MINIMALLY STRONG SUBGRAPH
$(k, \ell)$-ARC-CONNECTED DIGRAPHS

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Abstract

Let $D = (V, A)$ be a digraph of order $n$, $S$ a subset of $V$ of size $k$ and $2 \leq k \leq n$. A subdigraph $H$ of $D$ is called an $S$-strong subgraph if $H$ is strong and $S \subseteq V(H)$. Two $S$-strong subgraphs $D_1$ and $D_2$ are said to be arc-disjoint if $A(D_1) \cap A(D_2) = \emptyset$. Let $\lambda_S(D)$ be the maximum number of arc-disjoint $S$-strong digraphs in $D$. The strong subgraph $k$-arc-connectivity is defined as $\lambda_k(D) = \min \{\lambda_S(D) \mid S \subseteq V, |S| = k\}$. A digraph $D = (V, A)$ is called minimally strong subgraph $(k, \ell)$-arc-connected if $\lambda_k(D) \geq \ell$ but for any arc $e \in A$, $\lambda_k(D - e) \leq \ell - 1$. Let $\mathcal{G}(n, k, \ell)$ be the set of all minimally strong subgraph $(k, \ell)$-arc-connected digraphs with order $n$. We define $G(n, k, \ell) = \max \{|A(D)| \mid D \in \mathcal{G}(n, k, \ell)\}$ and $g(n, k, \ell) = \min \{|A(D)| \mid D \in \mathcal{G}(n, k, \ell)\}$.

In this paper, we study the minimally strong subgraph $(k, \ell)$-arc-connected digraphs. We give a characterization of the minimally strong subgraph $(3, n - 2)$-arc-connected digraphs, and then give exact values and bounds for the functions $g(n, k, \ell)$ and $G(n, k, \ell)$.

Keywords: strong subgraph $k$-connectivity, strong subgraph $k$-arc-connectivity, subdigraph packing.

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1. Introduction

1.1. Motivation and concepts

The generalized $k$-connectivity $\kappa_k(G)$ of a graph $G = (V,E)$ was introduced by Hager [8] in 1985 ($2 \leq k \leq |V|$). For a graph $G = (V,E)$ and a set $S \subseteq V$ of at least two vertices, an $S$-Steiner tree or, simply, an $S$-tree is a subgraph $T$ of $G$ which is a tree with $S \subseteq V(T)$. Two $S$-trees $T_1$ and $T_2$ are said to be edge-disjoint if $E(T_1) \cap E(T_2) = \emptyset$. Two edge-disjoint $S$-trees $T_1$ and $T_2$ are said to be internally disjoint if $V(T_1) \cap V(T_2) = S$. The generalized local connectivity $\lambda_s(G)$ is the maximum number of internally disjoint $S$-trees in $G$. For an integer $k$ with $2 \leq k \leq n$, the generalized $k$-connectivity is defined as

$$\kappa_k(G) = \min\{\kappa_S(G) \mid S \subseteq V(G), |S| = k\}.$$ 

Observe that $\kappa_2(G) = \kappa(G)$. Li, Mao and Sun [10] introduced the following concept of generalized $k$-edge-connectivity. The generalized local edge-connectivity $\lambda_s(G)$ is the maximum number of edge-disjoint $S$-trees in $G$. For an integer $k$ with $2 \leq k \leq n$, the generalized $k$-edge-connectivity is defined as

$$\lambda_k(G) = \min\{\lambda_S(G) \mid S \subseteq V(G), |S| = k\}.$$ 

Observe that $\lambda_2(G) = \lambda(G)$. Generalized connectivity of graphs has become a well-established area in graph theory, see a recent monograph [9] by Li and Mao on generalized connectivity of undirected graphs.

To extend generalized $k$-connectivity to directed graphs, Sun, Gutin, Yeo and Zhang [14] observed that in the definition of $\kappa_S(G)$, one can replace “an $S$-tree” by “a connected subgraph of $G$ containing $S$.” Therefore, Sun et al. [14] defined strong subgraph $k$-connectivity by replacing “connected” with “strongly connected” (or, simply, “strong”) as follows. Let $D = (V, A)$ be a digraph of order $n$, $S$ a subset of $V$ of size $k$ and $2 \leq k \leq n$. A subdigraph $H$ of $D$ is called an $S$-strong subgraph if $H$ is strong and $S \subseteq V(H)$. Two $S$-strong subgraphs $D_1$ and $D_2$ are said to be arc-disjoint if $A(D_1) \cap A(D_2) = \emptyset$. Two arc-disjoint $S$-strong subgraphs $D_1$ and $D_2$ are said to be internally disjoint if $V(D_1) \cap V(D_2) = S$. Let $\kappa_s(D)$ be the maximum number of internally disjoint $S$-strong subgraphs in $D$. The strong subgraph $k$-connectivity is defined as

$$\kappa_k(D) = \min\{\kappa_S(D) \mid S \subseteq V, |S| = k\}.$$ 

As a natural counterpart of the strong subgraph $k$-connectivity, Sun and Gutin [11] introduced the concept of strong subgraph $k$-arc-connectivity. Let $D = (V(D), A(D))$ be a digraph of order $n$, $S \subseteq V$ a $k$-subset of $V(D)$ and $2 \leq k \leq n$. Let $\lambda_s(D)$ be the maximum number of arc-disjoint $S$-strong subgraphs in $D$. The strong subgraph $k$-arc-connectivity is defined as

