A CLASSIFICATION OF CACTUS GRAPHS
ACCORDING TO THEIR DOMINATION NUMBER

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Abstract

A set $S$ of vertices in a graph $G$ is a dominating set of $G$ if every vertex not in $S$ is adjacent to some vertex in $S$. The domination number, $\gamma(G)$, of $G$ is the minimum cardinality of a dominating set of $G$. The authors proved in [A new lower bound on the domination number of a graph, J. Comb. Optim. 38 (2019) 721–738] that if $G$ is a connected graph of order $n \geq 2$ with $k \geq 0$ cycles and $\ell$ leaves, then $\gamma(G) \geq [(n - \ell + 2 - 2k)/3]$. As a consequence of the above bound, $\gamma(G) = (n - \ell + 2(1 - k) + m)/3$ for some integer $m \geq 0$. In this paper, we characterize the class of cactus graphs achieving equality here, thereby providing a classification of all cactus graphs according to their domination number.

Keywords: domination number, lower bounds, cycles, cactus graphs.

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1. Introduction

A dominating set of a graph $G$ is a set $S$ of vertices of $G$ such that every vertex not in $S$ has a neighbor in $S$, where two vertices are neighbors in $G$ if they are adjacent. The minimum cardinality of a dominating set is the domination number of $G$, denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is called a $\gamma$-set of $G$. As remarked in [5], the notion of domination and its variations in graphs has been studied a great deal; a rough estimate says that it occurs in more than 6000 papers to date. For fundamentals of domination theory in graphs we refer the reader to the so-called domination books by Haynes, Hedetniemi, and Slater [6, 7]. An updated glossary of domination parameters can be found in [4].

Two vertices $u$ and $v$ in a graph $G$ are connected if there exists a $(u, v)$-path in $G$. The graph $G$ is connected if every two vertices in $G$ are connected. A block of $G$ is a maximal connected subgraph of $G$ which has no cut-vertex of its own. A cactus is a connected graph in which every edge belongs to at most one cycle. Equivalently, a (nontrivial) cactus is a connected graph in which every block is an edge or a cycle. The distance between two vertices $u$ and $v$ in a connected graph $G$ is the minimum length of a $(u, v)$-path in $G$. The diameter, $\text{diam}(G)$, of $G$ is the maximum distance among pairs of vertices in $G$.

For notation and graph theory terminology we generally follow [8]. In particular, the order of a graph $G$ with vertex set $V(G)$ and edge set $E(G)$ is given by $n(G) = |V(G)|$ and its size by $m(G) = |E(G)|$. A neighbor of a vertex $v$ in $G$ is a vertex adjacent to $v$, and the open neighborhood of $v$ is the set of neighbors of $v$, denoted $N_G(v)$. The closed neighborhood of $v$ is the set $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex $v$ in $G$ is given by $d_G(v) = |N_G(v)|$.

For a set $S$ of vertices in a graph $G$, the subgraph induced by $S$ is denoted by $G[S]$. Further, the subgraph obtained from $G$ by deleting all vertices in $S$ and all edges incident with vertices in $S$ is denoted by $G - S$. If $S = \{v\}$, we simply denote $G - \{v\}$ by $G - v$. A leaf of a graph $G$ is a vertex of degree 1 in $G$, and its unique neighbor is called a support vertex. The set of all leaves of $G$ is denoted by $L(G)$, and we let $\ell(G) = |L(G)|$ be the number of leaves in $G$. We denote the set of support vertices of $G$ by $S(G)$. We call a vertex of degree at least 2 a non-leaf.

Following our notation in [5], we denote the path and cycle on $n$ vertices by $P_n$ and $C_n$, respectively. A complete graph on $n$ vertices is denoted by $K_n$, while a complete bipartite graph with partite sets of size $n$ and $m$ is denoted by $K_{n,m}$. A star is the graph $K_{1,k}$, where $k \geq 1$. Further if $k > 1$, the vertex of degree $k$ is called the center vertex of the star, while if $k = 1$, arbitrarily designate either vertex of $P_2$ as the center. A double star is a tree with exactly two (adjacent) non-leaf vertices.

A rooted tree $T$ distinguishes one vertex $r$ called the root. For each vertex
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\( v \neq r \) of \( T \), the parent of \( v \) is the neighbor of \( v \) on the unique \( (r,v) \)-path, while a child of \( v \) is any other neighbor of \( v \). A descendant of \( v \) is a vertex \( u \neq v \) such that the unique \( (r,u) \)-path contains \( v \). In particular, every child of \( v \) is a descendant of \( v \). We let \( D(v) \) denote the set of descendants of \( v \), and we define \( D[v] = D(v) \cup \{v\} \). The maximal subtree at \( v \) is the subtree of \( T \) induced by \( D[v] \), and is denoted by \( T_v \). We use the standard notation \([k] = \{1,\ldots,k\} \).

2. Main Result

Our aim in this paper is to provide a classification of all cactus graphs according to their domination number. For this purpose, we shall use a result of the authors in [5] (which we present in Section 4) that establishes a lower bound on the domination number of a graph in terms of its order, number of vertices of degree 1, and number of cycles. From this result, we prove our desired characterization below, where \( G^m_k \) is a family of graphs defined in Section 3.

