A CLASSIFICATION OF CACTUS GRAPHS
ACCORDING TO THEIR DOMINATION NUMBER

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Abstract

A set $S$ of vertices in a graph $G$ is a dominating set of $G$ if every vertex not in $S$ is adjacent to some vertex in $S$. The domination number, $\gamma(G)$, of $G$ is the minimum cardinality of a dominating set of $G$. The authors proved in [A new lower bound on the domination number of a graph, J. Comb. Optim. 38 (2019) 721–738] that if $G$ is a connected graph of order $n \geq 2$ with $k \geq 0$ cycles and $\ell$ leaves, then $\gamma(G) \geq [(n - \ell + 2 - 2k)/3]$. As a consequence of the above bound, $\gamma(G) = (n - \ell + 2(1 - k) + m)/3$ for some integer $m \geq 0$. In this paper, we characterize the class of cactus graphs achieving equality here, thereby providing a classification of all cactus graphs according to their domination number.

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1. Introduction

A dominating set of a graph $G$ is a set $S$ of vertices of $G$ such that every vertex not in $S$ has a neighbor in $S$, where two vertices are neighbors in $G$ if they are adjacent. The minimum cardinality of a dominating set is the domination number of $G$, denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is called a $\gamma$-set of $G$. As remarked in [5], the notion of domination and its variations in graphs has been studied a great deal; a rough estimate says that it occurs in more than 6000 papers to date. For fundamentals of domination theory in graphs we refer the reader to the so-called domination books by Haynes, Hedetniemi, and Slater [6, 7]. An updated glossary of domination parameters can be found in [4].

Two vertices $u$ and $v$ in a graph $G$ are connected if there exists a $(u, v)$-path in $G$. The graph $G$ is connected if every two vertices in $G$ are connected. A block of $G$ is a maximal connected subgraph of $G$ which has no cut-vertex of its own. A cactus is a connected graph in which every edge belongs to at most one cycle. Equivalently, a (nontrivial) cactus is a connected graph in which every block is an edge or a cycle. The distance between two vertices $u$ and $v$ in a connected graph $G$ is the minimum length of a $(u, v)$-path in $G$. The diameter, $\text{diam}(G)$, of $G$ is the maximum distance among pairs of vertices in $G$.

For notation and graph theory terminology we generally follow [8]. In particular, the order of a graph $G$ with vertex set $V(G)$ and edge set $E(G)$ is given by $n(G) = |V(G)|$ and its size by $m(G) = |E(G)|$. A neighbor of a vertex $v$ in $G$ is a vertex adjacent to $v$, and the open neighborhood of $v$ is the set of neighbors of $v$, denoted $N_G(v)$. The closed neighborhood of $v$ is the set $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex $v$ in $G$ is given by $d_G(v) = |N_G(v)|$.

For a set $S$ of vertices in a graph $G$, the subgraph induced by $S$ is denoted by $G[S]$. Further, the subgraph obtained from $G$ by deleting all vertices in $S$ and all edges incident with vertices in $S$ is denoted by $G - S$. If $S = \{v\}$, we simply denote $G - \{v\}$ by $G - v$. A leaf of a graph $G$ is a vertex of degree 1 in $G$, and its unique neighbor is called a support vertex. The set of all leaves of $G$ is denoted by $L(G)$, and we let $\ell(G) = |L(G)|$ be the number of leaves in $G$. We denote the set of support vertices of $G$ by $S(G)$. We call a vertex of degree at least 2 a non-leaf.

Following our notation in [5], we denote the path and cycle on $n$ vertices by $P_n$ and $C_n$, respectively. A complete graph on $n$ vertices is denoted by $K_n$, while a complete bipartite graph with partite sets of size $n$ and $m$ is denoted by $K_{n,m}$. A star is the graph $K_{1,k}$, where $k \geq 1$. Further if $k > 1$, the vertex of degree $k$ is called the center vertex of the star, while if $k = 1$, arbitrarily designate either vertex of $P_2$ as the center. A double star is a tree with exactly two (adjacent) non-leaf vertices.

A rooted tree $T$ distinguishes one vertex $r$ called the root. For each vertex
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v \neq r of T, the parent of v is the neighbor of v on the unique (r, v)-path, while a child of v is any other neighbor of v. A descendant of v is a vertex u \neq v such that the unique (r, u)-path contains v. In particular, every child of v is a descendant of v. We let D(v) denote the set of descendants of v, and we define D[v] = D(v) \cup \{v\}. The maximal subtree at v is the subtree of T induced by D[v], and is denoted by T_v. We use the standard notation \([k] = \{1, \ldots, k\}\).

2. Main Result

Our aim in this paper is to provide a classification of all cactus graphs according to their domination number. For this purpose, we shall use a result of the authors in [5] (which we present in Section 4) that establishes a lower bound on the domination number of a graph in terms of its order, number of vertices of degree 1, and number of cycles. From this result, we prove our desired characterization below, where G^m_k is a family of graphs defined in Section 3.

Theorem 1. Let m \geq 0 be an integer. If G is a cactus graph of order n \geq 2 with k \geq 0 cycles and \ell leaves, then \(\gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m)\), if and only if G \in G^m_k.

We proceed as follows. In Section 3 we define the families G^m_k of graphs for each integer k \geq 0 and m \geq 0. Known results on the domination number are given in Section 4. In Section 5 we present a proof of our main result.