$$\lambda_k(D) = \min\{\lambda_S(D) \mid S \subseteq V(D), |S| = k\}.$$
The strong subgraph \( k \)-\((arc-)\)connectivity is not only a natural extension of the concept of generalized \( k \)-(edge-)connectivity, but also relates to important problems in graph theory. For \( k = 2 \), \( \kappa_2(G) = \kappa(G) \) [14] and \( \lambda_2(G) = \lambda(G) \) [11]. Hence, \( \kappa_k(D) \) and \( \lambda_k(D) \) could be seen as generalizations of connectivity and edge-connectivity of undirected graphs, respectively. For \( k = n \), \( \kappa_n(D) = \lambda_n(D) \) is the maximum number of arc-disjoint spanning strong subgraphs of \( D \). Moreover, we know that \( \kappa_S(D) \) and \( \lambda_S(D) \) denote the number of internally-disjoint and arc-disjoint strong subgraphs containing a given set \( S \), respectively. Hence, these parameters are relevant to the subdigraph packing problem, see [2–5,7,13]. For a recent survey on the topic of strong subgraph connectivity, the readers can see [12].

A digraph \( D = (V(D), A(D)) \) is called \textit{minimally strong subgraph} \((k, \ell)\)-\textit{arc-connected} if \( \lambda_k(D) \geq \ell \) but for any arc \( e \in A(D) \), \( \lambda_k(D - e) \leq \ell - 1 \). Note that \( 2 \leq k \leq n, 1 \leq \ell \leq n - 1 \) by the definition of \( \lambda_k(D) \) and Theorem 3. Let \( \mathcal{G}(n, k, \ell) \) be the set of all minimally strong subgraph \((k, \ell)\)-arc-connected digraphs with order \( n \). We define

\[
G(n, k, \ell) = \max\{|A(D)| \mid D \in \mathcal{G}(n, k, \ell)\}
\]

and

\[
g(n, k, \ell) = \min\{|A(D)| \mid D \in \mathcal{G}(n, k, \ell)\}.
\]

We further define

\[
Ex(n, k, \ell) = \{D \mid D \in \mathcal{G}(n, k, \ell), |A(D)| = G(n, k, \ell)\}
\]

and

\[
ex(n, k, \ell) = \{D \mid D \in \mathcal{G}(n, k, \ell), |A(D)| = g(n, k, \ell)\}.
\]

In [11], Sun and Gutin first studied the minimally strong subgraph \((k, \ell)\)-arc-connected digraphs and gave some characterizations for some special cases (Proposition 7 and Theorem 8). In this paper, we continue to study the minimally strong subgraph \((k, \ell)\)-arc-connected digraphs. We first give a characterization of the minimally strong subgraph \((3, n - 2)\)-arc-connected digraphs (Theorem 4), then give exact values and bounds for the functions \( g(n, k, \ell) \) and \( G(n, k, \ell) \) (Theorem 6 and Proposition 10).

1.2. Preliminaries

We will use the following Tillson’s decomposition theorem.

\textbf{Theorem 1} [15]. The arcs of \( K_n \) can be decomposed into Hamiltonian cycles if and only if \( n \neq 4, 6 \).

The following proposition will also be used in our argument.
Proposition 2 [11]. Let $D$ be a digraph of order $n$, and let $k \geq 2$ be an integer. Then

1. $\lambda_{k+1}(D) \leq \lambda_k(D)$ for every $k \leq n - 1$,
2. $\lambda_k(D') \leq \lambda_k(D)$ where $D'$ is a spanning subgraph of $D$,
3. $\kappa_k(D) \leq \lambda_k(D) \leq \min\{\delta^+(D), \delta^-(D)\}$,

Sun and Gutin [11] obtained a sharp lower bound and a sharp upper bound of $\lambda_k(D)$ for $2 \leq k \leq n$.

Theorem 3. Let $2 \leq k \leq n$. For a strong digraph $D$ of order $n$, we have

$$1 \leq \lambda_k(D) \leq n - 1.$$ 

Moreover, both bounds are sharp, and the upper bound holds if and only if $D \cong \overrightarrow{K}_n$, where $k \not\in \{4, 6\}$, or, $k \in \{4, 6\}$ and $k < n$.

2. Characterization of the Minimally Strong Subgraph $(3, n - 2)$-Arc-Connected Digraphs

For a digraph $D$, its reverse $D^{rev}$ is a digraph with same vertex set and such that $xy \in A(D^{rev})$ if and only if $yx \in A(D)$.

Theorem 4. A digraph $D$ is minimally strong subgraph $(3, n - 2)$-arc-connected if and only if $D$ is a digraph obtained from the complete digraph $\overrightarrow{K}_n$ by deleting an arc set $M$ such that $\overrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all but at most one vertex of $\overrightarrow{K}_n$.

