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GRAPHS WITH ALL BUT TWO EIGENVALUES IN $[-2, 0]$

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Abstract

The eigenvalues of a graph are those of its adjacency matrix. Recently, Cioabă, Haemers and Vermette characterized all graphs with all but two eigenvalues equal to -2 and 0. In this article, we extend their result by characterizing explicitly all graphs with all but two eigenvalues in the interval $[-2, 0]$. Also, we determine among them those that are determined by their spectrum.

Keywords: graph spectrum, complete multipartite graph, graph determined by its spectrum.

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1. Introduction

Throughout the paper, assume that all graphs are simple, finite and undirected. We denote a graph by $G = (V, E)$, where $V(G)$ is the vertex set such that $|V(G)| = n$ and $E(G)$ is the edge set. The *adjacency matrix* of G, denoted by $A(G) = [a_{ij}]$, is the $(0, 1)$-matrix with entries $a_{ij} = 1$ if $\{v_i, v_j\} \in E(G)$ and $a_{ij} = 0$ otherwise, for $i, j = 1, \ldots, n$. Let $f_M(x) = \det(xI - M)$ be the characteristic polynomial of a matrix M. The characteristic polynomial of the graph G is the one of its adjacency matrix $A(G)$ and is denoted by $f_G(x)$. The zeros of $f_G(x)$ are said to be the *eigenvalues* of G. Since $A(G)$ is a real symmetric matrix, all eigenvalues of G are real. We arrange the eigenvalues of G in non-increasing order and denote them by $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$. The multiset of eigenvalues of $A(G)$ is the spectrum of G and is denoted by $spec(G) = \left\{\lambda_1^{(m(\lambda_1))}, \lambda_2^{(m(\lambda_2))}, \ldots, \lambda_s^{(m(\lambda_s))}\right\}$, where $m(\lambda_j)$ is the multiplicity of the eigenvalue λ_j, for $1 \leq j \leq s$. The second largest eigenvalue of a graph G, $\lambda_2(G)$, has been intensively studied in the literature. In particular, many papers have addressed the problem of characterizing graphs G such that $\lambda_2(G) \leq r$ for a real r, where $r \in \{1/3, \sqrt{2}-1, (\sqrt{5}-1)/2, 1, \sqrt{2}, \sqrt{3}, (\sqrt{5}+1)/2, 2\}$ as one can see in [2, 5, 7, 12, 13, 14, 15, 16, 18]. Here, we consider a more general problem:

"For given real numbers r_1 and r_2, find all graphs G such that $\lambda_2(G) \leq r_2$ and $\lambda_{n-1}(G) \geq r_1$".

Some work has been done for the case $r_2 = -r_1 = 1$. Cioabă, Haemers, Vermette and Wong in [4] determined the connected non-bipartite graphs with all but two eigenvalues in $\{-1, 1\}$ and de Lima, Mohammadian and Oliveira in [9] obtained all connected nonbipartite graphs with all but two eigenvalues in $[-1, 1]$. For $r_1 = -2$ and $r_2 = 0$, Cioabă, Haemers and Vermette [3] determined all graphs with all but two eigenvalues equal to -2 or 0. We present a generalization of their results by characterizing all graphs such that $\lambda_i \in [-2, 0]$, for $i = 2, \ldots, n-1$, and $\lambda_n < -2$. For simplicity, we group all graphs satisfying that property in the set $\mathcal{C}$.

In the light of Proposition 1.1, due to Smith [17].

Proposition 1.1 [17]. *Let G be a non-complete graph. Then*

$$\lambda_2(G) \geq 0$$

with equality if and only if G is a complete r-partite graph.

All graphs in $\mathcal{C}$ should be complete r-partite possibly with some isolated vertices. However, not all complete r-partite graphs are in $\mathcal{C}$, and we impose some constraints on the size of their parts.

In this paper, we prove that there are infinitely many families of complete multipartite graphs that belong to $\mathcal{C}$. Also, adding isolated vertices to any such graph yields to a graph that belongs to $\mathcal{C}$. Besides, we solve the problem of characterizing all complete multipartite graphs that are determined by their spectrum, which was posed by Ma and Ren in [11]. In particular, we prove that a complete r-partite graph in $\mathcal{C}$ with $r \geq 3$ is determined by its spectrum, i.e., is a DS Graph.

