CORRIGENDUM TO: INDEPENDENT TRANSVERSAL DOMINATION IN GRAPHS [DISCUSS. MATH. GRAPH THEORY 32 (2012) 5–17]

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Abstract

In [Independent transversal domination in graphs, Discuss. Math. Graph Theory 32 (2012) 5–17], Hamid claims that if $G$ is a connected bipartite graph with bipartition $\{X,Y\}$ such that $|X| \leq |Y|$ and $|X| = \gamma(G)$, then $\gamma_{it}(G) = \gamma(G) + 1$ if and only if every vertex $x$ in $X$ is adjacent to at least two pendant vertices. In this corrigendum, we give a counterexample for the sufficient condition of this sentence and we provide a right characterization. On the other hand, we show an example that disproves a construction which is given in the same paper.

Keywords: domination, independent, transversal, covering, matching.

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1. Introduction

Among the results that Hamid shows in [4] we find the following.

Theorem 1.1 [4]. Let $G$ be a connected bipartite graph with bipartition $\{X,Y\}$ such that $|X| \leq |Y|$ and $|X| = \gamma(G)$. Then $\gamma_{it}(G) = \gamma(G) + 1$ if and only if every vertex $x$ in $X$ is adjacent to at least two pendant vertices.

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We find a connected bipartite graph $G$ with bipartition $\{X, Y\}$ such that $|X| \leq |Y|$, $|X| = \gamma(G)$ and $\gamma_{iul}(G) = \gamma(G) + 1$. But there exists a vertex in $X$ which is not adjacent to at least two pendant vertices.

A problem that arises with Theorem 1.1 is that it is used in [1] in order to prove the following result.

**Corollary 1.2** [1]. Let $T$ be a tree with bipartition $\{X, Y\}$ such that $1 \leq |X| \leq |Y|$ and $\gamma(T) = |X|$. Then, $\gamma_{iul}(T) = \gamma(T)$ if and only if there is a vertex in $X$ which is adjacent to at most one pendant vertex.

In this corrigendum, we provide a right characterization for bipartite graphs $G$ with bipartition $\{X, Y\}$, $|X| \leq |Y|$ and $|X| = \gamma(G)$, such that $\gamma_{iul}(G) = \gamma(G) + 1$. As a consequence of the main result, we show the corrected version of Corollary 1.2.

Other result showed in [4] is the following.

**Theorem 1.3** [4]. Let $a$ and $b$ be two positive integers with $b \geq 2a - 1$. Then there exists a connected graph $G$ on $b$ vertices such that $\gamma_{iul}(G) = a$.

In order to prove Theorem 1.3, Hamid proposes the following construction: set $b = 2a + r$, with $r \geq -1$, and let $H$ be any connected graph on $a$ vertices. Let $V(H) = \{v_1, v_2, \ldots, v_a\}$ be the vertex set of $H$ and let $G$ be the graph obtained from $H$ by attaching $r + 1$ pendant edges at $v_1$ and one pendant edge at each $v_i$, for $i \geq 2$. Let $u_i$ be the pendant vertex in $G$ adjacent to $v_i$, for $i \geq 2$.

Hamid claims that $\gamma_{iul}(G) = a$ and $S = \{v_1, u_2, u_3, \ldots, u_a\}$ is a $\gamma_{iul}(G)$-set. Further, every maximum independent set of $G$ intersects $S$ and hence $\gamma_{iul}(G) = a$.

We find that, in some cases for $H$, $G$ holds $\gamma_{iul}(G) \neq a$ and there exists an $\alpha(G)$-set which does not intersect $S$.

In this corrigendum we provide a correct proof of Theorem 1.3 for $b \geq 2a$.