3. The Families G^m_k for m \geq 0 and k \geq 0

In this section, we define the families G^m_k of graphs for each integer k \geq 0 and m \geq 0. The families G^0_k, G^1_k, G^2_k, T^{1,1}_0, T^{2,1}_0 of graphs were defined by the authors in [5]. For completeness, we include these definitions in Sections 3.1 and 3.2. We first define the families G^0_k, G^1_k and G^2_k of graphs in the special case when k = 0.

3.1. The families G^0_0, G^1_0 and G^2_0

Hajian et al. [5] defined the class of trees G^0_0, G^1_0 and G^2_0 as follows.

• Let G^0_0 be the class of all trees T that can be obtained from a sequence T_1, \ldots, T_k of trees where k \geq 1 such that T_1 is a star with at least three vertices, T = T_k, and, if k \geq 2, then the tree T_{i+1} can be obtained from the tree T_i by applying Operation \(\mathcal{O}\) defined below for all i \in [k - 1].

Operation \(\mathcal{O}\). Add a vertex disjoint copy of a star Q_i with at least three vertices to the tree T_i and add an edge joining a leaf of Q_i and a leaf of T_i.
• Let $T_0^{1,1}$ be the class of all trees $T$ that can be obtained from a tree $T' \in \mathcal{G}_0^0$ by adding a vertex disjoint copy of a star with at least three vertices and adding an edge from a leaf of the added star to a non-leaf in $T'$. Now, let $\mathcal{G}_0^1$ be the class of all trees $T$ that can be obtained from a sequence $T_1, \ldots, T_k$ of trees where $k \geq 1$ such that $T_1 \in T_0^{1,1} \cup \{P_2\}$, $T = T_k$, and, if $k \geq 2$, then the tree $T_{i+1}$ can be obtained from the tree $T_i$ by applying Operation $O$ for all $i \in [k-1]$.

• Let $T_0^{2,1}$ be the class of all trees $T$ that can be obtained from a tree $T' \in \mathcal{G}_0^0$ by adding a vertex disjoint copy of a star (with at least two vertices) and adding an edge from the center of the added star to a non-leaf in $T'$. Let $T_0^{2,2}$ be the class of all trees $T$ that can be obtained from a tree $T' \in \mathcal{G}_0^0$ by adding a vertex disjoint copy of a star with at least three vertices and adding an edge from a leaf of the added star to a non-leaf in $T'$. Now, let $\mathcal{G}_0^2$ be the class of all trees $T$ that can be obtained from a sequence $T_1, \ldots, T_k$ of trees, where $k \geq 1$, such that $T_1 \in T_0^{2,1} \cup T_0^{2,2} \cup \{P_1\}$, $T = T_k$, and, if $k \geq 2$, then the tree $T_{i+1}$ can be obtained from the tree $T_i$ by applying Operation $O$ for all $i \in [k-1]$.

3.2. The families $\mathcal{G}_k^0$, $\mathcal{G}_k^1$ and $\mathcal{G}_k^2$ when $k \geq 1$

For $k \geq 1$, Hajian et al. [5] defined the families of graphs $\mathcal{G}_k^0$, $\mathcal{G}_k^1$ and $\mathcal{G}_k^2$ as follows.

• For $k \geq 1$, they recursively defined the family $\mathcal{G}_i^0$ of graphs for each $i \in [k]$ by the following procedure.

Procedure A. For $i \in [k]$, a graph $G_i$ belongs to the family $\mathcal{G}_i^0$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}_{i-1}^0$ and the vertices $x$ and $y$ are leaves in $G_i - e$ that are connected by a unique path in $G_i - e$.

• For $k \geq 1$, they recursively defined the family $\mathcal{G}_i^1$ of graphs for each $i \in [k]$ by the following two procedures.

Procedure B. For $i \in [k]$, a graph $G_i$ belongs to the family $\mathcal{G}_i^1$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}_{i-1}^1$ and the vertices $x$ and $y$ are leaves in $G_i - e$ that are connected by a unique path in $G_i - e$.

Procedure C. For $i \in [k]$, a graph $G_i$ belongs to the family $\mathcal{G}_i^1$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}_{i-1}^0$ and the vertices $x$ and $y$ are connected by a unique path in $G_i - e$. Further, exactly one of $x$ and $y$ is a leaf in $G_i - e$.

• For $k \geq 1$, they recursively defined the family $\mathcal{G}_i^2$ of graphs for each $i \in [k]$ by the following four procedures.

Procedure D. For $i \in [k]$, a graph $G_i$ belongs to the family $\mathcal{G}_i^2$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}_{i-1}^2$ and the vertices $x$ and $y$ are leaves in $G_i - e$ that are connected by a unique path in $G_i - e$. 
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Procedure E. For \( i \in [k] \), a graph \( G_i \) belongs to the family \( G_i^2 \) if it contains an edge \( e = xy \) such that the graph \( G_i - e \) belongs to the family \( G_{i-1}^1 \) and the vertices \( x \) and \( y \) are connected by a unique path in \( G_i - e \). Further, exactly one of \( x \) and \( y \) is a leaf in \( G_i - e \).

Procedure F. For \( i \in [k] \), a graph \( G_i \) belongs to the family \( G_i^2 \) if it contains an edge \( e = xy \) such that the graph \( G_i - e \) belongs to the family \( G_{i-1}^1 \) and the vertices \( x \) and \( y \) are connected by a unique path in \( G_i - e \). Further, both \( x \) and \( y \) are non-leaves in \( G_i - e \).