Theorem 1. Let \( m \geq 0 \) be an integer. If \( G \) is a cactus graph of order \( n \geq 2 \) with \( k \geq 0 \) cycles and \( \ell \) leaves, then \( \gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m) \), if and only if \( G \in G^m_k \).

We proceed as follows. In Section 3 we define the families \( G^m_k \) of graphs for each integer \( k \geq 0 \) and \( m \geq 0 \). Known results on the domination number are given in Section 4. In Section 5 we present a proof of our main result.

3. The Families \( G^m_k \) for \( m \geq 0 \) and \( k \geq 0 \)

In this section, we define the families \( G^m_k \) of graphs for each integer \( k \geq 0 \) and \( m \geq 0 \). The families \( G^0_k, G^1_k, G^2_k, T^1, T^2_0, T^2_1 \) of graphs were defined by the authors in [5]. For completeness, we include these definitions in Sections 3.1 and 3.2. We first define the families \( G^0_k, G^1_k \) and \( G^2_k \) of graphs in the special case when \( k = 0 \).

3.1. The families \( G^0_0, G^1_0 \) and \( G^2_0 \)

Hajian et al. [5] defined the class of trees \( G^0_0, G^1_0 \) and \( G^2_0 \) as follows.

• Let \( G^0_0 \) be the class of all trees \( T \) that can be obtained from a sequence \( T_1, \ldots, T_k \) of trees where \( k \geq 1 \) such that \( T_1 \) is a star with at least three vertices, \( T = T_k \), and, if \( k \geq 2 \), then the tree \( T_{i+1} \) can be obtained from the tree \( T_i \) by applying Operation \( \mathcal{O} \) defined below for all \( i \in [k - 1] \).

Operation \( \mathcal{O} \). Add a vertex disjoint copy of a star \( Q_i \) with at least three vertices to the tree \( T_i \) and add an edge joining a leaf of \( Q_i \) and a leaf of \( T_i \).
• Let $T^{1,1}_0$ be the class of all trees $T$ that can be obtained from a tree $T' \in \mathcal{G}_0^1$ by adding a vertex disjoint copy of a star with at least three vertices and adding an edge from a leaf of the added star to a non-leaf in $T'$. Now, let $\mathcal{G}_1^1$ be the class of all trees $T$ that can be obtained from a sequence $T_1, \ldots, T_k$ of trees where $k \geq 1$ such that $T_1 \in T^{1,1}_0 \cup \{P_2\}$, $T = T_k$, and, if $k \geq 2$, then the tree $T_{i+1}$ can be obtained from the tree $T_i$ by applying Operation $\mathcal{O}$ for all $i \in [k-1]$.

• Let $T^{2,1}_0$ be the class of all trees $T$ that can be obtained from a tree $T' \in \mathcal{G}_0^0$ by adding a vertex disjoint copy of a star (with at least two vertices) and adding an edge from the center of the added star to a non-leaf in $T'$. Let $T^{2,2}_0$ be the class of all trees $T$ that can be obtained from a tree $T' \in \mathcal{G}_0^0$ by adding a vertex disjoint copy of a star with at least three vertices and adding an edge from a leaf of the added star to a non-leaf in $T'$. Now, let $\mathcal{G}_0^2$ be the class of all trees $T$ that can be obtained from a sequence $T_1, \ldots, T_k$ of trees, where $k \geq 1$, such that $T_1 \in T^{2,1}_0 \cup T^{2,2}_0 \cup \{P_4\}$, $T = T_k$, and, if $k \geq 2$, then the tree $T_{i+1}$ can be obtained from the tree $T_i$ by applying Operation $\mathcal{O}$ for all $i \in [k-1]$.

3.2. The families $\mathcal{G}^0_k$, $\mathcal{G}^1_k$ and $\mathcal{G}^2_k$ when $k \geq 1$

For $k \geq 1$, Hajian et al. [5] defined the families of graphs $\mathcal{G}^0_k$, $\mathcal{G}^1_k$ and $\mathcal{G}^2_k$ as follows.

• For $k \geq 1$, they recursively defined the family $\mathcal{G}^0_i$ of graphs for each $i \in [k]$ by the following procedure.

Procedure A. For $i \in [k]$, a graph $G_i$ belongs to the family $\mathcal{G}^0_i$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}^0_{i-1}$ and the vertices $x$ and $y$ are leaves in $G_i - e$ that are connected by a unique path in $G_i - e$.

• For $k \geq 1$, they recursively defined the family $\mathcal{G}^1_i$ of graphs for each $i \in [k]$ by the following two procedures.

Procedure B. For $i \in [k]$, a graph $G_i$ belongs to the family $\mathcal{G}^1_i$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}^1_{i-1}$ and the vertices $x$ and $y$ are leaves in $G_i - e$ that are connected by a unique path in $G_i - e$.

Procedure C. For $i \in [k]$, a graph $G_i$ belongs to the family $\mathcal{G}^1_i$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}^1_{i-1}$ and the vertices $x$ and $y$ are connected by a unique path in $G_i - e$. Further, exactly one of $x$ and $y$ is a leaf in $G_i - e$.

• For $k \geq 1$, they recursively defined the family $\mathcal{G}^2_i$ of graphs for each $i \in [k]$ by the following four procedures.