Proof. Let $D$ be a digraph obtained from the complete digraph $\overrightarrow{K}_n$ by deleting an arc set $M$ such that $\overrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all but at most one vertex of $\overrightarrow{K}_n$. To prove the theorem it suffices to show that (a) $D$ is minimally strong subgraph $(3, n - 2)$-arc-connected, that is, $\lambda_3(D) \geq n - 2$ but for any arc $e \in A(D)$, $\lambda_3(D - e) \leq n - 3$, and (b) if a digraph $H$ is minimally strong subgraph $(3, n - 2)$-arc-connected then it must be constructed from $\overrightarrow{K}_n$ as the digraph $D$ above. Thus, the remainder of the proof has two parts.

Part (a). We just consider the case that $\overrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all vertices of $\overrightarrow{K}_n$, since the argument for the other case is similar. For any $e \in A(\overrightarrow{K}_n) \setminus M$, observe that $e$ must be adjacent to at least one element of $M$, so $\lambda_3(D - e) \leq \min\{\delta^+(D - e), \delta^-(D - e)\} = n - 3$ by (3). Hence, it suffices to show that $\lambda_3(D) = n - 2$ in the following. So we will show that for $S = \{x, y, z\} \subseteq V(D)$, there are at least $n - 2$ arc-disjoint $S$-strong subgraphs in $D$. 
Case 1. $x, y, z$ belong to the same cycle, say $C = u_1u_2 \cdots u_tu_1$, of $\overrightarrow{K}_n[M]$.

Subcase 1.1. $S$ induces a path of length two in $C$. Without loss of generality, assume that $x = u_1, y = u_2, z = u_3$.

For the case that $t = 3$, we construct the following $n - 2$ arc-disjoint $S$-strong subgraphs: let $D_1$ be the cycle $zyxz$; for any $u \in V(D) \setminus S$, let $D_u$ be the subdigraph of $D$ with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

For the case that $t = 4$, we construct the following $n - 2$ arc-disjoint $S$-strong subgraphs: let $D_1$ be the cycle $zyxz$; let $D_2$ be the subdigraph of $D$ with vertex set $V(C)$ and arc set $\{xu, zx, yu, uy, uz\}$; for any $u \in V(D) \setminus V(C)$, let $D_u$ be the subdigraph of $D$ with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

For the case that $t \geq 5$, we construct the following $n - 2$ arc-disjoint $S$-strong subgraphs: let $D_1$ be the cycle $zyxz$; let $D_2$ be the cycle $zyxz$; let $D_3$ be the subdigraph of $D$ with vertex set $V(C)$ and arc set $\{xu_3, u_3x, u_3y, yu_t, u_tz, zu_3\}$; for any $u \in V(D) \setminus V(C)$, let $D_u$ be the subdigraph of $D$ with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

Subcase 1.2. Exactly two elements of $S$ are adjacent. Without loss of generality, assume that $x = u_1, y = u_2$. We know $t \geq 5$ in this case.

If $t = 5$, then $z = u_4$. We construct the following $n - 2$ arc-disjoint $S$-strong subgraphs: let $D_1$ be the cycle $zyxz$; let $D_2$ be the cycle $zyxz$; let $D_3$ be the subdigraph of $D$ with vertex set $S \cup \{u_4, u_t\}$ and arc set $\{xu_3, u_3x, u_3y, yu_t, u_tz, zu_3\}$; for any $u \in V(D) \setminus V(C)$, let $D_u$ be the subdigraph of $D$ with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

We now consider the case that $t \geq 6$. If $z = u_4$, then we construct the following $n - 2$ arc-disjoint $S$-strong subgraphs: let $D_1$ be the cycle $zyxz$; let $D_2$ be the cycle $zyxz$; let $D_3$ be the subdigraph of $D$ with vertex set $\{x, y, u_3, z, u_t\}$ and arc set $\{xu_3, u_3x, u_3y, yu_t, u_tz, zu_3\}$; let $D_4$ be the subdigraph of $D$ with vertex set $\{x, y, z, u_3, u_t\}$ and arc set $\{xu_3, u_3x, u_3y, yu_3, u_3z, zu_3\}$; for any $u \in V(D) \setminus \{x, y, u_3, z, u_t\}$, let $D_u$ be the subdigraph of $D$ with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

If $z = u_{t-1}$, then we construct the following $n - 2$ arc-disjoint $S$-strong subgraphs: let $D_1$ be the cycle $zyxz$; let $D_2$ be the cycle $zyxz$; let $D_3$ be the subdigraph of $D$ with vertex set $\{x, y, u_3, u_{t-2}, z, u_t\}$ and arc set $\{xu_3, u_3x, u_3y, yu_t, u_tz, zu_3\}$; let $D_4$ be the subdigraph of $D$ with vertex set $\{x, y, u_3, u_{t-2}, z, u_t\}$ and arc set $\{xu_3, u_3x, u_3y, yu_t, u_tz, zu_3\}$; for any $u \in V(D) \setminus \{x, y, u_3, u_{t-2}, z, u_t\}$, let $D_u$ be the subdigraph of $D$ with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