Procedure G. For \( 2 \leq i \in [k] \), a graph \( G_i \) belongs to the family \( G_i^2 \) if it contains an edge \( e = xy \) such that the graph \( G_i - e \) belongs to the family \( G_{i-2}^1 \) and the vertices \( x \) and \( y \) are connected by exactly two paths in \( G_i - e \). Further, both \( x \) and \( y \) are leaves in \( G_i - e \).

3.3. The family \( G_0^m \) when \( m \geq 3 \)

In this section, we define a family of graphs \( G_0^m \) for each integer \( m \geq 3 \) as follows. We call a non-leaf \( x \) in a tree \( T \) a special vertex if \( \gamma(T - x) \geq \gamma(T) \). For \( m \geq 3 \), we first recursively define the class \( T_0^{m,1} \) and \( T_0^{m,2} \) of trees as follows.

- Let \( T_0^{m,1} \) be the class of all trees \( T \) that can be obtained from a tree \( T' \in G_0^{m-2} \) by adding a vertex disjoint copy of a star \( Q \) and joining the center of \( Q \) to a special vertex in \( T' \).
- Let \( T_0^{m,2} \) be the class of all trees \( T \) that can be obtained from a tree \( T' \in G_0^{m-1} \) by adding a vertex disjoint copy of a star \( Q \) with at least three vertices and joining a leaf of \( Q \) to a non-leaf in \( T' \).

For \( m \geq 3 \), we next recursively define the family \( G_0^m \) of graphs constructed from the families \( G_0^{m-1} \) and \( G_0^{m-2} \) as follows.

- Let \( G_0^m \) be the class of all trees \( T \) that can be obtained from a sequence \( T_1, \ldots, T_q \) of trees, where \( q \geq 1 \) and where the tree \( T_1 \in T_0^{m,1} \cup T_0^{m,2} \) and the tree \( T = T_q \). Further, if \( q \geq 2 \), then for each \( i \in [q] \setminus \{1\} \), the tree \( T_i \) can be obtained from the tree \( T_{i-1} \) by applying the Operation \( O \) defined in Section 3.1.

Operation \( O \). Add a vertex disjoint copy of a star \( Q_i \) with at least three vertices to the tree \( T_i \) and add an edge joining a leaf of \( Q_i \) and a leaf of \( T_i \).

3.4. The family \( G_k^m \) when \( m \geq 3 \) and \( k \geq 1 \)

For \( m \geq 3 \) and \( k \geq 1 \), we construct the family \( G_k^m \) from \( G_{k-1}^{m-2} \), \( G_{k-1}^{m-1} \) and \( G_{k-1}^m \), recursively, as follows.

Procedure H. For \( i \in [k] \), a graph \( G_i \) belongs to the family \( G_i^m \) if it contains an edge \( e = xy \) such that the graph \( G_i - e \) belongs to the family \( G_{i-1}^{m-1} \) and the vertices \( x \) and \( y \) are connected by a unique path in \( G_i - e \) and \( \gamma(G_i) = \gamma(G_i - e) \). Further, both \( x \) and \( y \) are leaves in \( G_i - e \).
**Procedure I.** For $i \in [k]$, a graph $G_i$ belongs to the family $G^m_i$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $G^{m-1}_{i-1}$ and the vertices $x$ and $y$ are connected by a unique path in $G_i - e$ and $\gamma(G_i) = \gamma(G_i - e)$. Further, exactly one of $x$ and $y$ is a leaf in $G_i - e$.

**Procedure J.** For $i \in [k]$, a graph $G_i$ belongs to the family $G^m_i$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $G^{m-2}_{i-1}$ and the vertices $x$ and $y$ are connected by a unique path in $G_i - e$ and $\gamma(G_i) = \gamma(G_i - e)$. Further, both $x$ and $y$ are non-leaves in $G_i - e$.

### 4. Known Results

In this section, we present some preliminary observations and known results. We begin with the following properties of graphs that belong to the families $G^0_k$, $G^1_k$ and $G^2_k$ for $k \geq 0$.

**Observation 1.** The following properties hold in a graph $G \in G^0_k \cup G^1_k \cup G^2_k$, where $k \geq 0$.

(a) The graph $G$ contains exactly $k$ cycles.

(b) The graph $G \in G^0_k \cup G^1_k$ is a cactus graph.

We shall also need the following elementary property of a dominating set in a graph.

**Observation 2.** If $G$ is connected graph of order at least 3, then there exists a $\gamma$-set of $G$ that contains no leaf of $G$.

The following lemma is established in [5].

**Lemma 2** [5]. If $G$ is a connected graph and $C$ is an arbitrary cycle in $G$, then there is an edge $e$ of $C$ such that $\gamma(G - e) = \gamma(G)$.

Several authors obtained bounds on the domination number in terms of different variants of graphs, see for example [1, 2, 3, 6, 9]. Let $\mathcal{R}$ be the family of all trees in which the distance between any two distinct leaves is congruent to 2 modulo 3. Lemańska [9] established the following lower bound on the domination number of a tree in terms of its order and number of leaves.

**Theorem 3** [9]. If $T$ is a tree of order $n \geq 2$ with $\ell$ leaves, then $\gamma(T) \geq (n - \ell + 2)/3$, with equality if and only if $T \in \mathcal{R}$.

Hajian et al. [5] showed that the family $\mathcal{R}$ is precisely the family $G^0_0$; that is, $\mathcal{R} = G^0_0$.