## 2. Definitions and Known Results

We use [2] and [3] for terminology and notation not defined here and consider finite and simple graphs only. For introductory notation, let $G$ be a graph. $n(G)$ denotes $|V(G)|$. Let $v$ be a vertex of $G$, the open neighborhood of $v$ in $G$, denoted by $N(v)$, is defined as the set $\{u \in V(G) : uv \in E(G)\}$. The degree of a vertex $v$, denoted by $\delta(v)$, is the number $|N(v)|$. We say that a vertex $u$ is a pendant vertex if $\delta(u) = 1$. For a graph $G$, the number $\min\{\delta(u) : u \in V(G)\}$ is denoted by $\delta(G)$. An edge of a graph is said to be a pendant edge if one of its vertices is a pendant vertex. A complete graph is a graph with $n$ vertices and an edge between every two vertices, denoted by $K_n$. A subset $I$ of $V(G)$ is said to be independent if every two vertices of $I$ are non-adjacent. We say that a graph $G$ is bipartite if...
there exists a partition \( \{X, Y\} \) of \( V(G) \) such that \( X \) and \( Y \) are independent sets (we call that partition a bipartition). If \( G \) contains every edge joining \( X \) and \( Y \), then \( G \) is a complete bipartite graph, denoted by \( K_{m,n} \) with \( |X| = m \) and \( |Y| = n \). The complete bipartite graph \( K_{1,n} \) is called a star.

A subset \( D \) of \( V(G) \) is said to be dominating if for every \( u \) in \( V(G) - D \) it holds \( N(u) \cap D \neq \emptyset \). The cardinality of a smallest dominating set is the domination number, denoted by \( \gamma(G) \), and we refer to such a set as a \( \gamma(G) \)-set.

The cardinality of a largest independent set in \( G \) is the independence number, denoted by \( \alpha(G) \), and an independent set having cardinality \( \alpha(G) \) is called a maximum independent set. We refer to such a set as an \( \alpha(G) \)-set. A subset \( M \) of \( E(G) \) is a matching if every two edges of \( M \) are non-adjacent. A maximum matching is one of largest cardinality in \( G \). The number of edges in a maximum matching of a graph \( G \) is called the matching number of \( G \), denoted by \( \beta(G) \). A subset \( K \) of \( V(G) \) such that every edge of \( G \) has at least one end in \( K \) is called a covering of \( G \). The number of vertices in a minimum covering of \( G \) is the covering number of \( G \), denoted by \( \delta(G) \). An independent transversal dominating set in \( G \) is a dominating set that intersects every maximum independent set in \( G \). The independent transversal domination number, denoted by \( \gamma_{it}(G) \), is the smallest cardinality of an independent transversal dominating set of \( G \). An independent transversal dominating set of cardinality \( \gamma_{it}(G) \) is called a minimum independent transversal dominating set. We refer to such a set as a \( \gamma_{it}(G) \)-set.

We need the following results.

**Theorem 2.1** [5]. For any tree \( T \), \( \gamma(T) = n(T) - \Delta(T) \) if and only if \( T \) is a wounded spider.

**Proposition 2.1** ([4], Example 3.1). \( \gamma_{it}(K_{m,n}) = 2 \).

**Theorem 2.2** [4]. For any graph \( G \), we have \( \gamma(G) \leq \gamma_{it}(G) \leq \gamma(G) + \delta(G) \).

**Lemma 2.3** ([2], page 74). Let \( M \) be a matching and \( K \) a covering such that \( |M| = |K| \). Then \( M \) is a maximum matching and \( K \) is a minimum covering.

**Lemma 2.4** ([2], page 101). Let \( G \) be a graph. Then \( \alpha(G) + \beta(G) = n(G) \).

3. A Counterexample for Theorem 1.1

Consider the graph \( G \) in Figure 1. Since \( M = \{x_1y_2, x_2y_5, x_3y_4, x_4y_6\} \) is a matching and \( X \) is a covering such that \( |M| = |X| \), it follows from Lemmas 2.3 and 2.4 that \( \alpha(G) = 7 \); also it is straightforward to see that \( \gamma(G) = 4 \). On the other hand, notice that \( X \) and \( (X - \{x_4\}) \cup \{y_6\} \) are the only one \( \gamma(G) \)-sets. Therefore, since \( Y \) and \( (Y - \{y_6\}) \cup \{x_4\} \) are \( \alpha(G) \)-sets such that \( X \cap Y = \emptyset \) and \( (Y - \{y_6\}) \cup \{x_4\} \)
∩ ((X − {x_4}) ∪ {y_6}) = ∅, we get from Theorem 2.2 that γ_{it}(G) = γ(G) + 1 (because δ(G) = 1). As x_4 is not adjacent to at least two pendant vertices, we obtain a counterexample for Theorem 1.1.

Figure 1. N(x_4) has no pendant vertices.

4. Right Characterization for Bipartite Graphs G Such That |X| ≤ |Y|, |X| = γ(G) and γ_{it}(G) = γ(G) + 1

We need the following results.