Procedure D. For $i \in [k]$, a graph $G_i$ belongs to the family $\mathcal{G}^2_i$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}^2_{i-1}$ and the vertices $x$ and $y$ are leaves in $G_i - e$ that are connected by a unique path in $G_i - e$. 

Procedure E. For $i \in [k]$, a graph $G_i$ belongs to the family $G_2^i$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $G_1^{i-1}$ and the vertices $x$ and $y$ are connected by a unique path in $G_i - e$. Further, exactly one of $x$ and $y$ is a leaf in $G_i - e$.

Procedure F. For $i \in [k]$, a graph $G_i$ belongs to the family $G_2^i$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $G_1^{i-1}$ and the vertices $x$ and $y$ are connected by a unique path in $G_i - e$. Further, both $x$ and $y$ are leaves in $G_i - e$.

Procedure G. For $2 \leq i \in [k]$, a graph $G_i$ belongs to the family $G_2^i$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $G_1^{i-2}$ and the vertices $x$ and $y$ are connected by exactly two paths in $G_i - e$. Further, both $x$ and $y$ are leaves in $G_i - e$.

3.3. The family $G_0^m$ when $m \geq 3$

In this section, we define a family of graphs $G_0^m$ for each integer $m \geq 3$ as follows. We call a non-leaf $x$ in a tree $T$ a special vertex if $\gamma(T - x) \geq \gamma(T)$. For $m \geq 3$, we first recursively define the class $T_0^{m,1}$ and $T_0^{m,2}$ of trees as follows.

- Let $T_0^{m,1}$ be the class of all trees $T$ that can be obtained from a tree $T' \in G_0^{m-2}$ by adding a vertex disjoint copy of a star $Q$ and joining the center of $Q$ to a special vertex in $T'$.
- Let $T_0^{m,2}$ be the class of all trees $T$ that can be obtained from a tree $T' \in G_0^{m-1}$ by adding a vertex disjoint copy of a star $Q$ with at least three vertices and joining a leaf of $Q$ to a non-leaf in $T'$.

For $m \geq 3$, we next recursively define the family $G_0^m$ of graphs constructed from the families $G_0^{m-1}$ and $G_0^{m-2}$ as follows.

- Let $G_0^m$ be the class of all trees $T$ that can be obtained from a sequence $T_1, \ldots, T_q$ of trees, where $q \geq 1$ and where the tree $T_i \in T_0^{m,1} \cup T_0^{m,2}$ and the tree $T = T_q$. Further, if $q \geq 2$, then for each $i \in [q] \setminus \{1\}$, the tree $T_i$ can be obtained from the tree $T_{i-1}$ by applying the Operation $\mathcal{O}$ defined in Section 3.1.

Operation $\mathcal{O}$. Add a vertex disjoint copy of a star $Q_i$ with at least three vertices to the tree $T_i$ and add an edge joining a leaf of $Q_i$ and a leaf of $T_i$.

3.4. The family $G_k^m$ when $m \geq 3$ and $k \geq 1$

For $m \geq 3$ and $k \geq 1$, we construct the family $G_k^m$ from $G_k^{m-2}$, $G_{k-1}^{m-1}$ and $G_k^{m-1}$, recursively, as follows.

Procedure H. For $i \in [k]$, a graph $G_i$ belongs to the family $G_k^m$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $G_{k-1}^{m-1}$ and the vertices $x$ and $y$ are connected by a unique path in $G_i - e$ and $\gamma(G_i) = \gamma(G_i - e)$. Further, both $x$ and $y$ are leaves in $G_i - e$. 

Procedure I. For $i \in [k]$, a graph $G_i$ belongs to the family $G^m_i$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $G^{m-1}_{i-1}$ and the vertices $x$ and $y$ are connected by a unique path in $G_i - e$ and $\gamma(G_i) = \gamma(G_i - e)$. Further, exactly one of $x$ and $y$ is a leaf in $G_i - e$.

Procedure J. For $i \in [k]$, a graph $G_i$ belongs to the family $G^m_i$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $G^{m-2}_{i-1}$ and the vertices $x$ and $y$ are connected by a unique path in $G_i - e$ and $\gamma(G_i) = \gamma(G_i - e)$. Further, both $x$ and $y$ are non-leaves in $G_i - e$.

4. Known Results

In this section, we present some preliminary observations and known results. We begin with the following properties of graphs that belong to the families $G^0_k, G^1_k$ and $G^2_k$ for $k \geq 0$.

Observation 1. The following properties hold in a graph $G \in G^0_k \cup G^1_k \cup G^2_k$, where $k \geq 0$.
(a) The graph $G$ contains exactly $k$ cycles.
(b) The graph $G \in G^0_k \cup G^1_k$ is a cactus graph.

We shall also need the following elementary property of a dominating set in a graph.

Observation 2. If $G$ is connected graph of order at least 3, then there exists a $\gamma$-set of $G$ that contains no leaf of $G$.

The following lemma is established in [5].

Lemma 2 [5]. If $G$ is a connected graph and $C$ is an arbitrary cycle in $G$, then there is an edge $e$ of $C$ such that $\gamma(G - e) = \gamma(G)$.