As a consequence of Theorem 3, we have the following result.
Corollary 4 [9]. If \( T \) is a tree of order \( n \geq 2 \) with \( \ell \) leaves, then \( \gamma(T) = \frac{1}{3}(n - \ell + 2 + m) \) for some integer \( m \geq 0 \).

Hajian et al. [5] strengthened the result in Theorem 3 as follows.

Theorem 5 [5]. If \( T \) is a tree of order \( n \geq 2 \) with \( \ell \) leaves, then the following holds.

(a) \( \gamma(T) \geq \frac{1}{3}(n - \ell + 2) \), with equality if and only if \( T \in G_0^0 \).

(b) \( \gamma(T) = \frac{1}{3}(n - \ell + 3) \) if and only if \( T \in G_0^1 \).

(c) \( \gamma(T) = \frac{1}{3}(n - \ell + 4) \) if and only if \( T \in G_0^2 \).

The result of Theorem 5 was generalized in [5] to connected graphs as follows.

Theorem 6 [5]. If \( G \) is a connected graph of order \( n \geq 2 \) with \( k \geq 0 \) cycles and \( \ell \) leaves, then the following holds.

(a) \( \gamma(G) \geq \frac{1}{3}(n - \ell + 2(1 - k)) \), with equality if and only if \( G \in G_k^0 \).

(b) \( \gamma(G) = \frac{1}{3}(n - \ell + 3 - 2k) \) if and only if \( G \in G_k^1 \).

(c) \( \gamma(G) = \frac{1}{3}(n - \ell + 4 - 2k) \) if and only if \( G \in G_k^2 \).

As a consequence of Theorem 6(a), we have the following.

Corollary 7 [5]. If \( G \) is a connected graph of order \( n \geq 2 \) with \( k \geq 0 \) cycles and \( \ell \) leaves, then \( \gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m) \) for some integer \( m \geq 0 \).

5. Proof of Main Result

In this section, we present a proof of our main result, namely Theorem 1. For this purpose, we first prove Theorem 1 in the special case when \( k = 0 \), that is, when the cactus is a tree.

Theorem 8. Let \( m \geq 0 \) be an integer. If \( T \) is a tree of order \( n \geq 2 \) with \( \ell \) leaves, then \( \gamma(T) = \frac{1}{3}(n - \ell + 2 + m) \) if and only if \( T \in G_0^m \).

Proof. Let \( T \) be a tree of order \( n \geq 2 \) with \( \ell \) leaves. We proceed by induction on \( m \geq 0 \), namely first-induction, to show that \( \gamma(T) = \frac{1}{3}(n - \ell + 2 + m) \), if and only if \( T \in G_0^m \). For the base step of the first-induction let \( m = 2 \). If \( m = 0 \), then the result follows by Theorem 5(a). If \( m = 1 \), then the result follows by Theorem 5(b). If \( m = 2 \), then the result follows by Theorem 5(c). This establishes the base step of the induction. Let \( m \geq 3 \) and assume that the result holds for all trees \( T_0 \) of order \( n_0 \) with \( \ell_0 \) leaves, for \( m_0 < m \). Let \( T \) be a tree of order \( n \) and with \( \ell \) leaves. We will show that \( \gamma(T) = \frac{1}{3}(n - \ell + 2 + m) \), if and only if \( T \in G_0^m \).

\((\Rightarrow)\) Assume that \( \gamma(T) = \frac{1}{3}(n - \ell + 2 + m) \), if and only if \( T \in G_0^m \). We show that \( T \in G_0^m \). If \( T = P_2 \), then by the definition of the family
\(G_0^1\), we have \(T \in G_0^1\). Then by Theorem 5(b), \(\gamma(T) = \frac{1}{3}(n - \ell + 2 + 1)\), and so \(m = 1\), a contradiction. Hence we may assume that \(\text{diam}(T) \geq 2\), for otherwise the desired result follows. If \(\text{diam}(T) = 2\), then \(T\) is a star, and by the definition of the family \(G_0^0\), we have \(T \in G_0^0\). Thus by Theorem 5(a), \(\gamma(T) = \frac{1}{3}(n - \ell + 2 + 1)\), and so \(m = 0\), a contradiction. If \(\text{diam}(T) = 2\), then \(T\) is a double star, and by definition of the family \(G_0^2\) we have \(T \in T_0^2 \subseteq G_0^2\). Thus by Theorem 5(c), \(\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)\), and so \(m = 2\), a contradiction. Hence, \(\text{diam}(T) \geq 4\) and \(n \geq 5\).

We now root the tree \(T\) at a vertex \(r\) at the end of a longest path \(P\) in \(T\).

Let \(u\) be a vertex at maximum distance from \(r\), and so \(d_T(u, r) = \text{diam}(T)\). Necessarily, \(r\) and \(u\) are leaves. Let \(v\) be the parent of \(u\), let \(w\) be the parent of \(v\), let \(x\) be the parent of \(w\), and let \(y\) be the parent of \(x\). Possibly, \(y = r\). Since \(u\) is a vertex at maximum distance from the root \(r\), every child of \(v\) is a leaf. By Observation 2, there exists a \(\gamma\)-set, say \(S\), of \(T\) that contains no leaf of \(T\); that is, \(L(T) \cap S = \emptyset\). In particular, we note that \(|S| = \gamma(T) = \frac{1}{3}(n - \ell + 2 + m)\). In order to dominate the vertex \(u\), we note therefore that \(v \in S\). Let \(d_T(v) = t\). We note that \(t \geq 2\).

