ of $D$. Sun and Gutin [11] gave the following characterizations.

**Proposition 7** [11]. The following assertions hold.

(i) A digraph $D$ is minimally strong subgraph $(k, 1)$-arc-connected if and only if $D$ is minimally strong digraph.

(ii) Let $2 \leq k \leq n$. If $k \notin \{4, 6\}$, or, $k \in \{4, 6\}$ and $k < n$, then a digraph $D$ is minimally strong subgraph $(k, n - 1)$-arc-connected if and only if $D \cong \overrightarrow{K}_n$.

**Theorem 8** [11]. A digraph $D$ is minimally strong subgraph $(2, n - 2)$-arc-connected if and only if $D$ is a digraph obtained from the complete digraph $\overrightarrow{K}_n$ by deleting an arc set $M$ such that $\overrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all but at most one vertex of $\overrightarrow{K}_n$.

To prove upper bounds on the number of arcs in a minimally strong subgraph $(k, \ell)$-arc-connected digraph, we will use the following result, see e.g. Corollary 5.3.6 of [1].

**Theorem 9.** Every strong digraph $D$ on $n$ vertices has a strong spanning subgraph $H$ with at most $2n - 2$ arcs and equality holds only if $H$ is a symmetric digraph whose underlying undirected graph is a tree.

**Proposition 10.** We have (i) $G(n, n, \ell) \leq 2\ell(n - 1)$; (ii) For every $k$ $(2 \leq k \leq n)$, $G(n, k, 1) = 2(n - 1)$ and $Ex(n, k, 1)$ consists of symmetric digraphs whose underlying undirected graphs are trees; (iii) $G(n, k, n - 2) = (n - 1)^2$ for $k \in \{2, 3\}$.

**Proof.** (i) Let $D = (V, A)$ be a minimally strong subgraph $(n, \ell)$-arc-connected
digraph, and let $D_1, \ldots, D_\ell$ be arc-disjoint strong spanning subgraphs of $D$. Since $D$ is minimally strong subgraph $(n,\ell)$-arc-connected and $D_1, \ldots, D_\ell$ are pairwise arc-disjoint, $|A(\sum_{i=1}^{\ell} A(D_i))|$. Thus, by Theorem 9, $\lambda_{\ell} = 2\ell(n-1)$.

(ii) In the proof of Proposition 7, Sun and Gutin [11] showed that a digraph $D$ is strong if and only if $\lambda_{\ell}(D) \geq 1$. Now let $\lambda_{\ell}(D) \geq 1$ and a digraph $D$ has a minimal number of arcs. By Theorem 9, we have that $|A(D)| \leq 2(n-1)$, and if $D \in Ex(n, k, 1)$ then $|A(D)| = 2(n-1)$ and $D$ is a symmetric digraph whose underlying undirected graph is a tree.

Part (iii) follows directly from Theorems 4 and 8.

By Theorems 4 and 8, we can get the following result on $ex(n, k, \ell)$ and $Ex(n, k, \ell)$.

**Proposition 11.** The following assertions hold.

(i) For $k \in \{2,3\}$, $Ex(n, k, n-2) = \{\overrightarrow{K}_n - M\}$ where $M$ is an arc set such that $\overrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all but exactly one vertex of $\overrightarrow{K}_n$.

(ii) For $k \in \{2,3\}$, $ex(n, k, n-2) = \{\overrightarrow{K}_n - M\}$ where $M$ is an arc set such that $\overrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all vertices of $\overrightarrow{K}_n$.

4. **Discussion**

In this paper, we give the characterization of minimally strong subgraph $(3, n-2)$-arc-connected digraphs. We determine the precise values for $g(n, k, \ell)$ completely and the precise values for $G(n, k, n-2)$ for $k \in \{2,3\}$. So it would be interesting to determine $G(n, k, n-2)$ for every value of $k \geq 2$, as obtaining characterizations of all $(k, n-2)$-arc-connected digraphs for $2 \leq k \leq n$ seems a very difficult problem. It would also be interesting to find a sharp upper bound for $G(n, k, \ell)$ for all $k \geq 2$ and $\ell \geq 2$.

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