Several authors obtained bounds on the domination number in terms of different variants of graphs, see for example [1, 2, 3, 6, 9]. Let $R$ be the family of all trees in which the distance between any two distinct leaves is congruent to 2 modulo 3. Lemańska [9] established the following lower bound on the domination number of a tree in terms of its order and number of leaves.

Theorem 3 [9]. If $T$ is a tree of order $n \geq 2$ with $\ell$ leaves, then $\gamma(T) \geq (n - \ell + 2)/3$, with equality if and only if $T \in R$.

Hajian et al. [5] showed that the family $R$ is precisely the family $G^0_0$; that is, $R = G^0_0$.

As a consequence of Theorem 3, we have the following result.
Corollary 4 [9]. If $T$ is a tree of order $n \geq 2$ with $\ell$ leaves, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$ for some integer $m \geq 0$.

Hajian et al. [5] strengthened the result in Theorem 3 as follows.

Theorem 5 [5]. If $T$ is a tree of order $n \geq 2$ with $\ell$ leaves, then the following holds.
(a) $\gamma(T) \geq \frac{1}{3}(n - \ell + 2)$, with equality if and only if $T \in G^0_0$.
(b) $\gamma(T) = \frac{1}{3}(n - \ell + 3)$ if and only if $T \in G^1_0$.
(c) $\gamma(T) = \frac{1}{3}(n - \ell + 4)$ if and only if $T \in G^2_0$.

The result of Theorem 5 was generalized in [5] to connected graphs as follows.

Theorem 6 [5]. If $G$ is a connected graph of order $n \geq 2$ with $k \geq 0$ cycles and $\ell$ leaves, then the following holds.
(a) $\gamma(G) \geq \frac{1}{3}(n - \ell + 2(1 - k))$, with equality if and only if $G \in G^0_k$.
(b) $\gamma(G) = \frac{1}{3}(n - \ell + 3 - 2k)$ if and only if $G \in G^1_k$.
(c) $\gamma(G) = \frac{1}{3}(n - \ell + 4 - 2k)$ if and only if $G \in G^2_k$.

As a consequence of Theorem 6(a), we have the following.

Corollary 7 [5]. If $G$ is a connected graph of order $n \geq 2$ with $k \geq 0$ cycles and $\ell$ leaves, then $\gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m)$ for some integer $m \geq 0$.

5. Proof of Main Result

In this section, we present a proof of our main result, namely Theorem 1. For this purpose, we first prove Theorem 1 in the special case when $k = 0$, that is, when the cactus is a tree.

Theorem 8. Let $m \geq 0$ be an integer. If $T$ is a tree of order $n \geq 2$ with $\ell$ leaves, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$ if and only if $T \in G^m_0$.

Proof. Let $T$ be a tree of order $n \geq 2$ with $\ell$ leaves. We proceed by induction on $m \geq 0$, namely first-induction, to show that $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$, if and only if $T \in G^m_0$. For the base step of the first-induction let $m \leq 2$. If $m = 0$, then the result follows by Theorem 5(a). If $m = 1$, then the result follows by Theorem 5(b). If $m = 2$, then the result follows by Theorem 5(c). This establishes the base step of the induction. Let $m \geq 3$ and assume that the result holds for all trees $T_0$ of order $n_0$ with $\ell_0$ leaves, for $m_0 < m$. Let $T$ be a tree of order $n$ and with $\ell$ leaves. We will show that $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$, if and only if $T \in G^m_0$.

$(\Rightarrow)$ Assume that $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$, if and only if $T \in G^m_0$. We show that $T \in G^m_0$. If $T = P_2$, then by the definition of the family...
We now root the tree $T$ at a vertex $r$ at the end of a longest path $P$ in $T$. Let $u$ be a vertex at maximum distance from $r$, and so $d_T(u,r) = \text{diam}(T)$. Necessarily, $r$ and $u$ are leaves. Let $v$ be the parent of $u$, let $w$ be the parent of $v$, let $x$ be the parent of $w$, and let $y$ be the parent of $x$. Possibly, $y = r$. Since $u$ is a vertex at maximum distance from the root $r$, every child of $v$ is a leaf. By Observation 2, there exists a $\gamma$-set, say $S$, of $T$ that contains no leaf of $T$; that is, $L(T) \cap S \neq \emptyset$. In particular, we note that $|S| = \gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$. In order to dominate the vertex $u$, we note therefore that $v \in S$. Let $d_T(v) = t$. We note that $t \geq 2$.

**Claim 1.** If $d_T(w) \geq 3$, then $T \in G_0^m$.

**Proof.** Suppose that $d_T(w) \geq 3$. In this case, we consider the tree $T' = T - V(T_v)$, where $T_v$ is the maximal subtree at $v$. Let $T'$ have order $n'$ and let $T'$ have $\ell'$ leaves. We note that $n' = n - t$. Since $w$ is not a leaf in $T'$, we have $\ell' = \ell - (t - 1) = \ell - t + 1$. By Corollary 4, $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m')$ for some integer $m' \geq 0$. If a child of $w$ is a leaf in $T'$, then since the dominating set $S$ contains no leaves, we have that $w \in S$. If no child of $w$ is a leaf in $T'$, then every child of $w$ is a support vertex and therefore belongs to the set $S$. In both cases, we note that the set $S \setminus \{v\}$ is a dominating set of $T'$, implying that $\gamma(T') \leq |S| - 1 = \gamma(T) - 1$. Every $\gamma$-set of $T'$ can be extended to a dominating set of $T$ by adding to it the vertex $v$, implying that $\gamma(T) \leq \gamma(T') + 1$. Consequently, $\gamma(T') = \gamma(T) - 1$. Thus,

