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TOTAL COLORING OF CLAW-FREE PLANAR GRAPHS

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Abstract

A total coloring of a graph is an assignment of colors to both its vertices and edges so that adjacent or incident elements acquire distinct colors. Let $\Delta(G)$ be the maximum degree of G. Vizing conjectured that every graph has a total $(\Delta + 2)$-coloring. This Total Coloring Conjecture remains open even for planar graphs, for which the only open case is $\Delta = 6$. Claw-free planar graphs have $\Delta \leq 6$. In this paper, we prove that the Total Coloring Conjecture holds for claw-free planar graphs.

Keywords: total coloring, total coloring conjecture, planar graph, claw.

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1. Introduction

All graphs considered here are finite, simple and nonempty. Let $G = (V, E)$ be a graph with *vertex set* V and *edge set* E. The number of vertices of G is called the *order* of G. For a vertex $v \in V$, the *open neighborhood* $N(v)$ of v is defined as the set of vertices adjacent to v, i.e., $N(v) = \{u \mid uv \in E\}$. The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. Every vertex in $N(v)$ is also called a *neighbor* of v. The *degree* of v is equal to $|N(v)|$, denoted by $d_G(v)$ or simply $d(v)$. By $\delta(G)$ and $\Delta(G)$, we denote the *minimum degree* and the *maximum degree* of the graph G, respectively. For a subset $S \subseteq V$, the *closed neighborhood* of S is $N[S] = \bigcup_{v \in S} N[v]$ and the *closed* 2*-neighborhood* of S is $N_2[S] = N[N[S]]$. For a subset $X \subseteq V$, the subgraph induced by X is denoted by $G[X]$. The set of edges between X and Y in E is denoted by $E(X, Y)$ for $X, Y \subseteq V$. As usual, $K_{m,n}$ denotes a complete bipartite graph with classes of cardinality m and n, and K_n and C_n denote the complete graph and cycle of order n. The graph $K_{1,3}$ is also called a *claw*, and K_3 a *triangle*. Given a graph F, a graph G is *F-free* if it does not contain F as an induced subgraph. In particular, a $K_{1,3}$-free graph is *claw-free*. For a family $\{F_1, \ldots, F_k\}$ of graphs, we say that G is $(F_1, \ldots, F_k)$-free if it is F_i-free for all i. By starting with a disjoint union of two graphs G and H and adding edges joining every vertex of G to every vertex of H, one obtains the *join* of G and H, denoted by $G \vee H$. The *n-wheel* is the graph $C_n \vee K_1$ and the *double n-wheel* is the graph $C_n \vee \overline{K}_2$ $(n \geq 4)$.

Given a graph G, an *element* of G is a member of $V(G) \cup E(G)$. Let two elements of a graph G be *adjacent* if they are either adjacent or incident in the traditional sense. Given a graph G, a *total k-coloring* of G is a function that takes each element to $\{1, 2, \ldots, k\}$ such that adjacent distinct elements receive distinct colors. In 1968, Vizing [17] (see also [2]) made the following conjecture, known as the Total Coloring Conjecture.

Conjecture. *Every graph has a total* $(\Delta + 2)$*-coloring.*

This conjecture is trivial for $\triangle \leq 2$. Rosenfeld [13] and Vijayaditya [16] proved it for $\Delta = 3$. Kostochka proved the $\Delta = 4$ [9] and $\Delta = 5$ [11] cases. The conjecture remains open even for planar graphs, but more is known. Borodin [4] proved it for planar graphs with $\Delta \geq 9$. The $\Delta = 8$ case was proved for planar graphs by Yap [21] and Andersen [1]. The $\Delta = 7$ case was proved for planar graphs by Sanders and Zhao [14]. Thus the only open case for planar graphs is the $\Delta = 6$ case. An extensive study on total coloring is done in [3, 5–8, 10, 18–20] and elsewhere. Claw-free planar graphs have maximum degree at most 6 [12]. In this paper, we prove that the Total Coloring Conjecture holds for claw-free planar graphs.