**Claim 1.** If \(d_T(w) \geq 3\), then \(T \in G_0^m\).

**Proof.** Suppose that \(d_T(w) \geq 3\). In this case, we consider the tree \(T' = T - V(T_v)\), where \(T_v\) is the maximal subtree at \(v\). Let \(T'\) have order \(n'\) and let \(T'\) have \(\ell'\) leaves. We note that \(n' = n - t\). Since \(w\) is not a leaf in \(T'\), we have \(\ell' = \ell - (t - 1) = \ell - t + 1\). By Corollary 4, \(\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m')\) for some integer \(m' \geq 0\). If a child of \(w\) is a leaf in \(T'\), then since the dominating set \(S\) contains no leaves, we have that \(w \in S\). If no child of \(w\) is a leaf in \(T'\), then every child of \(w\) is a support vertex and therefore belongs to the set \(S\). In both cases, we note that the set \(S \setminus \{v\}\) is a dominating set of \(T'\), implying that \(\gamma(T') \leq |S| - 1 = \gamma(T) - 1\). Every \(\gamma\)-set of \(T'\) can be extended to a dominating set of \(T\) by adding to it the vertex \(v\), implying that \(\gamma(T) \leq \gamma(T') + 1\). Consequently, \(\gamma(T') = \gamma(T) - 1\). Thus,

\[
\gamma(T') = \gamma(T) - 1
\]
\[
= \frac{1}{3}(n - \ell + 2 + m) - 1
\]
\[
= \frac{1}{3}(n - \ell + m - 1)
\]
\[
= \frac{1}{3}((n' + t) - (\ell' + t - 1) + m - 1)
\]
\[
= \frac{1}{3}(n' - \ell' + m).
\]

As observed earlier, \(\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m')\) for some integer \(m' \geq 0\). Thus, \(m' = m - 2\). Applying the inductive hypothesis to the tree \(T'\), we have \(T' \in G_0^{m-2}\). Let \(v'\) be a child of \(w\) different from \(v\). We note that the tree \(T_{v'}\) is a component of \(T' - w\) and this component is dominated by the vertex \(v'\). We
can therefore choose a \( \gamma \)-set of \( T' - w \) to contain the vertex \( v' \). Such a \( \gamma \)-set of \( T' - w \) is also a dominating set of \( T' \), implying that \( \gamma(T') \leq \gamma(T' - w) \); that is, the vertex \( w \) is a special vertex of \( T' \). Thus, the tree \( T \) is obtained from the tree \( T' \in G_0^{m-2} \) by adding a vertex disjoint copy of a star \( T_v \) and joining the center \( v \) of \( T_v \) to a special vertex \( w \) in \( T' \). Thus \( T \in T_0^{m,1} \). Consequently, \( T \in G_0^m \). This completes the proof of Claim 1. \( \square \)

By Claim 1, we may assume that \( d_T(w) = 2 \), for otherwise \( T \in G_0^m \) as desired.

We now consider the tree \( \gamma \)

By Corollary 4, \( \gamma(T') = \frac{1}{3}(n' - \ell + 2 + m') \) for some integer \( m' \geq 0 \).

As observed earlier, the vertex \( v \) belongs to the dominating set \( S \). If \( w \in S \), then we can replace \( w \) in \( S \) with the vertex \( x \) to produce a new \( \gamma \)-set of \( T \) that contains no leaf of \( T \). Hence we may assume that \( w \notin S \), implying that the set \( S \setminus \{v\} \) is a dominating set of \( T' \) and therefore \( \gamma(T') \leq |S| - 1 = \gamma(T) - 1 \). Every \( \gamma \)-set of \( T' \) can be extended to a dominating set of \( T \) by adding to it the vertex \( v \), implying that \( \gamma(T) \leq \gamma(T') + 1 \). Consequently, \( \gamma(T') = \gamma(T) - 1 \).

Claim 2. If \( d_T(x) \geq 3 \), then \( T \in G_0^m \).

Proof. Suppose that \( d_T(x) \geq 3 \). In this case, the vertex \( x \) is not a leaf of \( T' \), implying that \( \ell' = \ell - (t - 1) = \ell - t + 1 \). Thus,

\[
\gamma(T') = \gamma(T) - 1 \\
= \frac{1}{3}(n - \ell + m - 1) \\
= \frac{1}{3}(n' + t + 1) - (\ell' + t - 1) + m - 1 \\
= \frac{1}{3}(n' - \ell' + m + 1).
\]

As observed earlier, \( \gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m') \) for some integer \( m' \geq 0 \).

Thus, \( m' = m - 1 \). Applying the inductive hypothesis to the tree \( T' \), we have \( T' \in G_0^{m-1} \). Thus, the tree \( T \) is obtained from the tree \( T' \in G_0^{m-1} \) by adding a vertex disjoint copy of a star \( T_v \) with at least three vertices and joining a leaf of the star \( T_v \) to the non-leaf \( x \) of \( T' \). Thus \( T \in T_0^{m,2} \). Consequently, \( T \in G_0^m \). \( \square \)

By Claim 2, we may assume that \( d_T(x) = 2 \), for otherwise \( T \in G_0^m \) as desired.