Corollary 4.1 [4]. If G has an isolated vertex, then γ_{it}(G) = γ(G).

Theorem 4.2 shows the right version of Theorem 1.1. Moreover, Theorem 4.2 allows disconnected graphs.

Theorem 4.2. Let G be a bipartite graph with bipartition \{X, Y\} such that |X| ≤ |Y| and |X| = γ(G). Then γ_{it}(G) = γ(G) + 1 if and only if

1. every vertex x in X, such that δ(x) ≠ 1, is adjacent to at least two pendant vertices,
2. Y has no isolated vertices.

Proof. If |V(G)| = 2, hypothesis |X| = γ(G) implies that G = K_2 and therefore G satisfies Theorem 4.2. Assume that |V(G)| ≥ 3.

Suppose that γ_{it}(G) = γ(G) + 1. It follows from Corollary 4.1 that G has no isolated vertices, which implies that δ(G) ≥ 1. Therefore, in particular Y has no isolated vertices. Thus, it remains to prove that every vertex x in X, such that δ(x) ≠ 1, is adjacent to at least two pendant vertices. Suppose that there exists a vertex w in X such that δ(w) ≥ 2.

Notice that X is a γ(G)-set (because for every u in (V(G) − X) = Y, δ(u) ≥ 1, and |X| = γ(G)).
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Consider the following claims.

Claim 1. \( \alpha(G) = |Y| \).

Given that \( Y \) is an independent set in \( G \), we get that \( \alpha(G) \geq |Y| \). On the other hand, the hypotheses \( \gamma_{it}(G) = \gamma(G) + 1 \) and \( |X| = \gamma(G) \) imply that there exists an \( \alpha(G) \)-set \( S \) such that \( X \cap S = \emptyset \). Since \( S \subseteq Y \), then \( \alpha(G) = |S| \leq |Y| \). Therefore, \( \alpha(G) = |Y| \).

Claim 2. \( \delta(G) = 1 \).

Proceeding by contradiction, suppose that \( \delta(G) \geq 2 \). Let \( u \) and \( v \) be two vertices in \( G \) such that \( u \in X \) and \( v \in N(u) \). Set \( S = (X - \{u\}) \cup \{v\} \).

Claim 2.1. \( S \) is a dominating set in \( G \).

Since \( \delta(w) \geq 2 \) for every \( w \) in \( Y - \{v\} \), there exists \( x_w \) in \( X - \{u\} \) such that \( wx_w \in E(G) \). Therefore, \( S \) is a dominating set in \( G \) (consider the choice of \( v \)).

Claim 2.2. \( S \cap J \neq \emptyset \) for every \( \alpha(G) \)-set \( J \).

Let \( J \) be an \( \alpha(G) \)-set. If \( v \in J \), then \( S \cap J \neq \emptyset \). Suppose that \( v \notin J \). Given that \( |J| = \alpha(G) = |Y| \) (by Claim 1) and \( v \notin J \), it follows that \( X \cap J \neq \emptyset \). If \( u \notin J \), we get \( (X - \{u\}) \cap J \neq \emptyset \) (because \( X \cap J \neq \emptyset \)), which implies that \( S \cap J \neq \emptyset \). Thus, suppose that \( u \in J \). Since \( \delta(u) \geq 2 \), there exists \( z \) in \( Y - \{v\} \) such that \( uz \in E(G) \), which implies that \( |J \cap Y| \leq |Y| - 2 \) (because \( u \in J \), \( \{uv, uz\} \subseteq E(G) \) and \( J \) is an independent set). Therefore, \( 2 \leq |X \cap J| \), which implies that \( (X - \{u\}) \cap J \neq \emptyset \). Thus, \( S \cap J \neq \emptyset \).

We get from Claims 2.1, 2.2, the definition of \( S \) and the hypothesis that \( \gamma_{it}(G) \leq |S| = |X| = \gamma(G) \), a contradiction with \( \gamma_{it}(G) = \gamma(G) + 1 \). Therefore, \( \delta(G) = 1 \).

Let \( u \) be a vertex in \( X \) such that \( \delta(u) \geq 2 \). We will prove that \( u \) is adjacent to at least two pendant vertices. Proceeding by contradiction, suppose that \( N(u) \) contains at most one pendant vertex. If \( N(u) \) contains a pendant vertex \( v \), choose \( v \), otherwise let \( v \) be any vertex in \( N(u) \). Set \( S = (X - \{u\}) \cup \{v\} \).