2. Total Coloring on the Claw-Free Planar Graphs

First we introduce some notation and lemmas which are useful for the total coloring of claw-free planar graphs. Let a *k-vertex* be a vertex of degree k. Let an *at most k-vertex*, or simply a k^--vertex, be a vertex of degree at most k. Given a graph and integers $j_1, j_2, \ldots, j_i$, let a $(j_1^-, j_2^-, \ldots, j_i^-)$-vertex be an i-vertex v of G such that, for each $1 \leq m \leq i$, there is a j_m^--vertex y_m of G and the vertices $y_1, y_2, \ldots, y_i$ are distinct neighbors of v.

Lemma 1 [15]**.** *If G is a* (*claw,* K_4)*-free planar graph, then* $\Delta(G) \leq 5$ *and for every vertex v of degree* 5 *in* G, $G[N[v]]$ *is a* 5*-wheel.*

Lemma 2. *Let v be a* $(6^-, 6^-, 5^-)$*-vertex, or a* $(6, 6, 6)$*-vertex such that* $G[N[v]]$ *is not a claw, in a graph G. If* $G - v$ *has a total* 8*-coloring, then G is total* 8*-colorable.*

Proof. Let v be a $(6^-, 6^-, 6^-)$-vertex of a graph G. Given a total 8-coloring ϕ of $G - v$, we will attempt to extend ϕ to a total 8-coloring of G. Let $N(v) = \{y_1, y_2, y_3\}$, and for $i \in \{1, 2, 3\}$ let L_{vy_i} be the set of colors that are not used on y_i or its incident edges and so are available for use on the edge vy_i. Then $|L_{vy_i}| \geq 2$ for each i, since $d_{G-v}(y_i) \leq 5$. Clearly, these edges can be properly colored unless L_{vy_i} is the same set of two colors, say $L_{vy_i} = \{1, 2\}$, for each i. This is impossible if v is a $(6^-, 6^-, 5^-)$-vertex, so assume that v is a $(6, 6, 6)$-vertex such that $G[N[v]]$ is not a claw. Without loss of generality, y_2y_3 is an edge of G and $\phi(y_2y_3) = 3$. Recolor y_2y_3 with color 1, and then color vy_1, vy_2 and vy_3 with colors 1, 2 and 3, respectively. Once the edges incident with v are colored, there are at most six colors that cannot be used on v, and so v can be colored. This gives a total 8-coloring of G, as required. ■

Lemma 3. *Let v be a $(7^-, 6^-, 5^-, 4^-)$-vertex in a graph G such that $G[N[v]]$ contains neither a claw nor a K_4. If $G - v$ has a total 8-coloring, then G is total 8-colorable.*

Proof. Let $N(v) = \{y_1, y_2, y_3, y_4\}$ where $d(y_i) \leq 8 - i$ for each i. Given a total coloring ϕ of $G - v$ using a set C of eight colors, let F be the set of colors that are used on the vertices in $N(v)$, and for $i \in \{1, 2, 3, 4\}$ let L_{vy_i} be the set of colors that are available for use on the edge vy_i, as in the previous proof. Then $|L_{vy_i}| \geq i$ for each i, and so the edges vy_1, vy_2, vy_3, vy_4 can be properly colored in this order. If possible, do this so that at least one of these edges has a color in F. Call the new coloring ϕ'.