2. ALL GRAPHS IN $\mathcal{C}$

Let $K_{n_1,n_2,\ldots,n_r}$ be the complete r-partite graph with class sizes $n_1 \leq n_2 \leq \cdots \leq n_r$. A subgraph H is forbidden if $H \notin \mathcal{C}$ and any other graph G that contains H as an induced subgraph also does not belong to $\mathcal{C}$. We start by finding one forbidden subgraph among many. Considering the graph $K_{2,3,3}$, its second smallest eigenvalue is given by $\lambda_7(K_{2,3,3}) = \frac{3-\sqrt{57}}{2} \approx -2.275 < -2$ and thus $K_{2,3,3}$ is not in $\mathcal{C}$. By using the well-known Interlacing Theorem, it is possible to confirm that if a graph G has $K_{2,3,3}$ as an induced subgraph, then G does not belong to $\mathcal{C}$. Therefore, that $K_{2,3,3}$ is a forbidden subgraph. As a consequence, we obtain that any complete r-partite graph $G \in \mathcal{C}$ cannot have three or more classes of size greater than or equal to 3, since otherwise it is a supergraph of $K_{2,3,3}$. Also, by explicit enumeration, it is easy to check that none of the subgraphs of $K_{2,3,3}$ belongs to $\mathcal{C}$ except $K_{1,2,3}$. These facts imply that our investigation should be reduced to graphs with classes of size $1, 2$ and 3 such that not more than two classes are of the size greater than or equal to 3. These observations can be summarized in the following three cases.

(A) $r = 2$,

(B) $r \geq 3, n_{r-1} \leq 2$,

(C) $r \geq 3, n_{r-1} \geq 3$, and $n_i = 1$, for $i = 1, \ldots, r-2$.

The general statements (A), (B) and (C) have been proved in Propositions 2.1, 2.2, 2.3, 2.4 and 2.5. Our results show that there are some infinite families of complete r-partite graphs belonging to $\mathcal{C}$ and most of them are DS.

Next, we consider the case where G is a complete bipartite graph.

Proposition 2.1. *Let G be a graph isomorphic to the complete bipartite graph K_{n_1,n_2}. Then $G \in \mathcal{C}$ if and only if G is not isomorphic to one of the following graphs: $K_2, K_{1,2}, K_{1,3}, K_{1,4}$ and $K_{2,2}$.*

Proof. It is well-known that $spec(K_{n_1,n_2}) = \left\{-\sqrt{n_1 n_2},\ 0^{(n_1+n_2-2)}, \sqrt{n_1 n_2}\right\}$. So, $\lambda_{n_1+n_2}(G) \geq -2$ if and only if $n_1 n_2 \leq 4$ and the result follows. ■

Proposition 2.2. *Let G be a graph isomorphic to $K_{n_1,\dots,n_r}$ such that $r \geq 3$ and $n_{r-1} = 1$. Then $G \in \mathcal{C}$ if and only if either $r = 3$ and $n_r \geq 4$ or $r \geq 4$ and $n_r \geq 3$.*

Proof. The graph G can be written as a join of two regular graphs, that is, $G \simeq K_{r-1} \vee n_r K_1$. According to [6], the characteristic polynomial of G is given by

$$f_G(\lambda) = \lambda^{n_r-1}(\lambda+1)^{r-2}(\lambda^2 - (r-2)\lambda - (r-1)n_r).$$

Since $\lambda_n(G) = \frac{r-2-\sqrt{(r-2)^2+4(r-1)n_r}}{2}$, we obtain $\lambda_n(G) < -2$ if and only if $(r-1)(n_r-2) > 2$, and the result follows. ∎

Proposition 2.3. *Let G be a complete r-partite graph with parts $n_1 = n_p = 1$ and $n_{p+1} = n_{r-1} = 2$ such that $p \geq 1$ and $r - p \geq 2$. Then $G \in \mathcal{C}$ if and only if $n_r \geq 3$.*

Proof. Let $G \cong K_{1,\dots,1,2,\dots,2,n_r}$, where the number of ones and twos are $p \geq 1$ and $r - p \geq 2$, respectively. With a convenient vertex labeling, $A(G)$ can be written in the following form

$$A(G) = \left[\begin{array}{c|c|c} J_p - I_p & J_{p\times 2(r-p-1)} & J_{p\times n_r} \\ \hline J_{2(r-p-1)\times p} & R_{2(r-p-1)} & J_{2(r-p-1)\times n_r} \\ \hline J_{n_r\times p} & J_{n_r\times 2(r-p-1)} & \mathbf{0}_{n_r} \end{array}\right],$$