In this case, the vertex \( x \) is a leaf of \( T' \), implying that \( \ell' = \ell - (t - 1) + 1 = \ell - t + 2 \). Thus,

\[
\frac{1}{3}(n' - \ell' + 2 + m') = \gamma(T') = \gamma(T) - 1 \\
= \frac{1}{3}(n - \ell + m - 1) \\
= \frac{1}{3}(n' + t + 1) - (\ell' + t - 2) + m - 1 \\
= \frac{1}{3}(n' - \ell' + m + 2),
\]
and so \( m = m' \). Applying the inductive hypothesis to the tree \( T' \), we have \( T' \in G^m_0 \). Thus, the tree \( T \) is obtained from the tree \( T' \in G^m_0 \) by adding a vertex disjoint copy of a star \( T_v \) with at least three vertices and adding the edge \( xw \) joining a leaf \( w \) of \( T_v \) and a leaf \( x \) of \( T' \); that is, \( T \) is obtained from \( T' \) by Operation \( O \). Hence, by definition of the family \( G^m_0 \), we have \( T \in G^m_0 \), as desired. This completes the necessity part of the proof of Theorem 8.

\((\Leftarrow\Rightarrow)\) Conversely, assume that \( T \in G^m_0 \), where \( m \geq 0 \). Recall that \( T \) is a tree of order \( n \geq 2 \) with \( \ell \) leaves. Thus, \( T \) is obtained from a sequence \( T_1, \ldots, T_q \) of trees, where \( q \geq 1 \) and where the tree \( T_1 \in T^m_0 \cup T^m_0 \), and the tree \( T = T_q \). Further, if \( q \geq 2 \), then for each \( i \in [q] \setminus \{1\} \), the tree \( T_i \) can be obtained from the tree \( T_{i-1} \) by applying the following Operation \( O \). We proceed by induction on \( q \geq 1 \), namely second-induction, to show that \( \gamma_i(T) = \frac{1}{3}(n - \ell + 2 + m) \).

Claim 3.1. If \( q = 1 \), then \( \gamma_i(T) = \gamma(T) = \frac{1}{3}(n - \ell + 2 + m) \).

**Proof.** Suppose that \( q = 1 \). Thus, \( T_1 \in T^m_0 \cup T^m_0 \). We consider the two possibilities in turn, and in both cases we will show that the tree \( T \in G^m_0 \) satisfies \( \gamma(T) = \frac{1}{3}(n - \ell + 2 + m) \).

Claim 3.1. If \( T \in T^m_0 \), then \( \gamma_i(T) = \frac{1}{3}(n - \ell + 2 + m) \).

**Proof.** Suppose that \( T \in T^m_0 \). Thus, \( T \) is obtained from a tree \( T' \in G^m_0 \) by adding a vertex disjoint copy of a star \( Q \) with \( t \geq 2 \) vertices and joining the center of \( Q \), say \( y \), to a special vertex \( x \) in \( T' \). Let \( T' \) have order \( n' \), and so \( n' = n - t \). Further, let \( T' \) have \( \ell' \) leaves. Since \( x \) is a non-leaf of \( T' \), we have \( \ell' = \ell - (t - 1) \). Applying the first-induction hypothesis to the tree \( T' \in G^m_0 \), we have \( \gamma_i(T') = \frac{1}{3}(n' - \ell' + 2 + (m - 2)) = \frac{1}{3}(n' - \ell' + m) \).

We show next that \( \gamma(T) = \gamma(T') + 1 \). Since \( x \) is a special vertex of \( T' \), we note that \( \gamma(T' - x) \geq \gamma(T') \). Every \( \gamma \)-set of \( T' \) can be extended to a dominating set of \( T \) by adding to it the vertex \( y \), implying that \( \gamma(T) \leq \gamma(T') + 1 \). Conversely, we can choose a \( \gamma \)-set, say \( D \), of \( T \) to contain the vertex \( y \) which dominates the star \( Q \). If \( x \in D \), then \( D \setminus \{y\} \) is a dominating set of \( T' \), and so \( \gamma(T') \leq |D| - 1 \). If \( x \notin D \), then \( D \setminus \{y\} \) is a dominating set of \( T' - x \), and so \( \gamma(T') \leq |T' - x| \leq |D| - 1 \). In both cases, \( \gamma(T') \leq |D| - 1 = \gamma(T) - 1 \). Consequently, \( \gamma(T) = \gamma(T') + 1 \). Thus,

\[
\gamma(T) = \gamma(T') + 1
= \frac{1}{3}(n' - \ell' + m) + 1
= \frac{1}{3}((n - t) - (\ell - t + 1) + m) + 1
= \frac{1}{3}(n - \ell + 2 + m).
\]

This completes the proof of Claim 3.1. \( \square \)
Claim 3.2. If $T \in T_0^{m,2}$, then $\gamma_{t}(T) = \frac{1}{3}(n - \ell + 2 + m)$.

Proof. Suppose that $T \in T_0^{m,2}$. Thus, $T$ is obtained from a tree $T' \in G_0^{m-1}$ by adding a vertex disjoint copy of a star $Q$ with $t \geq 3$ vertices and joining a leaf, say $v$, of $Q$ to a non-leaf, say $w$, in $T'$. Let $u$ be the center of the star $Q$. Let $T'$ have order $n'$, and so $n' = n - t$. Further, let $T'$ have $\ell'$ leaves. Since $w$ is a non-leaf of $T'$, we have $\ell' = \ell - (t - 2)$. Applying the first-induction hypothesis to the tree $T' \in G_0^{m-1}$, we have $\gamma_{t}(T') = \frac{1}{3}(n' - \ell' + 2 + m) = \frac{1}{3}(n' - \ell' + m + 1)$.