Claim 3. \( S \) is a dominating set in \( G \).

Given that \( \delta(w) \geq 1 \) for every \( w \) in \( Y - N(u) \), it follows that there exists \( x_w \) in \( X - \{u\} \) such that \( wx_w \in E(G) \). On the other hand, since for every \( z \) in \( N(u) - \{v\} \) it holds that \( \delta(z) \geq 2 \), then there exists \( x_z \) in \( X - \{u\} \) such that \( zx_z \in E(G) \). Therefore, \( S \) is a dominating set in \( G \).

Claim 4. If \( J \) is an \( \alpha(G) \)-set, then \( S \cap J \neq \emptyset \).

The proof is the same as the proof of Claim 2.2.
We get from Claims 3, 4, the definition of $S$ and the hypothesis that $\gamma_{it}(G) \leq |S| = |X| = \gamma(G)$, a contradiction with $\gamma_{it}(G) = \gamma(G) + 1$. Hence, $u$ is adjacent to at least two pendant vertices.

Therefore, every vertex $x$ in $X$, such that $\delta(x) \neq 1$, is adjacent to at least two pendant vertices.

Suppose that for every vertex $w$ in $X$, such that $\delta(w) \neq 1$, $N(w)$ contains at least two pendant vertices and $Y$ has no isolated vertices. Notice that it follows from the hypothesis that $\delta(G) \geq 1$. Consider the following claims.

**Claim A.** $\alpha(G) = |Y|$.

Given that $Y$ is an independent set, we get that $\alpha(G) \geq |Y|$. Proceeding by contradiction, suppose that $\alpha(G) > |Y|$ and let $J$ be an $\alpha(G)$-set.

Since $\alpha(G) > |Y|$ and $|X| \leq |Y|$, we get that $J \cap X \neq \emptyset$ and $J \cap Y \neq \emptyset$. Set $X' = J \cap X$, $Y' = J \cap Y$, $X_1 = \{x \in X' : \delta(x) \geq 2\}$ and $X_2 = \{x \in X' : \delta(x) = 1\}$.

**Claim A.1.** $|X_1| \geq 1$.

As $|Y| = |Y'| + |Y - Y'|$, $|J| = |X'| + |Y'|$ and $|J| > |Y|$, it follows that $|X'| > |Y - Y'|$, which implies that there exist two vertices in $X'$, say $u_1$ and $u_2$, and there exists a vertex $y$ in $Y - Y'$ such that $\{u_1y, u_2y\} \subseteq E(G)$.

Proceeding by contradiction, suppose that $X_1 = \emptyset$. Since $\delta(u_1) = 1$ and $\delta(u_2) = 1$, then for every $z$ in $Y - (Y' \cup \{y\})$ there exists $x_z$ in $X - \{u_1, u_2\}$ such that $z x_z \in E(G)$ (recall that $\delta(G) \geq 1$). On the other hand, given that $J$ is an independent set, we get that for every $w$ in $Y'$ there exists $x_w$ in $X - X'$ such that $w x_w \in E(G)$. Hence, $(X - \{u_1, u_2\}) \cup \{y\}$ is a dominating set, a contradiction with $|X| = \gamma(G)$. Therefore, $|X_1| \geq 1$.

Since $N(X') \subseteq Y - Y'$ and every vertex of $X_1$ is adjacent to at least two pendant vertices, we get from the definition of $X_2$ that $|Y - Y'| \geq 2|X_1| + |X_2|$; that is, $|Y - Y'| \geq |X'| + |X_1|$, which implies that $|X_1| + |X'| + |Y'| \leq |Y|$. Hence, since $|X_1| + |J| \leq |Y|$, $1 \leq |X_1|$ (by Claim A.1) and $|Y| < |J|$, we get a contradiction.

Therefore, $\alpha(G) = |Y|$.