\[
\begin{align*}
\gamma(T') &= \gamma(T) - 1 \\
&= \frac{1}{3}(n - \ell + 2 + m) - 1 \\
&= \frac{1}{3}(n - \ell + m - 1) \\
&= \frac{1}{3}((n' + t) - (\ell' + t - 1) + m - 1) \\
&= \frac{1}{3}(n' - \ell' + m).
\end{align*}
\]

As observed earlier, $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m')$ for some integer $m' \geq 0$. Thus, $m' = m - 2$. Applying the inductive hypothesis to the tree $T'$, we have $T' \in G_0^{m-2}$. Let $v'$ be a child of $w$ different from $v$. We note that the tree $T_{v'}$ is a component of $T' - w$ and this component is dominated by the vertex $v'$. We
can therefore choose a \( \gamma \)-set of \( T' - w \) to contain the vertex \( v' \). Such a \( \gamma \)-set of \( T' - w \) is also a dominating set of \( T' \), implying that \( \gamma(T') \leq \gamma(T' - w) \); that is, the vertex \( w \) is a special vertex of \( T' \). Thus, the tree \( T \) is obtained from the tree \( T' \in G^{m-2}_0 \) by adding a vertex disjoint copy of a star \( T_v \) and joining the center \( v \) of \( T_v \) to a special vertex \( w \) in \( T' \). Thus \( T \in T^{m-1}_0 \). Consequently, \( T \in G^m_0 \). This completes the proof of Claim 1.

By Claim 1, we may assume that \( d_T(w) = 2 \), for otherwise \( T \in G^m_0 \) as desired. We now consider the tree \( T' = T - V(T_w) \), where \( T_w \) is the maximal subtree at \( w \). Let \( T' \) have order \( n' \) and let \( T' \) have \( \ell' \) leaves. We note that \( n' = n - t - 1 \).

By Corollary 4, \( \gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m') \) for some integer \( m' \geq 0 \).

As observed earlier, the vertex \( v \) belongs to the dominating set \( S \). If \( w \in S \), then we can replace \( w \) in \( S \) with the vertex \( x \) to produce a new \( \gamma \)-set of \( T \) that contains no leaf of \( T \). Hence we may assume that \( w \notin S \), implying that the set \( S \setminus \{v\} \) is a dominating set of \( T' \) and therefore \( \gamma(T') \leq |S| - 1 = \gamma(T) - 1 \). Every \( \gamma \)-set of \( T' \) can be extended to a dominating set of \( T \) by adding to it the vertex \( v \), implying that \( \gamma(T) \leq \gamma(T') + 1 \). Consequently, \( \gamma(T') = \gamma(T) - 1 \).

Claim 2. If \( d_T(x) \geq 3 \), then \( T \in G^m_0 \).

Proof. Suppose that \( d_T(x) \geq 3 \). In this case, the vertex \( x \) is not a leaf of \( T' \), implying that \( \ell' = \ell - (t - 1) = \ell - t + 1 \). Thus,

\[
\gamma(T') = \gamma(T) - 1 = \frac{1}{3}(n - \ell + m - 1) = \frac{1}{3}(n' + t + 1 - (\ell' + t - 1) + m - 1) = \frac{1}{3}(n' - \ell' + m + 1).
\]

As observed earlier, \( \gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m') \) for some integer \( m' \geq 0 \). Thus, \( m' = m - 1 \). Applying the inductive hypothesis to the tree \( T' \), we have \( T' \in G^{m-1}_0 \). Thus, the tree \( T \) is obtained from the tree \( T' \in G^{m-1}_0 \) by adding a vertex disjoint copy of a star \( T_v \) with at least three vertices and joining a leaf of the star \( T_v \) to the non-leaf \( x \) of \( T' \). Thus \( T \in T^{m-2}_0 \). Consequently, \( T \in G^m_0 \).

By Claim 2, we may assume that \( d_T(x) = 2 \), for otherwise \( T \in G^m_0 \) as desired. In this case, the vertex \( x \) is a leaf of \( T' \), implying that \( \ell' = \ell - (t - 1) + 1 = \ell - t + 2 \). Thus,

\[
\frac{1}{3}(n' - \ell' + 2 + m') = \gamma(T') = \gamma(T) - 1 = \frac{1}{3}(n - \ell + m - 1) = \frac{1}{3}(n' + t + 1 - (\ell' + t - 2) + m - 1) = \frac{1}{3}(n' - \ell' + m + 2),
\]
and so \( m = m' \). Applying the inductive hypothesis to the tree \( T' \), we have \( T' \in G_0^m \). Thus, the tree \( T \) is obtained from the tree \( T' \in G_0^m \) by adding a vertex disjoint copy of a star \( T_v \) with at least three vertices and adding the edge \( xw \) joining a leaf \( w \) of \( T_v \) and a leaf \( x \) of \( T' \); that is, \( T \) is obtained from \( T' \) by Operation \( O \). Hence, by definition of the family \( G_0^m \), we have \( T \in G_0^m \), as desired. This completes the necessity part of the proof of Theorem 8.