where $R_{2(r-p-1)}$ is the adjacency matrix of the subgraph induced by partitions of size 2. We find that $\mathbf{e}_{n-n_r+1} - \mathbf{e}_j$, for $j = n-n_r+2,\dots,n$ and $\mathbf{e}_{p+2k+1} - \mathbf{e}_{p+2k+2}$, for $k = 0,\dots,r-p-2$ are eigenvectors for the eigenvalue 0 which has multiplicity at least $n_r + r - p - 2$. Also, we find that $\mathbf{e}_1 - \mathbf{e}_j$ for $j = 2,\dots,p$ are eigenvectors for the eigenvalue -1 which has multiplicity at least $p-1$. If $r - p \geq 3$, $\mathbf{e}_{p+1} + \mathbf{e}_{p+2} - \mathbf{e}_{p+2k+2} - \mathbf{e}_{p+2k+3}$ are eigenvectors for -2 with multiplicity at least $r-p-2$ for $k = 1,\dots,r-p-2$. Since $A(G)$ has an equitable partition, the remaining 3 eigenvalues of $A(G)$ are the eigenvalues of the matrix

$$M = \begin{bmatrix} p-1 & 2(r-p-1) & n_r \\ p & 2(r-p-1)-2 & n_r \\ p & 2(r-p-1) & 0 \end{bmatrix}.$$

The characteristic polynomial of M is given by

$$g(x) = x^3 + (p - 2r + 5)x^2 + (n_r p - 2n_r r + 2n_r - 2r + 4)x + 2n_r - 2n_r r$$

and the characteristic polynomial of G is

$$f_G(x) = x^{(r-p)+n_r-2}(x+1)^{p-1}(x+2)^{(r-p)-2}g(x).$$

By Proposition 1.1, $g(x)$ has exactly one positive root which is the index of G. Next, we localize the other two roots. Suppose that $n_r = 2$. The polynomial $g(x)$

can be rewritten as $g(x) = (x+2)h(x)$, where $h(x) = x^2 + (p - 2r + 3)x - 2r + 2$. Since $h(-2) = 2(r - p) > 0$ and $h(-1) = -p < 0$, there is a root in $(-2, -1)$. So, if $n_r = 2$, $G \notin \mathcal{C}$. Now, suppose that $n_r \geq 3$. Since $g(-2) = 2(n_r - 2)(r - p - 1) > 0$ and $g(-1) = -p(n_r - 1) < 0$, we obtain that $g(x)$ has one root in $(-\infty, -2)$ and another root in $(-2, -1)$. So, $G \in \mathcal{C}$ whenever $n_r \geq 3$. If $r - p = 2$, $A(G)$ has an equitable partition and the remaining 3 eigenvalues of $A(G)$ are the eigenvalues of the matrix

$$M_1 = \begin{bmatrix} p-1 & 2 & n_r \\ p & 0 & n_r \\ p & 2 & 0 \end{bmatrix}.$$

The characteristic polynomial of M_1 is given by

$$f_{M_1}(x) = -x^3 + (p-1)x^2 + (2p + p\,n_r + 2\,n_r)x + 2\,n_r\,p + 2\,n_r$$

and the characteristic polynomial of G is

$$f_G(x) = x^{n_r}(x+1)^{p-1} f_{M_1}(x).$$

By Proposition 1.1, $f_{M_1}(x)$ has exactly one positive root which is the index of G. If $n_r = 2$, then

$$f_{M_1}(x) = (x+2)h_1(x),$$

where $h_1(x) = -x^2 + (p+1)x + 2p + 2$. Since $h_1(-2) = -4$ and $h_1(-1) = p$, $G \notin \mathcal{C}$. Now suppose $n_r \geq 3$. Since $f_{M_1}(-n_r) = n_r(n_r - 1)(n_r - 2)$, $f_{M_1}(-2) = 4 - 2n_r$ and $f_{M_1}(-1) = p(n_r - 1)$, we conclude that $G \in \mathcal{C}$ for $n_r \geq 3$ and the result follows. ■

Proposition 2.4. *Let G be a complete r-partite graph such that $r \geq 3$ and $n_1 = n_{r-1} = 2$. Then $G \in \mathcal{C}$ if and only if $n_r \geq 3$.*

Proof. Let $n_r = 2$. In this case, $G \cong K_{2,\ldots,2}$ is a cocktail party graph with the eigenvalues $2r - 2, 0$ and -2 with multiplicities $1, r$ and $r - 1$, respectively which implies that $G \notin \mathcal{C}$. Let $n_r \geq 3$. In this case, $G \cong K_{2,2,\ldots,2} \vee n_r K_1$. From [6] on page 72 and the equitable partition properties, on page 25 of [1], the characteristic polynomial of G is

$$f_G(x) = x^{r+n_r-2}(x+2)^{r-2} g(x),$$

where $g(x) = x^2 - (2r-4)x - 2(r-1)n_r$ with roots $\left(2r - 4 \pm \sqrt{4(r-2)^2 - 8(n_r - rn_r)}\right)$ $/2$. It is easy to verify that $\lambda_n \in (-n_r, -2)$ and therefore $G \in \mathcal{C}$, which completes the proof. ■