**Claim B.** If $D$ is a $\gamma(G)$-set, then $V(G) - D$ is an $\alpha(G)$-set.

Let $D$ be a $\gamma(G)$-set. Since $|D| = \gamma(G) = |X|$, then $|V(G) - D| = (|V(G)| - |X|) = |Y| = \alpha(G)$ (by Claim A). It remains to prove that $V(G) - D$ is an independent set. It is clear that $V(G) - D$ is an independent set if either $(V(G) - D) \subseteq X$ or $(V(G) - D) \subseteq Y$. Hence, suppose that $(V(G) - D) \cap X \neq \emptyset$ and $(V(G) - D) \cap Y \neq \emptyset$. Let $u$ and $v$ be two vertices in $V(G) - D$; we will prove that $uv \notin E(G)$. Suppose that $u \in (V(G) - D) \cap X$ and $v \in (V(G) - D) \cap Y$.

**Claim B.1.** $\delta(u) = 1$. 

Proceeding by contradiction, suppose that \( \delta(u) \geq 2 \). It follows from the hypothesis that \( N(u) \) has at least two pendant vertices, say \( w \) and \( z \). Since \( u \notin D \), we get that \( \{w, z\} \subseteq D \) (because \( D \) is a dominating set).

We will see that \( S = (D - \{w, z\}) \cup \{u\} \) is a dominating set. Notice that \( V(G) - S = (((V(G) - D) \cap X) - \{u\}) \cup (((V(G) - D) \cap Y) \cup \{w, z\}) \). \( D = (D \cap X) \cup (D \cap Y) \) and \( S = (D \cap X) \cup ((D \cap Y) - \{w, z\}) \cup \{u\} \). Given that \( D \) is a dominating set, we get that for every \( y \) in \( V(G) - D \cap Y \) there exists \( x_y \in D \cap X \) such that \( xy \in E(G) \). In the same way for every \( x \) in \( ((V(G) - D) \cap X) - \{u\} \) there exists \( y_x \in D \cap Y \) such that \( xy_x \in E(G) \) (because \( w \) and \( z \) are pendant vertices which are adjacent to \( u \)). Hence, we conclude that \( S \) is a dominating set. Since \( |S| = |X| - 1 \), we get a contradiction with \( |X| = \gamma(G) \).

Therefore, \( \delta(u) = 1 \).

Given that \( \delta(u) = 1 \), \( u \notin D \) and \( D \) is a dominating set, it follows that \( N(u) \subseteq D \), which implies that \( uv \notin E(G) \). Therefore, \( V(G) - D \) is an independent set. Hence, \( V(G) - D \) is an \( \alpha(G) \)-set.

\begin{claim}
\( \delta(G) = 1 \).
\end{claim}

Recall that \( \delta(G) \geq 1 \). If \( X \) has a pendant vertex, then we are done; otherwise, it follows from the hypothesis that for \( u \) in \( X \) there exists a pendant vertex in \( N(u) \). Therefore, \( \delta(G) = 1 \).

It follows from Claim B that \( \gamma_{ul}(G) \neq \gamma(G) \). Therefore, we get from Claim C and Theorem 2.2 that \( \gamma_{ul}(G) = \gamma(G) + 1 \).

5. Some Consequences of Theorem 4.2

A subdivision of an edge \( uv \) is obtained by replacing the edge \( uv \) with a path \((u, w, v)\), where \( w \) is a new vertex. For a positive integer \( t \), a wounded spider is a star \( K_{1,t} \) with at most \( t - 1 \) of its edges subdivided. Similarly, for an integer \( t \geq 2 \), a healthy spider is a star \( K_{1,t} \) with all of its edges subdivided.

\begin{remark}
It is straightforward to see that if \( G \) is a healthy spider, then \( \gamma(G) = \Delta(G) \). On the other hand, if \( G \) is a healthy spider, it follows from Theorem 4.2 that \( \gamma_{ul}(G) = \gamma(G) \).
\end{remark}

\begin{remark}
Let \( G \) be a wounded spider which is not a star. Suppose that \( G \) is obtained from \( K_{1,t} \) by subdividing \( r \) of its edges, with \( 1 \leq r \leq t - 1 \) and \( t \geq 2 \).
\begin{enumerate}
\item If \( r \leq t - 2 \), then \( \gamma_{ul}(G) = \gamma(G) + 1 = r + 2 \).
\item If \( r = t - 1 \), then \( \gamma_{ul}(G) = \gamma(G) = t \).
\end{enumerate}
\end{remark}