\( (\Leftarrow) \) Conversely, assume that \( T \in G_0^m \), where \( m \geq 0 \). Recall that \( T \) is a tree of order \( n \geq 2 \) with \( \ell \) leaves. Thus, \( T \) is obtained from a sequence \( T_1, \ldots, T_q \) of trees, where \( q \geq 1 \) and where the tree \( T_1 \in T_0^{m,1} \cup T_0^{m,2} \), and the tree \( T = T_q \). Further, if \( q \geq 2 \), then for each \( i \in [q] \setminus \{1\} \), the tree \( T_i \) can be obtained from the tree \( T_{i-1} \) by applying the following Operation \( O \). We proceed by induction on \( q \geq 1 \), namely second-induction, to show that \( \gamma_i(T) = \frac{1}{3}(n - \ell + 2 + m) \).

**Claim 3.** If \( q = 1 \), then \( \gamma_1(T) = \gamma(T) = \frac{1}{3}(n - \ell + 2 + m) \).

**Proof.** Suppose that \( q = 1 \). Thus, \( T_1 \in T_0^{m,1} \cup T_0^{m,2} \). We consider the two possibilities in turn, and in both cases we will show that the tree \( T \in G_0^m \) satisfies \( \gamma(T) = \frac{1}{3}(n - \ell + 2 + m) \).

**Claim 3.1.** If \( T \in T_0^{m,1} \), then \( \gamma_1(T) = \frac{1}{3}(n - \ell + 2 + m) \).

**Proof.** Suppose that \( T \in T_0^{m,1} \). Thus, \( T \) is obtained from a tree \( T' \in G_0^{m-2} \) by adding a vertex disjoint copy of a star \( Q \) with \( t \geq 2 \) vertices and joining the center of \( Q \), say \( y \), to a special vertex \( x \) in \( T' \). Let \( T' \) have order \( n' \), and so \( n' = n - t \). Further, let \( T' \) have \( \ell' \) leaves. Since \( x \) is a non-leaf of \( T' \), we have \( \ell' = \ell - (t - 1) \). Applying the first-induction hypothesis to the tree \( T' \in G_0^{m-2} \), we have \( \gamma_1(T') = \frac{1}{3}(n' - \ell' + 2 + (m - 2)) = \frac{1}{3}(n' - \ell' + m) \).

We show next that \( \gamma(T) = \gamma(T') + 1 \). Since \( x \) is a special vertex of \( T' \), we note that \( \gamma(T'-x) \geq \gamma(T') \). Every \( \gamma \)-set of \( T' \) can be extended to a dominating set of \( T \) by adding to it the vertex \( y \), implying that \( \gamma(T) \leq \gamma(T') + 1 \). Conversely, we can choose a \( \gamma \)-set, say \( D \), of \( T \) to contain the vertex \( y \) which dominates the star \( Q \). If \( x \in D \), then \( D \setminus \{y\} \) is a dominating set of \( T' \), and so \( \gamma(T') \leq |D| - 1 \). If \( x \notin D \), then \( D \setminus \{y\} \) is a dominating set of \( T' \setminus x \), and so \( \gamma(T') \leq |T' \setminus x| \leq |D| - 1 \). In both cases, \( \gamma(T') \leq |D| - 1 = \gamma(T) - 1 \). Consequently, \( \gamma(T) = \gamma(T') + 1 \). Thus,

\[
\gamma(T) = \gamma(T') + 1 \\
= \frac{1}{3}(n' - \ell' + m) + 1 \\
= \frac{1}{3}((n - t) - (\ell - t + 1) + m) + 1 \\
= \frac{1}{3}(n - \ell + 2 + m).
\]

This completes the proof of Claim 3.1. \( \square \)
Claim 3.2. If $T \in T_{0}^{m,2}$, then $\gamma_{t}(T) = \frac{1}{3}(n - \ell + 2 + m)$.

Proof. Suppose that $T \in T_{0}^{m,2}$. Thus, $T$ is obtained from a tree $T' \in G_{0}^{m-1}$ by adding a vertex disjoint copy of a star $Q$ with $t \geq 3$ vertices and joining a leaf, say $v$, of $Q$ to a non-leaf, say $w$, in $T'$. Let $u$ be the center of the star $Q$. Let $T'$ have order $n'$, and so $n' = n - t$. Further, let $T'$ have $\ell'$ leaves. Since $w$ is a non-leaf of $T'$, we have $\ell' = \ell - (t - 2)$. Applying the first-induction hypothesis to the tree $T' \in G_{0}^{m-1}$, we have $\gamma_{t}(T') = \frac{1}{3}(n' - \ell' + 2 + (m - 1)) = \frac{1}{3}(n' - \ell' + m + 1)$. We show next that $\gamma(T) = \gamma_{t}(T') + 1$. Every $\gamma$-set of $T'$ can be extended to a dominating of $T$ by adding to it the vertex $u$, implying that $\gamma(T) \leq \gamma(T') + 1$. By Observation 2, there exists a $\gamma$-set $D$ of $T$ that contains no leaf of $G$. Thus, $u \in D$. If $v \in D$, then we can replace $v$ in $D$ with the vertex $w$. Hence we may assume that $v \notin D$, implying that $D \setminus \{u\}$ is a dominating set of $T'$, and so $\gamma(T') \leq |D| - 1 = \gamma(T) - 1$. Consequently, $\gamma(T) = \gamma(T') + 1$. Thus, $\gamma(T) = \gamma(T') + 1 = \frac{1}{3}(n' - \ell' + m + 1) + 1 = \frac{1}{3}((n - t) - (\ell - t + 2) + m) + 1 = \frac{1}{3}(n - \ell + 2 + m)$. This completes the proof of Claim 3.2. □