If $z \notin \{u_4, u_{t-1}\}$, say $z = u_5$, then we construct the following $n - 2$ arc-disjoint $S$-strong subgraphs: let $D_1$ be the cycle $zyxz$; let $D_2$ be the cycle $zyxz$; let $D_3$ be the subdigraph of $D$ with vertex set $\{x, y, u_3, u_4, z, u_{t-1}\}$ and arc set $\{xu_3, u_3x, u_3y, yu_{t-1}, u_{t-1}z, zu_4, u_4u_3\}$; let $D_4$ be the subdigraph of $D$ with ver-
tex set \( \{x, y, z, u_t-1, u_t\} \) and arc set \( \{xu_{t-1}, u_{t-1}x, u_tz, zu_t, yu_t, u_tu_{t-1}, u_{t-1}y\} \); let \( D_5 \) be the subdigraph of \( D \) with vertex set \( \{x, y, u_3, z, u_{t-1}\} \) and arc set \( \{xu_4, u_4x, u_4y, yu_4, u_4u_{t-1}, u_{t-1}u_4, u_{t-1}u_4, u_{t-1}u_3, u_3u_{t-1}, u_3z, zu_t\} \); for any \( u \in V(D) \setminus \{x, y, u_3, u_{t-1}, u_1\} \), let \( D_u \) be the subdigraph of \( D \) with vertex set \( S \cup \{u\} \) and arc set \( \{xu, ux, yu, uy, zu, uz\} \).

Subcase 1.3. Any two elements of \( S \) are nonadjacent. Without loss of generality, assume that \( x = u_1 \). We know \( t \geq 6 \) in this case.

If \( t = 6 \), then we can assume that \( y = u_3, z = u_5 \). We construct the following \( n - 2 \) arc-disjoint \( S \)-strong subgraphs: let \( D_1 \) be the cycle \( yzyzz \); let \( D_2 = D_1^{rev} \); let \( D_3 \) be the subdigraph of \( D \) with vertex set \( S \cup \{u_2, u_t\} \) and arc set \( \{xu_6, u_6y, yu_2, u_2x, u_2z, zu_2\} \); let \( D_4 \) be the subdigraph of \( D \) with vertex set \( S \cup \{u_4, u_t\} \) and arc set \( \{zu_4, u_4y, yu_6, u_6z, u_4x, xu_4\} \); for any \( u \in V(D) \setminus (S \cup \{u_2, u_4, u_t, u_1\}) \), let \( D_u \) be the subdigraph of \( D \) with vertex set \( S \cup \{u\} \) and arc set \( \{xu, ux, yu, uy, zu, uz\} \).

In the following we assume that \( t \geq 7 \). We consider the case that exactly one pair of elements, say \( x \) and \( z \), of \( S \) does not have a common neighbor in the cycle \( C \). Without loss of generality, assume that \( y = u_3, z = u_5 \) (observe that \( x \) and \( y \) have a common neighbor \( u_2 \), \( y \) and \( z \) have a common neighbor \( u_4 \), but \( z \) and \( x \) do not have a common neighbor in the cycle \( C \)). We construct the following \( n - 2 \) arc-disjoint \( S \)-strong subgraphs: let \( D_1 \) be the cycle \( yzyzz \); let \( D_2 = D_1^{rev} \); let \( D_3 \) be the subdigraph of \( D \) with vertex set \( S \cup \{u_2, u_t\} \) and arc set \( \{xu_6, u_6y, yu_2, u_2x, u_2z, zu_2\} \); let \( D_4 \) be the subdigraph of \( D \) with vertex set \( S \cup \{u_4, u_t\} \) and arc set \( \{zu_4, u_4y, yu_6, u_6z, u_4x, xu_4\} \); let \( D_5 \) be the subdigraph of \( D \) with vertex set \( S \cup \{u_6, u_t\} \) and arc set \( \{xu_6, u_6x, u_6y, yu_6, zzu_t, uu_tu_6, uu_6z\} \); for any \( u \in V(D) \setminus (S \cup \{u_2, u_4, u_6, u_t, u_1\}) \), let \( D_u \) be the subdigraph of \( D \) with vertex set \( S \cup \{u\} \) and arc set \( \{xu, ux, yu, uy, zu, uz\} \).

We now consider the case that exactly one pair of elements, say \( x \) and \( y \), of \( S \) has a common neighbor in the cycle \( C \). Without loss of generality, assume that \( y = u_3, z = u_6 \) (we know \( x \) and \( y \) have a common neighbor \( u_2 \), \( y \) and \( z \) do not have a common neighbor, \( z \) and \( x \) do not have a common neighbor in the cycle \( C \)). We construct the following \( n - 2 \) arc-disjoint \( S \)-strong subgraphs: let \( D_1 \) be the cycle \( zyxxz \); let \( D_2 = D_1^{rev} \); let \( D_3 \) be a subdigraph of \( D \) with vertex set \( S \cup \{u_2, u_t\} \) and arc set \( \{xu_6, u_6y, yu_2, u_2x, u_2z, zu_2\} \); let \( D_4 \) be a subdigraph of \( D \) with vertex set \( S \cup \{u_4, u_t\} \) and arc set \( \{zu_4, u_4y, yu_6, u_6z, u_4x, xu_4\} \); let \( D_5 \) be a subdigraph of \( D \) with vertex set \( S \cup \{u_6, u_t\} \) and arc set \( \{zu_6, u_6x, u_6y, yu_6, zzu_t, uu_tu_6, uu_6z\} \); let \( D_6 \) be a subdigraph of \( D \) with vertex set \( S \cup \{u_7, u_t\} \) and arc set \( \{zu_7, yu_7, u_7u_4, uu_4, u_4z, zu_4\} \); for any \( u \in V(D) \setminus (S \cup \{u_2, u_4, u_6, u_7, u_t\}) \), let \( D_u \) be the subdigraph of \( D \) with vertex set \( S \cup \{u\} \) and arc set \( \{xu, ux, yu, uy, zu, uz\} \).