\begin{proof}
Suppose that \( V(G) = \{u_1, v_2, \ldots, v_t, v_{t+1}\} \cup \{u_2, \ldots, u_r, u_{r+1}\}, E(G) = \{u_1v_j : j \in \{2, \ldots, t+1\}\} \cup \{u_iv_i : i \in \{2, \ldots, r+1\}\}. \) Set \( X = \{u_1, u_2, \ldots, u_r, u_{r+1}\} \) and \( Y = \{v_2, \ldots, v_t, v_{t+1}\} \).
\end{proof}
1. Suppose that \( r \leq t - 2 \). It follows from Theorem 2.1 that \( \gamma(G) = (t + 1) + r \) which implies that \( |X| = \gamma(G) \). Therefore, we get from Theorem 4.2 that \( \gamma_{it}(G) = \gamma(G) + 1 = (r + 1) + 1 \).

2. Suppose that \( r = t - 1 \). It follows from Theorem 2.1 that \( \gamma(G) = t \). Since \( |X| = \gamma(G) \) and \( u_1 \) is not adjacent to at least two pendant vertices in \( G \), it follows from Theorem 4.2 that \( \gamma_{it}(G) \neq \gamma(G) + 1 \). Therefore, given that \( \delta(G) = 1 \), we get from Theorem 2.2 that \( \gamma_{it}(G) = \gamma(G) \). Hence, \( \gamma_{it}(G) = t \).

**Corollary 5.1.** Let \( T \) be a tree with bipartition \( \{X, Y\} \) such that \( 1 \leq |X| \leq |Y| \) and \( \gamma(T) = |X| \). Then, \( \gamma_{it}(T) = \gamma(T) \) if and only if there is a vertex \( x \) in \( X \), with \( \delta(x) \neq 1 \), which is adjacent to at most one pendant vertex.

### 6. Example Disproving Construction in Theorem 1.3

Recall that, in order to prove Theorem 1.3, Hamid proposes the following construction: set \( b = 2a + r \), with \( r \geq -1 \), and let \( H \) be any connected graph on \( a \) vertices. Let \( V(H) = \{v_1, v_2, \ldots, v_a\} \) be the vertex set of \( H \) and let \( G \) be the graph obtained from \( H \) by attaching \( r + 1 \) pendant edges at \( v_1 \) and one pendant edge at each \( v_i \), for \( i \geq 2 \). Let \( u_i (i \geq 2) \) be the pendant vertex in \( G \) adjacent to \( v_i \).

Hamid claims that \( \gamma_{it}(G) = a \) and \( S = \{v_1, u_2, u_3, \ldots, u_a\} \) is a \( \gamma_{it}(G) \)-set. Further, every maximum independent set of \( G \) intersects \( S \) and hence \( \gamma_{it}(G) = a \).

- We find that, when \( r = -1 \) and \( a \geq 3 \), for the graph \( H = K_a \), the associated graph \( G \) does not hold the conclusion of Theorem 1.3, see Figure 2.

![Figure 2](image-url)
independent transversal dominating set in $G$, which implies that $\gamma_{it}(G) \leq a - 1$. On the other hand, let $S$ be a $\gamma_{it}(G)$-set. Given that $S$ is a dominating set, then $\{v_i, u_i\} \cap S \neq \emptyset$ for every $i$ in $\{2, \ldots, a\}$, which implies that $a - 1 \leq |S|$. Therefore, $\gamma_{it}(G) = a - 1$

- We find that, when $r > 0$ and $a \geq 2$, for the graph $H = K_{1,a-1}$, the associated graph $G$ is a wounded spider and this does not hold the conclusion of Theorem 1.3, see Figure 3.

\[ \text{Figure 3} \]

Notice that $G$ is also obtained from $K_{1,a+r}$ by subdividing exactly $a - 1$ of its edges, where $a - 1 \leq (a + r) - 2$. Therefore, it follows from Remark 5.2 that $\gamma_{it}(G) = \gamma(G) + 1 = (a - 1) + 2 = a + 1$.

- When $r > 0$ and $a = 1$ we have that $G = K_{1,r+1}$ and in this case we get from Proposition 2.1 that $\gamma_{it}(G) = 2 = a + 1$, see Figure 4.

\[ \text{Figure 4} \]

- When $H = K_{1,a-1}$, for $r \geq 0$ and $a \geq 2$, there exists an $\alpha(G)$-set in $G$ which does not intersect $S = \{v_1, v_2, u_3, \ldots, u_a\}$.