By Claims 3.1 and 3.2, if $T \in T_{0}^{m,1} \cup T_{0}^{m,2}$, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$. This completes the proof of Claim 3. □

By Claim 3, if $q = 1$, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$. This establishes the base step of the second-induction. Let $q \geq 2$ and assume that if $q'$ is an integer where $1 \leq q' < q$ and if $T' \in G_{0}^{n'}$ is a tree of order $n' \geq 2$ with $\ell'$ leaves obtained from a sequence of $q'$ trees, then $\gamma(T) = \frac{1}{3}((n' - \ell') + 2 + m)$. Recall that $T$ is obtained from a sequence $T_{1}, \ldots, T_{q}$ of trees, where $q \geq 1$ and where the tree $T_{1} \in T_{0}^{m,1} \cup T_{0}^{m,2}$, and the tree $T = T_{q}$. Further for each $i \in [q] \setminus \{1\}$, the tree $T_{i}$ can be obtained from the tree $T_{i-1}$ by applying the Operation $O$.

We now consider the tree $T' = T_{q-1}$. Thus, the tree $T \in G_{0}^{n'}$ is obtained from the tree $T'$ by adding a vertex disjoint copy of a star $Q$ with $t \geq 3$ vertices and adding an edge joining a leaf of $Q$ to a leaf of $T'$. Let $T'$ have order $n'$ and let $T'$ have $\ell'$ leaves. We note that $n' = n - t$ and $\ell' = \ell - (t - 2) + 1 = \ell - t + 3$. Applying the second-induction hypothesis to the tree $T' \in G_{0}^{n'}$, we have $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m)$. Analogous arguments as before show that $\gamma(T) = \gamma_{t}(T') + 1$. Thus, $\gamma(T) = \gamma(T') + 1 = \frac{1}{3}(n' - \ell' + 2 + m) + 1 = \frac{1}{3}((n - t) - (\ell - t + 3) + 2 + m) + 1 = \frac{1}{3}(n - \ell + 2 + m)$. □
Hence we have shown that if \( T \in G_0^m \), where \( m \geq 0 \) and where \( T \) has order \( n \geq 2 \) with \( \ell \) leaves, then \( \gamma(T) = \frac{1}{3}(n - \ell + 2 + m) \). This completes the proof of Theorem 8.

We are now in a position to prove our main result, namely Theorem 1. Recall its statement.

**Theorem 1.** Let \( m \geq 0 \) be an integer. If \( G \) is a cactus graph of order \( n \geq 2 \) with \( k \geq 0 \) cycles and \( \ell \) leaves, then \( \gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m) \), if and only if \( G \in G_k^m \).

**Proof.** Let \( m \geq 0 \) be an integer, and let \( G \) be a cactus graph of order \( n \geq 2 \) with \( k \geq 0 \) cycles and \( \ell \) leaves. We proceed by induction on \( k \) to show that \( \gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m) \) if and only if \( G \in G_k^m \). If \( k = 0 \), then the result follows from Theorem 8. This establishes the base case. Let \( k \geq 1 \) and assume that if \( G' \) is a cactus graph of order \( n' \geq 2 \) with \( k' \geq 0 \) cycles and \( \ell' \) leaves where \( 0 \leq k' < k \), then \( \gamma(G) = \frac{1}{3}(n' - \ell' + 2(1 - k') + m') \) if and only if \( G \in G_{k'}^{m'} \). Let \( G \) be a cactus graph of order \( n \geq 2 \) with \( k \geq 0 \) cycles and \( \ell \) leaves. We will show that \( \gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m) \), if and only if \( G \in G_k^m \). If \( m = 0 \), then the result follows by Theorem 6(a). If \( m = 1 \), then the result follows by Theorem 6(b). If \( m = 2 \), then the result follows by Theorem 6(c). Thus, we may assume that \( m \geq 3 \), for otherwise the desired result follows.