We show next that $\gamma(T) = \gamma(T') + 1$. Every $\gamma$-set of $T'$ can be extended to a dominating of $T$ by adding to it the vertex $u$, implying that $\gamma(T) \leq \gamma(T') + 1$. By Observation 2, there exists a $\gamma$-set $D$ of $T$ that contains no leaf of $G$. Thus, $u \in D$. If $v \in D$, then we can replace $v$ in $D$ with the vertex $w$. Hence we may assume that $v \notin D$, implying that $D \setminus \{u\}$ is a dominating set of $T'$, and so $\gamma(T') \leq |D| - 1 = \gamma(T) - 1$. Consequently, $\gamma(T) = \gamma(T') + 1$. Thus,

$$
\gamma(T) = \gamma(T') + 1 \\
= \frac{1}{3}(n' - \ell' + m + 1) + 1 \\
= \frac{1}{3}((n - t) - (\ell - t + 2) + m + 1) + 1 \\
= \frac{1}{3}(n - \ell + 2 + m).
$$

This completes the proof of Claim 3.2. 

By Claims 3.1 and 3.2, if $T \in T_0^{m,1} \cup T_0^{m,2}$, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$. This completes the proof of Claim 3. 

By Claim 3, if $q = 1$, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$. This establishes the base step of the second-induction. Let $q \geq 2$ and assume that if $q'$ is an integer where $1 \leq q' < q$ and if $T' \in G_0^{m}$ is a tree of order $n' \geq 2$ with $\ell'$ leaves obtained from a sequence of $q'$ trees, then $\gamma(T) = \frac{1}{3}(n' - \ell' + 2 + m)$. Recall that $T$ is obtained from a sequence $T_1, \ldots, T_q$ of trees, where $q \geq 1$ and where the tree $T_1 \in T_0^{m,1} \cup T_0^{m,2}$, and the tree $T = T_q$. Further for each $i \in [q] \setminus \{1\}$, the tree $T_i$ can be obtained from the tree $T_{i-1}$ by applying the Operation $O$.

We now consider the tree $T' = T_{q-1}$. Thus, the tree $T \in G_0^{m}$ is obtained from the tree $T'$ by adding a vertex disjoint copy of a star $Q$ with $t \geq 3$ vertices and adding an edge joining a leaf of $Q$ to a leaf of $T'$. Let $T'$ have order $n'$ and let $T'$ have $\ell'$ leaves. We note that $n' = n - t$ and $\ell' = \ell - (t - 2) + 1 = \ell - t + 3$. Applying the second-induction hypothesis to the tree $T' \in G_0^{m}$, we have $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m)$. Analogous arguments as before show that $\gamma(T) = \gamma_{t}(T') + 1$. Thus,

$$
\gamma(T) = \gamma(T') + 1 \\
= \frac{1}{3}(n' - \ell' + 2 + m) + 1 \\
= \frac{1}{3}((n - t) - (\ell - t + 3) + 2 + m) + 1 \\
= \frac{1}{3}(n - \ell + 2 + m).
$$
Hence we have shown that if \( T \in \mathcal{G}_m^0 \), where \( m \geq 0 \) and where \( T \) has order \( n \geq 2 \) with \( \ell \) leaves, then \( \gamma(T) = \frac{1}{3}(n - \ell + 2 + m) \). This completes the proof of Theorem 8.

We are now in a position to prove our main result, namely Theorem 1. Recall its statement.

**Theorem 1.** Let \( m \geq 0 \) be an integer. If \( G \) is a cactus graph of order \( n \geq 2 \) with \( k \geq 0 \) cycles and \( \ell \) leaves, then \( \gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m) \), if and only if \( G \in \mathcal{G}_m^k \).

**Proof.** Let \( m \geq 0 \) be an integer, and let \( G \) be a cactus graph of order \( n \geq 2 \) with \( k \geq 0 \) cycles and \( \ell \) leaves. We proceed by induction on \( k \) to show that \( \gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m) \) if and only if \( G \in \mathcal{G}_m^k \). If \( k = 0 \), then the result follows from Theorem 8. This establishes the base case. Let \( k \geq 1 \) and assume that if \( G' \) is a cactus graph of order \( n' \geq 2 \) with \( k' \geq 0 \) cycles and \( \ell' \) leaves where \( 0 \leq k' < k \), then \( \gamma(G') = \frac{1}{3}(n' - \ell' + 2(1 - k') + m') \) if and only if \( G' \in \mathcal{G}_m^{k'} \).

Let \( G \) be a cactus graph of order \( n \geq 2 \) with \( k \geq 0 \) cycles and \( \ell \) leaves. We will show that \( \gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m) \), if and only if \( G \in \mathcal{G}_m^k \). If \( m = 0 \), then the result follows by Theorem 6(a). If \( m = 1 \), then the result follows by Theorem 6(b). If \( m = 2 \), then the result follows by Theorem 6(c). Thus, we may assume that \( m \geq 3 \), for otherwise the desired result follows.