For every $i$ in $\{1, \ldots, r+1\}$ let $x_i$ be the pendant vertex adjacent to $v_1$. Since $M = \{v_1x_1, v_2x_2, \ldots, v_ax_a\}$ is a matching and $K = V(H)$ is a covering such that $|M| = |K|$, then we get from Lemma 2.3 that $K$ is a minimum covering. On the other hand, it follows from Lemma 2.4 that $2a + r = |V(G)| = \alpha(G) + \beta(G) = \alpha(G) + a$, which implies that $\alpha(G) = a + r$.

Therefore $(V(H) - \{v_1\}) \cup \{x_1, \ldots, x_{r+1}\}$ is an $\alpha(G)$-set in $G$ which does not intersect $S$. 

- When $H = K_{1,a-1}$, for $r \geq 0$ and $a \geq 2$, there exists an $\alpha(G)$-set in $G$ which does not intersect $S = \{v_1, v_2, u_3, \ldots, u_a\}$.
For \( b \geq 2a \) we proceed to prove the following.

**Theorem 6.1.** Let \( a \) and \( b \) be two positive integers with \( b \geq 2a \). Then there exists a connected graph \( G \) on \( b \) vertices such that \( \gamma_{it}(G) = a \).

**Proof.** Suppose that \( b = 2a + r \), for some \( r \) in \( \mathbb{N} \). Let \( H \) be a connected graph of order \( a \), such that \( H \not\cong K_{1,a-1} \), with vertex set \( V(H) = \{v_1, \ldots, v_a\} \). Let \( \{x_1, \ldots, x_{r+1}\} \) and \( \{u_2, \ldots, u_a\} \) be two sets such that \( \{x_1, \ldots, x_{r+1}\} \cap \{u_2, \ldots, u_a\} = \emptyset, \{x_1, \ldots, x_{r+1}\} \cap V(H) = \emptyset \) and \( V(H) \cap \{u_2, \ldots, u_a\} = \emptyset \). Let \( G \) be the graph with \( V(G) = V(H) \cup \{x_1, \ldots, x_{r+1}\} \cup \{u_2, \ldots, u_a\} \) and \( E(G) = E(H) \cup \{v_1u_i : i \in \{2, \ldots, a\}\} \cup \{v_1x_i : i \in \{1, \ldots, r+1\}\} \).

**Claim 1.** \( a \leq \gamma_{it}(G) \).

We will prove that \( \gamma(G) = a \). Since \( V(H) \) is a dominating set in \( G \), then \( \gamma(G) \leq a \). On the other hand, let \( S \) be a \( \gamma(G) \)-set. Given that \( \{u_2, v_1\} \cap S \neq \emptyset \) (because \( S \) is a dominating set) for every \( i \) in \( \{2, \ldots, a\} \) and \( r+1 \geq 1 \) we get that \( |S| \geq r+1 \). Hence, \( \gamma(G) = a \). Therefore, it follows from Theorem 2.2 that \( a \leq \gamma_{it}(G) \).

**Claim 2.** \( \alpha(G) = r + a \).

Since \( K = V(H) \) is a covering and \( M = \{v_1u_i : i \in \{2, \ldots, a\}\} \cup \{v_1x_1\} \) is a matching such that \( |K| = a = |M| \), it follows from Lemma 2.3 that \( |K| = \beta(G) \). Hence, we get from Lemma 2.4 that \( \alpha(G) = r + a \).

**Claim 3.** \( S = \{v_1, u_2, \ldots, u_a\} \) is an independent transversal dominating set in \( G \).

Given that \( S \) is a dominating set, it remains to prove that \( S \) intersects every maximum independent set in \( G \). Since \( H \not\cong K_{1,a-1} \) and \( H \) is connected, we get that \( V(H) \setminus \{v_1\} \) is not an independent set in \( G \), which implies that \( (V(H) \setminus \{v_1\}) \cup \{x_1, \ldots, x_{r+1}\} \) is not an independent set in \( G \). Since \( |(V(H) \setminus \{v_1\}) \cup \{x_1, \ldots, x_{r+1}\}| = a + r \), it follows that \( S \) intersects every maximum independent set in \( G \).

Therefore, we get from Claims 1 and 3 that \( a \leq \gamma_{it}(G) \leq a \).

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**References**


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