We consider the remaining case that any pair of elements of \( S \) does not have a common neighbor in the cycle \( C \). Without loss of generality, assume that \( y = u_4, z = u_7 \) (we know \( x \) and \( y \) do not a common neighbor \( u_2 \), \( y \) and \( z \) do not have a common neighbor, \( z \) and \( x \) do not have a common neighbor in the cycle \( C \)). We
construct the following $n - 2$ arc-disjoint $S$-strong subgraphs: let $D_1$ be the cycle $zxyzx$; let $D_2 = D_1^{\text{ev}}$; let $D_3$ be a subdigraph of $D$ with vertex set $S \cup \{u_2, u_1\}$ and arc set $\{xu_1, u_1u_2, u_2x, u_2y, yu_2, u_2z, zu_2\}$; let $D_4$ be a subdigraph of $D$ with vertex set $S \cup \{u_3, u_1\}$ and arc set $\{u_3u_1, u_1y, yu_3, u_3x, xu_3, uz_3\}$; let $D_5$ be a subdigraph of $D$ with vertex set $S \cup \{u_3, u_2\}$ and arc set $\{u_3y, yu_1, u_1u_5, u_5xz, xu_5, uz_5\}$; let $D_6$ be a subdigraph of $D$ with vertex set $S \cup \{u_6, u_1\}$ and arc set $\{u_6u_1, u_1z, zu_6, xu_6, u_6x, yu_6, u_6y\}$; let $D_7$ be a subdigraph of $D$ with vertex set $S \cup \{u_8, u_1\}$ and arc set $\{u_8z, zu_1, u_1u_8, xu_8, u_8x, yu_8, uz_8\}$; for any $u \in V(D) \setminus (S \cup \{u_2, u_3, u_5, u_6, u_8, u_1\})$, let $D_u$ be the subdigraph of $D$ with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

Case 2. Exactly two elements of $S$ belong to the same cycle, say $C_1 = u_1u_2 \cdots u_{11}$, of $\overrightarrow{K}_n[M]$, and the remaining element belongs to the other cycle $C_2 = v_1v_2 \cdots v_{11}$. Without loss of generality, assume that $x, y \in V(C_1), z = v_1$.

Subcase 2.1. $x$ and $y$ are adjacent. Without loss of generality, assume that $x = u_1, y = u_2$. We just consider the case that $t \geq 4$ and $h \geq 3$, since the arguments for the other cases are similar and simpler. We construct the following $n - 2$ arc-disjoint $S$-strong subgraphs: let $D_1$ be the cycle $zxyzx$; let $D_2 = D_1^{\text{ev}}$; let $D_3$ be a subdigraph of $D$ with vertex set $S \cup \{u_1\}$ and arc set $\{xu_1, u_1y, yx, xz, xu_2\}$; let $D_4$ be a subdigraph of $D$ with vertex set $S \cup \{u_3, u_1\}$ and arc set $\{yu_1, u_1u_3, u_3y, xu_3, uz_3\}$; let $D_5$ be a subdigraph of $D$ with vertex set $S \cup \{v_2, v_1\}$ and arc set $\{zv_1, v_1v_2, v_2z, v_2x, xv_2, yv_2, yv_1\}$; for any $u \in V(D) \setminus (S \cup \{u_3, u_1, v_2, v_1\})$, let $D_u$ be the subdigraph of $D$ with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, xu, uy, zu, uz\}$.

Subcase 2.2. $x$ and $y$ are nonadjacent. Without loss of generality, assume that $x = u_1$. We first consider the case that $t = 4$, and observe that $y = u_3$ now. Furthermore, assume that $h \geq 3$ since the argument for the remaining case is similar and simpler. We construct the following $n - 2$ arc-disjoint $S$-strong subgraphs: let $D_1$ be the cycle $zxyzx$; let $D_2 = D_1^{\text{ev}}$; let $D_3$ be a subdigraph of $D$ with vertex set $S \cup \{u_2, u_1\}$ and arc set $\{xu_1, u_1y, yu_2, u_2x, u_2z, zu_2\}$; let $D_4$ be a subdigraph of $D$ with vertex set $S \cup \{v_2, u_1\}$ and arc set $\{xv_2, v_2x, yv_2, v_2y, u_1z, zu_2, v_2u_1\}$; let $D_5$ be a subdigraph of $D$ with vertex set $S \cup \{v_2, v_1\}$ and arc set $\{xv_1, v_1x, yv_1, yv_2, v_2z, zu_1\}$; for any $u \in V(D) \setminus (S \cup \{u_2, u_1, v_2, v_1\})$, let $D_u$ be the subdigraph of $D$ with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, xu, uy, zu, uz\}$.