Proposition 2.5. *Let G be a graph isomorphic to $K_{n_1,\ldots,n_r}$ such that $r \geq 3$, $n_{r-2} = 1$ and $n_{r-1} \geq 3$. Then $G \in \mathcal{C}$ if and only if one of the following holds.*

(i) $r = 3$,

(ii) $r = 4$ *and either* $n_3 \in \{3, 4\}$ *or* ($n_3 = 5$ *and* $n_4 \in \{5, 6, 7, 8\}$) *or* $n_4 = n_3 = 6$,

(iii) $r = 5$ *and either* $n_4 = 3$ *or* $n_4 = n_5 = 4$,

(iv) $r = 6$ *and* ($n_5 = 3$ *and* $n_6 \in \{3, 4\}$),

(v) $r = 7$ *and* $n_7 = n_6 = 3$.

Proof. Corresponding to the vertex sets of K_{r-2}, $n_{r-1}K_1$ and n_rK_1, the adjacency matrix of G can be written in the following form

$$A(G) = \left[\begin{array}{c|c|c} J_{r-2} - I_{r-2} & J_{(r-2)\times n_{r-1}} & J_{(r-2)\times n_r} \\ \hline J_{n_{r-1}\times(r-2)} & \mathbf{0}_{n_{r-1}} & J_{n_{r-1}\times n_r} \\ \hline J_{n_r\times(r-2)} & J_{n_r\times n_{r-1}} & \mathbf{0}_{n_r} \end{array}\right].$$

We find that $\mathbf{e}_{n-n_{r-1}-n_r+1} - \mathbf{e}_j$, for $j = n - n_{r-1} - n_r + 2, \ldots, n - n_r$ and $\mathbf{e}_{n-n_r+1} - \mathbf{e}_j$, for $j = n - n_r + 2, \ldots, n$ are the eigenvectors for the eigenvalue 0 which has multiplicity at least $n_{r-1} + n_r - 2$. Also, we find that $\mathbf{e}_1 - \mathbf{e}_j$ for $j = 2, \ldots, n - n_r - n_{r-1}$ are the eigenvectors for the eigenvalue -1 which has multiplicity at least $n - n_{r-1} - n_r - 1$. The remaining 3 eigenvalues are the eigenvalues of the matrix M

$$M = \begin{bmatrix} r-3 & n_{r-1} & n_r \\ r-2 & 0 & n_r \\ r-2 & n_{r-1} & 0 \end{bmatrix},$$

with characteristic polynomial

$$f_{r,n_{r-1},n_r}(x) = x^3 - (r-3)x^2 - (n_{r-1}n_r + (r-2)n_{r-1} + (r-2)n_r)x - (r-1)n_{r-1}n_r.$$

Consequently, the characteristic polynomial of G is

$$f_G(x) = x^{n_{r-1}+n_r-2}(x+1)^{n-n_{r-1}-n_r-1} f_{r,n_{r-1},n_r}(x).$$

Now, let $n_{r-1} \geq 3$. Under this assumption, we get $f_{r,n_{r-1},n_r}(-n_r) = -n_r(n_r - 1)(n_r - n_{r-1}) \leq 0$ and $f_{r,n_{r-1},n_r}(-1) = -(r-2)(n_{r-1} - 1)(n_r - 1) \leq 0$. In order to obtain $\lambda_n(G) < -2 \leq \lambda_{n-1}(G)$, we find values for r, n_r and n_{r-1} such that

$$f_{r,n_{r-1},n_r}(-2) = -(r-2)(n_{r-1} - 2)(n_r - 2) + n_{r-1}n_r - 4 \geq 0.$$

We consider the following cases.

Case 1. Let $r = 3$. Then, $f_{3,n_2,n_3}(-2) = 2(n_2 + n_3 - 4) > 0$ and the proof of item (i) is completed.