\[ (\Rightarrow) \text{ Assume that } \gamma(G) = \frac{1}{3}(n - \ell + 2 + m - 2k) \text{ (where we recall that here } m \geq 3). \text{ We will show that } T \in G_k^m. \text{ By Lemma 2, the graph } G \text{ contains a cycle edge } e \text{ such that } \gamma(G - e) = \gamma(G). \text{ Let } e = uv, \text{ and consider the graph } G' = G - e. \text{ Let } G' \text{ have order } n' \text{ with } k' \geq 0 \text{ cycles and } \ell' \text{ leaves. We note that } n' = n. \text{ Further, since } G \text{ is a cactus graph, } k' = k - 1. \text{ Removing the cycle edge } e \text{ from } G \text{ produces at most two new leaves, namely the ends of the edge } e, \text{ implying that } \ell' - 2 \leq \ell \leq \ell'. \text{ By Corollary 7, we have } \gamma(G') = \frac{1}{3}(n' - \ell' + 2 + m' - 2k') \text{ for some integer } m' \geq 0. \text{ Applying the inductive hypothesis to the cactus graph } G', \text{ we have that } G' \in G_k^{m'} = G_{k-1}^{m'}. \text{ Our earlier observations imply that } \]

\[
\frac{1}{3}(n - \ell + 2 + m - 2k) = \gamma(G) = \gamma(G') = \frac{1}{3}(n' - \ell' + 2 + m' - 2k') = \frac{1}{3}(n - \ell + 2 + m' - 2(k - 1)),
\]

and so \( m - \ell = m' - \ell' + 2 \). Since \( G \) is a cactus, the vertices \( u \) and \( v \) are connected in \( G' = G - e \) by a unique path. As observed earlier, \( \ell - 2 \leq \ell \leq \ell' \).

Suppose that \( \ell = \ell' \). In this case, neither \( u \) nor \( v \) is a leaf of \( G' \), implying that both \( u \) and \( v \) have degree at least 2 in \( G' \). Further, the equation \( m - \ell = m' - \ell' + 2 \) simplifies to \( m' = m - 2 \). Thus, \( G' \in G_{k-1}^{m-2} \). Hence, the graph \( G \) is obtained from \( G' \) by Procedure J and therefore \( G \in G_k^m \).
Suppose that \( \ell = \ell' - 1 \). In this case, exactly one of \( u \) and \( v \) is a leaf of \( G' \). Further, the equation \( m - \ell = m' - \ell' + 2 \) simplifies to \( m' = m - 1 \). Thus, \( G' \in \mathcal{G}_{k-1}^{m-1} \). Hence, the graph \( G \) is obtained from \( G' \) by Procedure I, and therefore \( G \in \mathcal{G}_k^{m} \).

Suppose that \( \ell = \ell' - 2 \). In this case, both \( u \) and \( v \) are leaves in \( G' \). Further, the equation \( m - \ell = m' - \ell' + 2 \) simplifies to \( m' = m \). Thus, \( G' \in \mathcal{G}_{k-1}^{m-1} \). Hence, the graph \( G \) is obtained from \( G' \) by Procedure H, and therefore \( G \in \mathcal{G}_k^{m} \). This completes the necessity part of the proof of Theorem 1.

\((\Leftarrow)\) Conversely, assume that \( G \in \mathcal{G}_k^{m} \). Recall that by our earlier assumptions, \( m \geq 3 \) and \( k \geq 1 \). Thus, the graph \( G \) is obtained from either a graph \( G' \in \mathcal{G}_{k-1}^{m} \) by Procedure H or from a graph \( G' \in \mathcal{G}_{k-1}^{m-1} \) by Procedure I or from a graph \( G' \in \mathcal{G}_{k-1}^{m-2} \) by Procedure J. In all three cases, let \( G' \) have order \( n' \) with \( k' \geq 0 \) cycles and \( \ell' \) leaves. Further, in all cases we note that \( n' = n \) and \( k' = k-1 \).

We consider the three possibilities in turn.

Suppose firstly that \( G \) is obtained from a graph \( G' \in \mathcal{G}_{k-1}^{m} \) by Procedure H. In this case, \( \ell = \ell' - 2 \) and \( \gamma(G) = \gamma(G') \). Applying the inductive hypothesis to the graph \( G' \in \mathcal{G}_{k-1}^{m} \), we have \( \gamma(G) = \gamma(G') = \frac{1}{3}(n' - \ell' + 2 + m - 2(k - 1)) = \frac{1}{3}(n - \ell + 2 + m - 2k) \).

Suppose next that \( G \) is obtained from a graph \( G' \in \mathcal{G}_{k-1}^{m-1} \) by Procedure I. In this case, \( \ell = \ell' - 1 \) and \( \gamma(G) = \gamma(G') \). Applying the inductive hypothesis to the graph \( G' \in \mathcal{G}_{k-1}^{m-1} \), we have \( \gamma(G) = \gamma(G') = \frac{1}{3}(n' - \ell' + 2 + (m - 1) - 2(k - 1)) = \frac{1}{3}(n - \ell + 2 + m - 2k) \).

Suppose finally that \( G \) is obtained from a graph \( G' \in \mathcal{G}_{k-1}^{m-2} \) by Procedure J. In this case, \( \ell = \ell' \) and \( \gamma(G) = \gamma(G') \). Applying the inductive hypothesis to the graph \( G' \in \mathcal{G}_{k-1}^{m-2} \), we have \( \gamma(G) = \gamma(G') = \frac{1}{3}(n' - \ell' + 2 + (m - 2) - 2(k - 1)) = \frac{1}{3}(n - \ell + 2 + m - 2k) \). In all three cases, \( \gamma(G) = \frac{1}{3}(n - \ell + 2 + m - 2k) \). This completes the proof of Theorem 1.

\[ \square \]

References


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