Now we assume that $t \geq 5$. We first consider the case that $x$ and $y$ have exactly one common neighbor in the cycle $C_1$. With a similar argument to that of the case that $t \geq 7$ and exactly one pair of elements, say $x$ and $y$, of $S$ has a common neighbor in the cycle $C$ in Subcase 1.3, we can construct $n - 2$ arc-disjoint $S$-strong subgraphs.

We next consider the case that $x$ and $y$ do not have a common neighbor in
the cycle $C_1$. If $h \geq 3$, then with a similar argument to that of the case that $t \geq 7$ and any pair of elements of $S$ does not have a common neighbor in the cycle $C$ in Subcase 1.3, we can construct $n - 2$ arc-disjoint strong subgraphs containing $S$. Otherwise, we have $h = 2$. Without loss of generality, assume that $y = u_4$. We construct the following $n - 2$ arc-disjoint $S$-strong subgraphs: let $D_1$ be the cycle $x_1y_1z_1x_2$; let $D_2 = D_1^{rev}$; let $D_3$ be a subdigraph of $D$ with vertex set $S \cup \{u_2, u_4\}$ and arc set $\{xu_4, u_2u_2, u_2y_2, u_2u_2, u_2z_2, z_2u_2\}$; let $D_4$ be a subdigraph of $D$ with vertex set $S \cup \{u_3, u_4\}$ and arc set $\{u_3u_4, u_3u_3, u_3x_3, xu_3, u_3z_3, zu_3\}$; let $D_5$ be a subdigraph of $D$ with vertex set $S \cup \{u_5, u_4\}$ and arc set $\{u_5y_2, u_5u_5, u_5x_5, xu_5, u_5z_5, zu_5\}$; let $D_6$ be a subdigraph of $D$ with vertex set $S \cup \{u_6, v_6\}$ and arc set $\{xv_6, v_6x, v_6u_6, u_6v_6, v_6y_6, y_6v_6, u_6v_6, v_6u_6\}$; for any $u \in V(D) \setminus (S \cup \{u_3, u_5, u_6, v_6\})$, let $D_u$ be the subdigraph of $D$ with vertex set $S \cup \{u\}$ and arc set $\{xu, xu, yu, uy, zu, uz\}

Case 3. The elements of $S$ belong to distinct cycles, say $x \in V(C_1)$, $y \in V(C_2)$, $z \in V(C_3)$, of $\overrightarrow{K}_n[M]$.

Subcase 3.1. $|V(C_i)| \geq 3$ for all $1 \leq i \leq 3$. With a similar argument to the case that $t \geq 7$ and exactly one pair of elements, say $x$ and $y$, of $S$ has a common neighbor in the cycle $C$ in Subcase 1.3, we can construct $n - 2$ arc-disjoint $S$-strong subgraphs.

Subcase 3.2. $|V(C_{i_0})| = 2$ for some $1 \leq i_0 \leq 3$. With a similar argument to the case that $x, y$ do not have a common neighbor in the cycle $C_1$ and $h = 2$ in last paragraph of Subcase 2.2, we can construct $n - 2$ arc-disjoint $S$-strong subgraphs.

Subcase 3.3. $|V(C_{i_0})| = |V(C_{j_0})| = 2$ for some $1 \leq i_0, j_0 \leq 3$. Without loss of generality, we assume that $i_0 = 2, j_0 = 3$ and furthermore, $u_1x, xu_2 \in E(C_1)$, $u_3y, yu_3 \in E(C_2)$, $u_4z, zu_4 \in E(C_3)$. We construct the following $n - 2$ arc-disjoint $S$-strong subgraphs: let $D_1$ be the cycle $x_1y_1z_1x_2$; let $D_2 = D_1^{rev}$; let $D_3$ be a subdigraph of $D$ with vertex set $S \cup \{u_1, u_2\}$ and arc set $\{u_1u_2, u_2x, xu_1, u_2y, yu_2, u_2z, z_2u_2\}$; let $D_4$ be the cycle $xu_4y_1zu_3x$; let $D_5 = D_4^{rev}$; for any $u \in V(D) \setminus (S \cup \{u, u_3, u_4\})$, let $D_u$ be a subdigraph of $D$ with vertex set $S \cup \{u\}$ and arc set $\{xu, xu, yu, uy, zu, uz\}$.

Subcase 3.4. $|V(C_i)| = 2$ for all $1 \leq i \leq 3$. This case is easy and we omit the details.

Part (b). Let $H$ be minimally strong subgraph $(3, n - 2)$-arc-connected. By Theorem 3, we have that $H \not\cong \overrightarrow{K}_n$, that is, $H$ can be obtained from a complete digraph $\overrightarrow{K}_n$ by deleting a nonempty arc set $M$. To end our argument, we need the following claim. Let us start from a simple yet useful observation, which follows from (3).
Proposition 5. No pair of arcs in $M$ has a common head or tail.