It is now possible to color v unless every color is used on an element adjacent to v. For this to happen, it must be that $|F| = 4$ and all the lists L_{vy_i} are subsets of $C \setminus F$. In particular, $L_{vy_4} = C \setminus F$. Since $G[N[v]]$ contains neither a claw nor a K_4, G must contain an edge y_ry_4 for some $r \in \{1, 2, 3\}$. Clearly $\phi(y_ry_4) \notin L_{vy_4}$, and so $\phi(y_ry_4) \in F$, while $\phi'(vy_r) \in L_{vy_r} \subseteq L_{vy_4}$. Interchange the colors of vy_r and y_ry_4. Since vy_r now has a color in F, there are at most seven different colors that are unavailable for v, and so v can now be colored. This gives a total 8-coloring of G, as required. ■

Theorem 4. *Every claw-free planar graph is total 8-colorable.*

Proof. Let G be a claw-free planar graph. We prove the theorem by induction on the size $m = |E(G)|$. Suppose that the theorem holds when G has fewer than m edges. In the following, we will prove the theorem when G has m edges.

If G has a cut vertex v, we can easily see that the theorem holds. So we may assume that G is 2-connected. In addition, if G has no K_4, then $\Delta(G) \leq 5$ by Lemma 1, and so the theorem holds since the Total Coloring Conjecture holds when $\Delta(G) \leq 5$ [11]. So let $K = [x_1x_2x_3x_4]$ be a K_4 of G, where x_1 is inside the cycle $C = [x_2x_3x_4]$ in the embedding of G in the plane. Let G' be the plane graph induced by the vertices inside and on C, and choose K so that G' has as fewer vertices as possible. Then K is the only K_4 in G'.

Since every claw-free planar graph has maximum degree at most 6, the result follows from Lemma 2 if G contains a 3^--vertex. Thus we may assume that $\delta(G) \geq 4$. Without loss of generality, we may assume that x_1 has a neighbor u inside the triangle $T = [x_1x_2x_3]$ in the embedding of G in the plane. Let $V_{\text{in}}(T)$ denote the set of vertices inside the triangle T, and let $G_T = G[V_{\text{in}}(T) \cup V(T)]$, the plane graph induced by the vertices inside and on T. We will make frequent use of the following facts.

(F1) G_T is K_4-free. This is because K is the only K_4 in G'.

(F2) Every vertex of $V_{\text{in}}(T)$ is adjacent to at most two vertices of T, by (F1).

(F3) Every vertex of $V_{\text{in}}(T)$ has degree 4 or 5, since $\delta(G) \geq 4$ and $\Delta(G_T) \leq 5$ by Lemma 1.

(F4) We may assume that no two 4-vertices in $V_{\text{in}}(T)$ are adjacent. Indeed, if they are, then each of them is a $(6^-, 6^-, 5^-, 4^-)$-vertex by (F2) and (F3), and so the result follows by Lemma 3.

(F5) For $i = 1, 2, 3$, the neighbors of x_i in $V_{\text{in}}(T)$ induce a complete graph. Otherwise, there would be a claw centered on x_i (including the edge x_ix_4).

(F6) Every vertex of T is adjacent to 0, 1 or 2 adjacent vertices in $V_{\text{in}}(T)$. This follows from (F1) and (F5).

(F7) Every 5-vertex in $V_{\text{in}}(T)$ is adjacent to 0 or 2 vertices of T. Indeed, let v be a 5-vertex in $V_{\text{in}}(T)$ that is adjacent to x_1 (say). By Lemma 1, $G[N[v]]$ is a 5-wheel. Thus $G[N[v]]$ contains two vertices v_1, v_2 that are adjacent to both x_1 and v but not to each other. At least one of v_1, v_2 must be in T, by (F5). But the neighbors x_1, v_1, v_2 of v cannot all be in T, by (F2). Thus v is adjacent to exactly two vertices of T.

Recall that x_1 has a neighbor $u \in V_{\text{in}}(T)$. We consider two cases.

Case 1. $d(u) = 5$. By (F7), u is adjacent to exactly one of x_2 and x_3, say x_2. By Lemma 2, $G[N[u]]$ is a 5-wheel. Let $x_1x_2u_1u_2u_3x_1$ be the 5-cycle of $G[N[u]]$ (see Figure 1). Then u and u_3 are the two vertices of $V_{\text{in}}(T)$ that are adjacent to x_1, and u and u_1 are the two vertices of $V_{\text{in}}(T)$ that are adjacent to x_2.