Case 2. Let $r = 4$. Then, $f_{4,n_3,n_4}(-2) = (n_4 - 4)(4 - n_3) + 4$ is positive for $n_3 \leq 4$ and so $K_{1,1,2,n_4}, K_{1,1,3,n_4}$ and $K_{1,1,4,n_4} \in \mathcal{C}$ for any $n_4 \geq 3$. If $n_3 = 5$, then $f_{4,5,n_4}(-2) = -n_4 + 8 \geq 0$ when $5 \leq n_4 \leq 8$. So, the graphs $K_{1,1,5,5}, K_{1,1,5,6}, K_{1,1,5,7}, K_{1,1,5,8} \in \mathcal{C}$. If $n_3 = 6$, then $f_{4,n_3,n_4}(-2) = -2n_4 + 12 \geq 0$ since $n_4 \geq n_3 = 6$. Then, the graph $K_{1,1,6,n_4} \in \mathcal{C}$ for $n_4 = 6$. If $n_3 \geq n_4 \geq 7$, it is easy to see that $f_{4,n_3,n_4}(-2) < 0$ and this completes the proof of Case (ii).

Case 3. Let $r = 5$. Then, $f_{5,n_4,n_5}(-2) = -(n_5 - 3)(n_4 - 3) + 1$ is positive when $(n_5 - 3)(n_4 - 3) \leq 1$. Accordingly, $n_4 = 3$ or $n_4 = n_5 = 4$.

Case 4. Let $r = 6$. If $n_5 = 3$, $f_{6,3,n_6}(-2) > 0$ whenever $n_6 \in \{3, 4\}$. If $n_5 = 4+k$ for $k \geq 0$, then $f_{6,n_5,n_6}(-2) > 0$ if and only if $n_6 \leq (8k+12)/(3k+4)$. The latter inequality and $n_6 \geq 4+k$, imply that $3k^2 + 8k + 4 \leq 0$ is not possible. Hence, $f_{6,n_5,n_6}(-2)$ is never positive and the proof of (iv) is complete.

Case 5. Let $r = 7$. If $n_6 = 2$, then $f_{7,2,n_7}(-2) = 2n_7 - 4 > 0$. If $n_6 = 3$, then $f_{7,3,n_7}(-2) = -2n_7 + 6$, which is non-negative for $n_7 \leq 3$, and then $n_7 = 3$. If $n_6 = 4 + k$ for $k \geq 0$, we get $f(7, 4 + k, n_7) \geq 0$ when $n_7 \leq (10k + 16)/(4k + 6)$. Since $4 + k \leq n_7$, it follows that $k^2 + 3k + 2 \leq 0$, which is not possible for $k \geq 0$. So, there is no n_7 such that $f_{7,4+k,n_7}(-2) \geq 0$ and this completes the proof of (v).

Case 6. Let $r = 8 + k$ for $k \geq 0$. If $n_{r-1} = 3 + t$ for $t \geq 0$, we obtain $f_{8+k,3+t,n_r}(-2) = -(5\,n_r - 12)k – ((n_r - 2)k + n_r - 2)t - 3\,n_r + 8$. So, $f_{8+k,3+t,n_r}(-2) \geq 0$ implies

$$n_r \leq \frac{12k + 2kt + 2t + 8}{5k + kt + t + 3}.$$

Since $n_r \geq 3 + k$, we obtain

$$3 + k \leq n_r \leq \frac{12k + 2kt + 2t + 8}{5k + kt + t + 3},$$

which does not hold for $t, k \geq 0$ and the proof of the proposition follows. ■

Remark 1. Note that if $H \in \mathcal{C}$, then $G = H \cup sK_1$ is also in $\mathcal{C}$ for any integer $s \geq 1$.

3. Graphs in $\mathcal{C}$ Determined by Their Spectrum

Esser and Harary [10] proved that all complete multipartite graphs are DS among all connected graphs, i.e., such graphs have no cospectral mate amongst the complete multipartite graphs. In order to find all DS graphs in $\mathcal{C}$, including the disconnected ones, we should consider the complete multipartite graphs with some additional isolated vertices. For instance, $K_{1,35} \in \mathcal{C}$ is not DS since the graph $K_{5,7} \cup 24K_1$ is its cospectral mate. The characteristic polynomial of any complete multipartite graph can be written using the following well-known elementary symmetric function

$$S_j(x_1,\dots,x_k) = \sum_{1\le i_1<\cdots<i_j\le k} \prod_{l=1}^{j} x_{i_l}, \ \ 1\le j\le k,$$

where $S_0 = 1$.

Proposition 3.1 [8]**.** *The characteristic polynomial of $K_{n_1,\dots,n_k}$ is given by*

$$f_{K_{n_1,\dots,n_k}}(\lambda) = \sum_{i=0}^{k}(1-i)S_i\lambda^{n-i},$$

where $n = n_1+\cdots+n_k$ and S_i is the elementary symmetric function of order i of the numbers $n_1,\dots,n_k$.