Thus, $\overrightarrow{K}_n[M]$ must be a union of vertex-disjoint cycles or paths, otherwise, there are two arcs of $M$ such that they have a common head or tail, a contradiction with Proposition 5.

Claim 1. $\overrightarrow{K}_n[M]$ does not contain a path of order at least two.

Proof. Suppose that $\overrightarrow{K}_n[M]$ contains a path of order at least two. Let $M' \supseteq M$ be a set of arcs obtained from $M$ by adding some arcs from $\overrightarrow{K}_n - M$ such that the digraph $\overrightarrow{K}_n[M']$ contains no path of order at least two. For example, if $\overrightarrow{K}_n[M]$ contains a path $u_1, \ldots, u_\ell$ with $\ell \geq 2$, then add the arc $u_\ell u_1$ to $M'$. Note that $\overrightarrow{K}_n[M']$ is a supergraph of $\overrightarrow{K}_n[M]$ and is a union of vertex-disjoint cycles which cover all but at most one vertex of $\overrightarrow{K}_n$. By Part (a), we have that $\lambda_k(\overrightarrow{K}_n - M') = n - 2$, so $H$ is not minimally strong subgraph $(3,n - 2)$-arc-connected, a contradiction.

It follows from Claim 1 and its proof that $\overrightarrow{K}_n[M]$ must be a union of vertex-disjoint cycles which cover all but at most one vertex of $\overrightarrow{K}_n$, which completes the proof of Part (b).

3. Results for $g(n,k,\ell)$, $G(n,k,\ell)$, $ex(n,k,\ell)$ and $Ex(n,k,\ell)$

The following result concerns the precise value for $g(n,k,\ell)$.

Theorem 6. For any triple $(n,k,\ell)$ with $2 \leq k \leq n, 1 \leq \ell \leq n - 1$ such that $(n,k,\ell) \notin \{(4,4,3),(6,6,5)\}$, we have

$$g(n,k,\ell) = n\ell.$$ 

Proof. By Theorem 3 and the definition of $g(n,k,\ell)$, we have $(n,k,\ell) \notin \{(4,4,3),(6,6,5)\}$.

For all digraphs $D$ and $k \geq 2$, we have $\lambda_k(D) \leq \delta^+(D)$ and $\lambda_k(D) \leq \delta^-(D)$ by (3). Hence for each $D$ with $\lambda_k(D) = \ell$, we have that $\delta^+(D), \delta^-(D) \geq \ell$, so $|D| \geq n\ell$ and then $g(n,k,\ell) \geq n\ell$.

We first consider the case that $n \notin \{4,6\}$. Let $D \cong \overrightarrow{K}_n$. By Theorem 1, $D$ can be decomposed into $n - 1$ Hamiltonian cycles $H_i$ ($1 \leq i \leq n - 1$). Let $D_\ell$ be the spanning subdigraph of $D$ with arc sets $A(D_\ell) = \bigcup_{1 \leq i \leq \ell} A(H_i)$. Clearly, we have $\lambda_k(D_\ell) \geq \ell$ for $2 \leq k \leq n, 1 \leq \ell \leq n - 1$. Furthermore, by (3), we have $\lambda_k(D_\ell) \leq \ell$ since the in-degree and out-degree of each vertex in $D_\ell$ are both $\ell$. Hence, $\lambda_k(D_\ell) = \ell$ for $2 \leq k \leq n, 1 \leq \ell \leq n - 1$. For any $e \in A(D_\ell)$, we have $\delta^+(D_\ell - e) = \delta^-(D_\ell - e) = \ell - 1$, so $\lambda_k(D_\ell - e) \leq \ell - 1$ by (3). Thus,
$D_\ell$ is minimally strong subgraph $(k, \ell)$-arc-connected. As $|A(D_\ell)| = n\ell$, we have $g(n, k, \ell) \leq n\ell$. From the lower bound that $g(n, k, \ell) \geq n\ell$, we have $g(n, k, \ell) = n\ell$ for the case that $n \notin \{4, 6\}$.

Now we assume that $n \in \{4, 6\}$. We just consider the case that $n = 6$, since the remaining case is similar and simpler. Let $D$ be a digraph with vertex set $V(D) = \{u_i \mid 1 \leq i \leq 6\}$ such that $D$ is a union of four arc-disjoint cycles $C_i$, where $C_1 : u_1u_2u_3u_4u_5u_6u_1$, $C_2 = C_1^{\text{rev}}$, $C_3 : u_1u_3u_5u_2u_4u_6u_1$ and $C_4 = C_3^{\text{rev}}$.