Case 1.1. $d(u_1) = 5$. Note that u_1 is adjacent to x_2 but cannot be adjacent to x_1, and so u_1 is adjacent to x_3 by (F7). So let $x_2x_3wu_2ux_2$ be the 5-cycle of $G[N[u_1]]$ (see Figure 1, left). Each vertex of T is now adjacent to two vertices of $V_{\text{in}}(T)$, and so cannot be adjacent to any further vertex of $V_{\text{in}}(T)$, by (F6). Thus any further neighbor of u_3 cannot be adjacent to x_1 and so must be adjacent to u_2, to avoid a claw centered on u_3. If $d(u_2) = 4$, then the only possible further neighbor of u_3 is w, and so u_2, u_3 are adjacent 4-vertices, contrary to (F4). If however $d(u_2) = 5$, then the additional neighbor q of u_2 can be adjacent only to u_2, u_3, w of the vertices so far named (since it cannot be adjacent to x_1 or x_3), and so q is a 3-vertex by the claw-freeness, which is not permissible, by (F3).

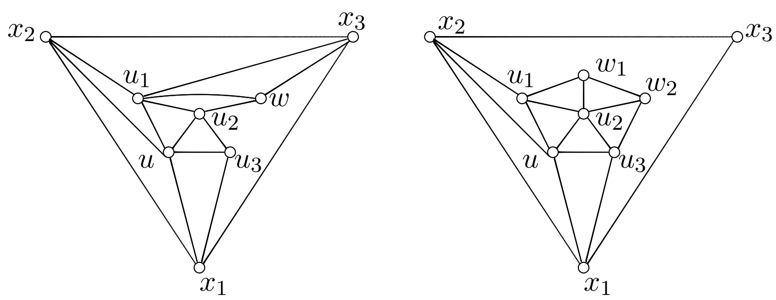

Figure 1. Cases 1.1 and 1.2 of Theorem 5.

Case 1.2. $d(u_1) = 4$. Then $d(u_2) = 5$, by (F4). Let $uu_1w_1w_2u_3u$ be the 5-cycle of $G[N[u_2]]$ (see Figure 1, right). Note that x_1 and x_2 each already have two neighbors in $V_{\text{in}}(T)$, and so u_2 has no neighbors in T, by (F6) and (F7). Thus $w_1 \in V_{\text{in}}(T)$ and so $d(w_1) = 5$, by (F4), since $d(u_1) = 4$. But it is impossible for $G[N[w_1]]$ to be a 5-wheel since $d(u_1) = 4$ and there is no edge x_2w_1.

Case 2. $d(u) = 4$. Let the neighbors of u be x_1, u_1, u_2 and u_3. We may assume that $u_2, u_3 \in V_{\text{in}}(T)$ and possibly $u_1 = x_2$ or $u_1 = x_3$, by (F2). Then $d(u_2) = 5$ and $d(u_3) = 5$, by (F4). If x_1 is adjacent to the 5-vertex u_2 or u_3, by choosing u_2 or u_3 instead of u, we are back in Case 1. In addition, if x_2 or x_3 is adjacent to the 5-vertex u_2 or u_3, by choosing x_2 or x_3 instead of x_1, we are also back in Case 1. Thus we assume that $N[\{u_2, u_3\}] \subseteq V_{\text{in}}(T)$. But since $G[N[u_2]]$ and $G[N[u_3]]$ are 5-wheels, u must be adjacent to at least four vertices of $N[\{u_2, u_3\}]$ as well as to x_1, and this is impossible since $d(u) = 4$.

This completes the proof of Theorem 4. ■

Note that the only open case for the Total Coloring Conjecture in planar graphs is $\Delta = 6$. By Theorem 4, immediately, we have the following theorem.

Theorem 5. *The Total Coloring Conjecture holds in claw-free planar graphs.*

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