From [8, 11], we have that two graphs $K_{n_1,\dots,n_k}\cup sK_1$ and $K_{m_1,\dots,m_k}\cup rK_1$ are cospectral if and only if

$$n_1+\cdots+n_k+s = m_1+\cdots+m_k+r, \tag{1}$$

$$S_j(n_1,\dots,n_k) = S_j(m_1,\dots,m_k), \ \ 2\le j\le k. \tag{2}$$

This is summarized in the following lemma.

Lemma 3.2. *Let G be a graph isomorphic to $K_{n_1,\dots,n_k}\cup sK_1$. If the system*

$$S_j(x_1,\dots,x_k) = S_j(n_1,\dots,n_k), \ \ 2\le j\le k \tag{3}$$

has only one positive integer solution, then G is a DS graph.

In [11], the authors proposed the problem of characterizing all complete multipartite graphs that are DS. Our next result is a solution for this problem for a given sequence of integers. Before presenting this result, we need the following: let $n_1,\dots,n_k$ be positive integers and define $\mathcal{L}$ to be the set of all positive integer solutions $(x_1,\dots,x_k)$ of the system $S_j(x_1,\dots,x_k) = S_j(n_1,\dots,n_k)$, for each $2\le j\le k$. Next, let $\mathcal{M} = \{a_1+a_2+\cdots+a_k \,|\, (a_1,a_2,\dots,a_k)\in\mathcal{L}\}$, $p = \min_{(a_1,a_2,\dots,a_k)\in\mathcal{L}}\mathcal{M}$ and $q = \min_{(a_1,a_2,\dots,a_k)\in\mathcal{L}}\{\mathcal{M}\setminus\{p\}\}$.

Theorem 3.3. *Let $n_1,\dots,n_k$ be positive integers and let $\mathcal{L}$ be the set of all positive integer solutions of the system $S_j(x_1,\dots,x_k) = S_j(n_1,\dots,n_k)$, for each $2\le j\le k$. Then the following properties are satisfied.*

(i) *If $|\mathcal{L}| = 1$, then $K_{n_1,\dots,n_k}\cup sK_1$ is DS for any non-negative integer s;*

(ii) *If $|\mathcal{L}|\ge 2$, then there is at least one solution $(x_1',\dots,x_k')\in\mathcal{L}$ such that $K_{x_1',\dots,x_k'}\cup sK_1$ is DS for each $s\in\{0,1,\dots,q-p-1\}$.*

Proof. If $|\mathcal{L}| = 1$, the proof follows immediately from Lemma 3.2. Now, assume $|\mathcal{L}| \geq 2$. Let $(x'_1, x'_2, \ldots, x'_k)$ and $(x''_1, x''_2, \ldots, x''_k)$ be the solutions that realize p and q, respectively. The corresponding graphs $K_{x'_1,x'_2,\ldots,x'_k} \cup sK_1$ and $K_{x''_1,x''_2,\ldots,x''_k}$ are cospectral if and only if

$$s = x''_1 + x''_2 + \cdots + x''_k - \left(x'_1 + x'_2 + \cdots + x'_k\right) = q - p.$$

Therefore, for $s \in \{0, \ldots, q-p-1\}$ the graph $K_{x'_1,x'_2,\ldots,x'_k} \cup sK_1$ is DS and the result follows. ■

Remark 2. Note that if $s = q-p+1$, then $K_{x'_1,x'_2,\ldots,x'_k} \cup sK_1$ is cospectral with $K_{x''_1,x''_2,\ldots,x''_k} \cup K_1$, which explains why s cannot be greater than $q-p$.

Lemma 3.4. *Let $K_{n_1,\ldots,n_r}$ and $K_{m_1,\ldots,m_r}$ be two r-complete multipartite graphs with the same non-null eigenvalues. Then $\{n_1, \ldots, n_r\} \cap \{m_1, \ldots, m_r\} = \emptyset$ or $K_{n_1,\ldots,n_r}$ and $K_{m_1,\ldots,m_r}$ are isomorphic.*

Proof. Since $K_{n_1,\ldots,n_r}$ and $K_{m_1,\ldots,m_r}$ have the same non-null eigenvalues, we have that

$$S_j(n_1, \ldots, n_r) = S_j(m_1, \ldots, m_r) = a_j, \;\; 2 \leq j \leq r. \tag{4}$$