Let $D_\ell = \bigcup_{1 \leq i \leq \ell} A(C_i)$. Let $D_\ell = \overrightarrow{K}_6$. Clearly, we have $\lambda_k(D_\ell) \geq \ell$ for $2 \leq k \leq 5, 1 \leq \ell \leq 5$. Furthermore, by (3), we have $\lambda_k(D_\ell) \leq \ell$ since the in-degree and out-degree of each vertex in $D_\ell$ are both $\ell$. Hence, $\lambda_k(D_\ell) = \ell$ for $2 \leq k \leq 5, 1 \leq \ell \leq 5$. For any $e \in A(D_\ell)$, we have $\delta^+(D_\ell-e) = \delta^-(D_\ell-e) = \ell-1$, so $\lambda_k(D_\ell-e) \leq \ell-1$ by (3). Thus, $D_\ell$ is minimally strong subgraph $(k, \ell)$-arc-connected. As $|A(D_\ell)| = n\ell$, we have $g(n, k, \ell) \leq n\ell$. Hence, $g(n, k, \ell) = n\ell$ holds for this case by the lower bound that $g(n, k, \ell) \geq n\ell$. For the case that $k = n = 6$, we have $1 \leq \ell \leq 4$, with a similar argument, we can also deduce that $g(n, k, \ell) = n\ell$.

A digraph $D$ is minimally strong if $D$ is strong but $D-e$ is not for every arc $e$ of $D$. Sun and Gutin [11] gave the following characterizations.

**Proposition 7** [11]. The following assertions hold.

(i) A digraph $D$ is minimally strong subgraph $(k, 1)$-arc-connected if and only if $D$ is minimally strong digraph.

(ii) Let $2 \leq k \leq n$. If $k \notin \{4, 6\}$, or, $k \in \{4, 6\}$ and $k < n$, then a digraph $D$ is minimally strong subgraph $(k, n-1)$-arc-connected if and only if $D \cong \overrightarrow{K}_n$.

**Theorem 8** [11]. A digraph $D$ is minimally strong subgraph $(2, n-2)$-arc-connected if and only if $D$ is a digraph obtained from the complete digraph $\overrightarrow{K}_n$ by deleting an arc set $M$ such that $\overrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all but at most one vertex of $\overrightarrow{K}_n$.

To prove upper bounds on the number of arcs in a minimally strong subgraph $(k, \ell)$-arc-connected digraph, we will use the following result, see e.g. Corollary 5.3.6 of [1].

**Theorem 9.** Every strong digraph $D$ on $n$ vertices has a strong spanning subgraph $H$ with at most $2n - 2$ arcs and equality holds only if $H$ is a symmetric digraph whose underlying undirected graph is a tree.

**Proposition 10.** We have (i) $G(n, n, \ell) \leq 2\ell(n-1)$; (ii) For every $k$ ($2 \leq k \leq n$), $G(n, k, 1) = 2(n-1)$ and $Ex(n, k, 1)$ consists of symmetric digraphs whose underlying undirected graphs are trees; (iii) $G(n, k, n-2) = (n-1)^2$ for $k \in \{2, 3\}$.

**Proof.** (i) Let $D = (V, A)$ be a minimally strong subgraph $(n, \ell)$-arc-connected
digraph, and let $D_1, \ldots, D_\ell$ be arc-disjoint strong spanning subgraphs of $D$. Since $D$ is minimally strong subgraph $(n, \ell)$-arc-connected and $D_1, \ldots, D_\ell$ are pairwise arc-disjoint, $|A| = \sum_{i=1}^\ell |A(D_i)|$. Thus, by Theorem 9, $|A| \leq 2\ell(n - 1)$.

(ii) In the proof of Proposition 7, Sun and Gutin [11] showed that a digraph $D$ is strong if and only if $\lambda_k(D) \geq 1$. Now let $\lambda_k(D) \geq 1$ and a digraph $D$ has a minimal number of arcs. By Theorem 9, we have that $|A(D)| \leq 2(n - 1)$, and if $D \in \text{Ex}(n, k, 1)$ then $|A(D)| = 2(n - 1)$ and $D$ is a symmetric digraph whose underlying undirected graph is a tree.

Part (iii) follows directly from Theorems 4 and 8.

By Theorems 4 and 8, we can get the following result on $ex(n, k, \ell)$ and $Ex(n, k, \ell)$.

**Proposition 11.** The following assertions hold.

(i) For $k \in \{2, 3\}$, $\text{Ex}(n, k, n - 2) = \{\overrightarrow{K_n} - M\}$ where $M$ is an arc set such that $\overrightarrow{K_n}[M]$ is a union of vertex-disjoint cycles which cover all but exactly one vertex of $\overrightarrow{K_n}$.

(ii) For $k \in \{2, 3\}$, $\text{ex}(n, k, n - 2) = \{\overrightarrow{K_n} - M\}$ where $M$ is an arc set such that $\overrightarrow{K_n}[M]$ is a union of vertex-disjoint cycles which cover all vertices of $\overrightarrow{K_n}$.

4. Discussion

In this paper, we give the characterization of minimally strong subgraph $(3, n - 2)$-arc-connected digraphs. We determine the precise values for $g(n, k, \ell)$ completely and the precise values for $G(n, k, n - 2)$ for $k \in \{2, 3\}$. So it would be interesting to determine $G(n, k, n - 2)$ for every value of $k \geq 2$, as obtaining characterizations of all $(k, n - 2)$-arc-connected digraphs for $2 \leq k \leq n$ seems a very difficult problem. It would also be interesting to find a sharp upper bound for $G(n, k, \ell)$ for all $k \geq 2$ and $\ell \geq 2$.

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