\[ (\implies) \] Assume that \( \gamma(G) = \frac{1}{3}(n - \ell + 2 + m - 2k) \) (where we recall that here \( m \geq 3 \)). We will show that \( T \in \mathcal{G}_m^k \). By Lemma 2, the graph \( G \) contains a cycle edge \( e \) such that \( \gamma(G - e) = \gamma(G) \). Let \( e = uv \), and consider the graph \( G' = G - e \). Let \( G' \) have order \( n' \) with \( k' \geq 0 \) cycles and \( \ell' \) leaves. We note that \( n' = n \). Further, since \( G \) is a cactus graph, \( k' = k - 1 \). Removing the cycle edge \( e \) from \( G \) produces at most two new leaves, namely the ends of the edge \( e \), implying that \( \ell' - 2 \leq \ell \leq \ell' \). By Corollary 7, we have \( \gamma(G') = \frac{1}{3}(n' - \ell' + 2 + m' - 2k') \) for some integer \( m' \geq 0 \). Applying the inductive hypothesis to the cactus graph \( G' \), we have that \( G' \in \mathcal{G}_m^{m'} = \mathcal{G}_m^{k-1} \). Our earlier observations imply that

\[
\frac{1}{3}(n - \ell + 2 + m - 2k) = \gamma(G) = \gamma(G') = \frac{1}{3}(n' - \ell' + 2 + m' - 2k') = \frac{1}{3}(n - \ell' + 2 + m' - 2(k - 1)),
\]

and so \( m - \ell = m' - \ell' + 2 \). Since \( G \) is a cactus, the vertices \( u \) and \( v \) are connected in \( G' = G - e \) by a unique path. As observed earlier, \( \ell' - 2 \leq \ell \leq \ell' \).

Suppose that \( \ell = \ell' \). In this case, neither \( u \) nor \( v \) is a leaf of \( G' \), implying that both \( u \) and \( v \) have degree at least 2 in \( G' \). Further, the equation \( m - \ell = m' - \ell' + 2 \) simplifies to \( m' = m - 2 \). Thus, \( G' \in \mathcal{G}_m^{m-2} \). Hence, the graph \( G \) is obtained from \( G' \) by Procedure J and therefore \( G \in \mathcal{G}_m^k \).
Suppose that \( \ell = \ell' - 1 \). In this case, exactly one of \( u \) and \( v \) is a leaf of \( G' \). Further, the equation \( m - \ell = m' - \ell' + 2 \) simplifies to \( m' = m - 1 \). Thus, \( G' \in \mathcal{G}_{k-1}^{m-1} \). Hence, the graph \( G \) is obtained from \( G' \) by Procedure I, and therefore \( G \in \mathcal{G}_{k}^{m} \).

Suppose that \( \ell = \ell' - 2 \). In this case, both \( u \) and \( v \) are leaves in \( G' \). Further, the equation \( m - \ell = m' - \ell' + 2 \) simplifies to \( m' = m \). Thus, \( G' \in \mathcal{G}_{k-1}^{m} \). Hence, the graph \( G \) is obtained from \( G' \) by Procedure H, and therefore \( G \in \mathcal{G}_{k}^{m} \). This completes the necessity part of the proof of Theorem 1.

\((\Leftarrow)\) Conversely, assume that \( G \in \mathcal{G}_{k}^{m} \). Recall that by our earlier assumptions, \( m \geq 3 \) and \( k \geq 1 \). Thus, the graph \( G \) is obtained from either a graph \( G' \in \mathcal{G}_{k-1}^{m} \) by Procedure H or from a graph \( G' \in \mathcal{G}_{k-1}^{m-1} \) by Procedure I or from a graph \( G' \in \mathcal{G}_{k-1}^{m-2} \) by Procedure J. In all three cases, let \( G' \) have order \( n' \) with \( k' \geq 0 \) cycles and \( \ell' \) leaves. Further, in all cases we note that \( n' = n \) and \( k' = k - 1 \).

We consider the three possibilities in turn.

Suppose firstly that \( G \) is obtained from a graph \( G' \in \mathcal{G}_{k-1}^{m} \) by Procedure H. In this case, \( \ell = \ell' - 2 \) and \( \gamma(G) = \gamma(G') \). Applying the inductive hypothesis to the graph \( G' \in \mathcal{G}_{k-1}^{m} \), we have \( \gamma(G) = \gamma(G') = \frac{1}{3}(n' - \ell + 2 + m - 2(k - 1)) = \frac{1}{3}(n - \ell + 2 + m - 2k) \).

Suppose next that \( G \) is obtained from a graph \( G' \in \mathcal{G}_{k-1}^{m-1} \) by Procedure I. In this case, \( \ell = \ell' - 1 \) and \( \gamma(G) = \gamma(G') \). Applying the inductive hypothesis to the graph \( G' \in \mathcal{G}_{k-1}^{m-1} \), we have \( \gamma(G) = \gamma(G') = \frac{1}{3}(n' - \ell' + 2 + (m - 1) - 2(k - 1)) = \frac{1}{3}(n - \ell + 3 + m - 2k) \).

Suppose finally that \( G \) is obtained from a graph \( G' \in \mathcal{G}_{k-1}^{m-2} \) by Procedure J. In this case, \( \ell = \ell' \) and \( \gamma(G) = \gamma(G') \). Applying the inductive hypothesis to the graph \( G' \in \mathcal{G}_{k-1}^{m-2} \), we have \( \gamma(G) = \gamma(G') = \frac{1}{3}(n' - \ell' + 2 + (m - 2) - 2(k - 1)) = \frac{1}{3}(n - \ell + 2 + m - 2k) \). In all three cases, \( \gamma(G) = \frac{1}{3}(n - \ell + 2 + m - 2k) \). This completes the proof of Theorem 1.

\[ \square \]

References


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