Working towards a contradiction, suppose that $w \in \{n_1, \ldots, n_r\} \cap \{m_1, \ldots, m_r\} \neq \emptyset$ and that $K_{n_1,\ldots,n_r}$ and $K_{m_1,\ldots,m_r}$ are not isomorphic. Let us say that $w = n_1 = m_1$. Then

$$S_j(n_2, \ldots, n_r) = \sum_{i=1}^{r-j} -\frac{S_{j+i}(n_1, \ldots, n_r)}{(-w)^i} = \sum_{i=1}^{r-j} -\frac{a_{j+i}}{(-w)^i}, \;\; 1 \leq j \leq r-1 \tag{5}$$

and

$$S_j(m_2, \ldots, m_r) = \sum_{i=1}^{r-j} -\frac{S_{j+i}(m_1, \ldots, m_r)}{(-w)^i} = \sum_{i=1}^{r-j} -\frac{a_{j+i}}{(-w)^i}, \;\; 1 \leq j \leq r-1. \tag{6}$$

Both systems (5) and (6) are of the same form and have the same unique solution $n_i = m_i$, for $i = 2, \ldots, r$. Since $m_1 = n_1$, we have that $K_{n_1,\ldots,n_r}$ and $K_{m_1,\ldots,m_r}$ are isomorphic, which is a contradiction and the result follows. ■

In [11], the complete bipartite graphs which are DS were characterized as showed in Proposition 3.5.

Proposition 3.5 [11]**.** *Let s, t be the positive integers. The complete bipartite graph $K_{s,t}$ is DS if and only if the equality $n = st$ is the decomposition of two factors s and t with the smallest sum $s + t$.*

From Proposition 3.5, we can conclude that for $r = 2$, there are many graphs in $\mathcal{C}$ which are not DS. For instance, $K_{1,35} \in \mathcal{C}$ is not DS since $K_{5,7} \cup 24K_1$ is its cospectral mate. Next, for $r \geq 3$, we prove that all complete r-partite graphs in $\mathcal{C}$ are DS.

Lemma 3.6. *The graph K_{1,n_2,n_3} is DS.*

Proof. For the sake of simplicity, take $n_2 = p$ and $n_3 = q$. Suppose that there exist positive integers x_1, x_2, x_3, s such that $K_{x_1,x_2,x_3} \cup sK_1$ and $K_{1,p,q}$ are cospectral. By the equations (1) and (2) the following hold

$$x_1 + x_2 + x_3 < 1 + p + q \tag{7}$$
$$x_1x_2 + x_1x_3 + x_2x_3 = p + q + pq \tag{8}$$
$$x_1x_2x_3 = pq. \tag{9}$$

By dividing (8) by (9) we obtain

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 1 + \frac{1}{q} + \frac{1}{p}. \tag{10}$$

We distinguish three cases.

Case 1. If $p = q = 1$, then $K_{1,1,1} = K_3$ which is DS.

Case 2. Let $p = 1$ and $q \geq 2$. Assume that $x_1 = 1$. By (7), we get $x_2 + x_3 < q + 1$. Since $x_2 + x_3 = q + 1$ by equation (10), we get a contradiction. Now, let $x_1 \geq 2$. Note that $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \leq \frac{3}{2}$. Since $2 + \frac{1}{q} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$, we get a contradiction. So, $K_{1,1,q}$ is DS.

Case 3. Let $q \geq p \geq 2$ and let $x_1 = 1$. From (7), $x_2 + x_3 < p + q$. By (10), $x_2 + x_3 = p + q$, which is a contradiction. Now, let $x_1 = 2$. From (7), $x_2 + x_3 < p + q - 1$ and by (8) and (9), we obtain

$$2x_2 + 2x_3 + x_2x_3 = p + q + pq \tag{11}$$
$$2x_2x_3 = pq. \tag{12}$$

Subtracting (12) from (11), we get

$$(2 - x_2)(x_3 - 2) = p + q - 4.$$

If $x_2 = 2$, then $p + q = 4$ which implies $p = q = 2$ and $x_3 = 1$, and we get a contradiction since $x_3 \geq 2$.

If $x_1 \geq 3$, then

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \leq 1 \leq 1 + \frac{1}{p} + \frac{1}{q},$$

which is a contradiction by (10). The proof is complete. ■

Proposition 3.7. *Let G be a complete r-partite graph with $n_1 = \cdots = n_{r-2} = 1$. Then G is DS.*

Proof. Let $K_{1,\ldots,1,n_{r-1},n_r}$ and $K_{x_1,\ldots,x_r} \cup sK_1$ be cospectral graphs. Note that (1) and (2) hold. In particular, from the last two equations, we get:

$$S_{r-1} = \sum_{1 \leq i_1 < \cdots < i_{r-1} \leq r} x_{i_1} x_{i_2} \cdots x_{i_{r-1}} = p + (r-2)pq, \tag{13}$$

$$x_1 x_2 \cdots x_r = n_{r-1} n_r. \tag{14}$$

Dividing (13) by (14),

$$\frac{1}{x_1} + \cdots + \frac{1}{x_r} = \frac{1}{n_r} + r - 2.$$

If $r = 3$, then G is DS from Lemma 3.6. For $r \geq 4$ and $x_1 \geq 2$, we get

$$\frac{1}{x_1} + \cdots + \frac{1}{x_r} \leq \frac{r}{2} \leq r - 2 < r - 2 + \frac{1}{n_r} = \frac{1}{x_1} + \cdots + \frac{1}{x_r},$$

which is a contradiction. Now, let $r \geq 4$ and $x_1 = 1$. Since $K_{1,\ldots,1,n_{r-1},n_r}$, and $K_{x_1,x_2,\ldots,x_r} \cup sK_1$ are cospectral graphs, by Lemma 3.4, we may conclude that those graphs are isomorphic with $s = 0$, which means that $K_{1,\ldots,1,n_{r-1},n_r}$ is DS. ∎

Theorem 3.8. *Let G be a complete r-partite graph with $n_1 = n_p = 1$ and $n_{p+1} = n_{r-2} = 2$ such that $p \geq 1$ and $r - p \geq 3$. Then G is DS.*

Proof. Let $K_{1,\ldots,1,2,\ldots,2,n_{r-1},n_r}$, and $K_{x_1,\ldots,x_r} \cup sK_1$, $x_1 \leq \cdots \leq x_r$ be cospectral graphs. Note that (1) and (2) hold. In particular, from the last two equations, we get

$$\begin{aligned} S_{r-1} &= \sum_{1 \leq i_1 < \cdots < i_{r-1} \leq r} x_{i_1} x_{i_2} \cdots x_{i_{r-1}} \\ &= n_{r-1} 2^{r-p-2} + p n_{r-1} n_r 2^{r-p-2} + (r-p-2) 2^{r-p-3} n_r n_{r-1} + n_r 2^{r-p-2}, \end{aligned} \tag{15}$$

$$x_1 x_2 \cdots x_r = 2^{r-p-2} n_{r-1} n_r. \tag{16}$$

Dividing (15) by (16),

$$\frac{1}{x_1} + \cdots + \frac{1}{x_r} = p + \frac{1}{n_r} + \frac{r-p-2}{2} + \frac{1}{n_{r-1}}.$$

If $x_1 \geq 3$, then

$$\frac{1}{x_1} + \cdots + \frac{1}{x_r} \leq \frac{r}{3} \leq p + \frac{r-p-2}{2} < p + \frac{r-p-2}{2} + \frac{1}{n_{r-1}} + \frac{1}{n_r} = \frac{1}{x_1} + \cdots + \frac{1}{x_r},$$

which is a contradiction. So, $x_1 \in \{1, 2\}$. Since $K_{1,\ldots,1,2,\ldots,2,n_{r-1},n_r}$ and $K_{x_1,x_2,\ldots,x_r} \cup sK_1$ are cospectral graphs, by Lemma 3.4, we may conclude that those graphs are isomorphic with $s = 0$, which means that $K_{1,\ldots,1,2,\ldots,2,n_{r-1},n_r}$ is DS. ■

As a consequence of Theorem 3.8, the graphs $K_{1,\ldots,1,2,\ldots,2,n_{r-1},n_r} \cup sK_1$ with exactly $p \geq 1$ parts of size 1 and $r - p \geq 3$ parts of size 2 are DS for any non negative integer s.

Proposition 3.9. *Let $r \geq 3$, $s \geq 0$, H be a complete r-partite graph with $n_1 = n_{r-1} = 2$ and $n_r \geq 3$. Then $G = H \cup sK_1$ is DS.*

Proof. From Proposition 2.4 and Remark 1, $G \in \mathcal{C}$. Any graph G' cospectral to G also belongs to $\mathcal{C}$, so G' is a complete multipartite graph with isolated vertices. Therefore $G' = K_{x_1,\ldots,x_r} \cup s'K_1$. According to Theorem 3.8, if $x_1 = 1$, then G' is DS. So $x_1 \geq 2$. From Propositions 2.1 to 2.5 there is no graphs in $\mathcal{C}$ such that $x_1 \geq 3$. Therefore, $x_1 = 2$ and from Lemma 3.4, we conclude that G and G' are isomorphic, which implies that G is DS. ■

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