ON THE $\alpha$-SPECTRAL RADIUS OF UNIFORM HYPERGRAPHS

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Abstract

For $0 \leq \alpha < 1$ and a uniform hypergraph $G$, the $\alpha$-spectral radius of $G$ is the largest $H$-eigenvalue of $\alpha D(G) + (1-\alpha)A(G)$, where $D(G)$ and $A(G)$ are the diagonal tensor of degrees and the adjacency tensor of $G$, respectively. We give upper bounds for the $\alpha$-spectral radius of a uniform hypergraph, propose some transformations that increase the $\alpha$-spectral radius, and determine the unique hypergraphs with maximum $\alpha$-spectral radius in some classes of uniform hypergraphs.

Keywords: $\alpha$-spectral radius, $\alpha$-Perron vector, adjacency tensor, uniform hypergraph, extremal hypergraph.

2010 Mathematics Subject Classification: 05C50, 05C65.

1. Introduction

Let $G$ be a hypergraph on $n$ vertices with vertex set $V(G)$ and edge set $E(G)$. If $|e| = k$ for each $e \in E(G)$, then $G$ is said to be a $k$-uniform hypergraph. For a vertex $v \in V(G)$, the set of the edges containing $v$ in $G$ is denoted by $E_G(v)$, and the degree of $v$ in $G$, denoted by $d_G(v)$ or $d_v$, is the size of $E_G(v)$. We say that $G$ is regular if all vertices of $G$ have equal degrees. Otherwise, $G$ is irregular.

For $u, v \in V(G)$, a walk from $u$ to $v$ in $G$ is defined to be an alternating sequence of vertices and edges $(v_0, e_1, v_1, \ldots, v_{s-1}, e_s, v_s)$ with $v_0 = u$ and $v_s = v$ such that edge $e_i$ contains vertices $v_{i-1}$ and $v_i$, and $v_{i-1} \neq v_i$ for $i = 1, \ldots, s$. The value $s$ is the length of this walk. A path is a walk with all $v_i$ distinct and all $e_i$

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distinct. A cycle is a walk containing at least two edges, all \( e_i \) are distinct and all \( v_i \) are distinct except \( v_0 = v_k \). If there is a path from \( u \) to \( v \) for any \( u, v \in V(G) \), then we say that \( G \) is connected. A hypertree is a connected hypergraph with no cycles. For \( k \geq 2 \), the number of vertices of a \( k \)-uniform hypertree with \( m \) edges is \( 1 + (k - 1)m \).

The distance between vertices \( u \) and \( v \) in a connected hypergraph \( G \) is the length of a shortest path from \( u \) to \( v \) in \( G \). The diameter of connected hypergraph \( G \) is the maximum distance between any two vertices of \( G \).

For positive integers \( k \) and \( n \), a tensor \( T = (T_{i_1 \cdots i_k}) \) of order \( k \) and dimension \( n \) is a multidimensional array with entries \( T_{i_1 \cdots i_k} \in \mathbb{C} \) for \( i_j \in [n] = \{1, \ldots, n\} \) and \( j \in [k] \), where \( \mathbb{C} \) is the complex field.

Let \( M \) be a tensor of order \( k \geq 2 \) and dimension \( n \), and \( N \) a tensor of order \( \ell \geq 1 \) and dimension \( n \). The product \( MN \) is the tensor of order \( (k - 1)(\ell - 1) + 1 \) and dimension \( n \) with entries [22]

\[
(MN)_{i_1 \cdots j_{k-1}} = \sum_{i_2, \ldots, i_k \in [n]} M_{ii_2 \cdots i_k} N_{i_2 j_1} \cdots N_{i_k j_{k-1}},
\]

with \( i \in [n] \) and \( j_1, \ldots, j_{k-1} \in [n]^{\ell-1} \). Then for a tensor \( T \) of order \( k \) and dimension \( n \) and an \( n \)-dimensional vector \( x = (x_1, \ldots, x_n)^\top \), \( Tx \) is an \( n \)-dimensional vector whose \( i \)-th entry is

\[
(Tx)_i = \sum_{i_2, \ldots, i_k = 1}^n T_{i_2 \cdots i_k} x_{i_2} \cdots x_{i_k},
\]

where \( i \in [n] \). For some complex \( \lambda \), if there is a nonzero vector \( x \) such that

\[
Tx = \lambda \left(x_1^{k-1}, \ldots, x_n^{k-1}\right)^\top,
\]

then \( \lambda \) is called an eigenvalue of \( T \), and \( x \) is called an eigenvector of \( T \) corresponding to \( \lambda \). Moreover, if both \( \lambda \) and \( x \) are real, then we call \( \lambda \) an \( H \)-eigenvalue and \( x \) an \( H \)-eigenvector of \( T \). See [10, 18, 20] for more details. The spectral radius of \( T \) is the largest modulus of its eigenvalues, denoted by \( \rho(T) \).

Let \( G \) be a \( k \)-uniform hypergraph with vertex set \( V(G) = \{v_1, \ldots, v_n\} \), where \( k \geq 2 \). The adjacency tensor of \( G \) is defined in [1] as the tensor \( A(G) \) of order \( k \) and dimension \( n \) whose \((i_1, \ldots, i_k)\)-entry is \( \frac{1}{(k-1)!} \) if \( \{v_{i_1}, \ldots, v_{i_k}\} \in E(G) \), and 0 otherwise. The degree tensor of \( G \) is the diagonal tensor \( D(G) \) of order \( k \) and dimension \( n \) with \((i, \ldots, i)\)-entry to be the degree of vertex \( v_i \in [n] \). Then \( Q(G) = D(G) + A(G) \) is the signless Laplacian tensor of \( G \) [20]. Motivated by work of Nikiforov [14] (see also [5, 15]), Lin et al. [11] proposed to study the convex linear combinations \( A_\alpha(G) \) of \( D(G) \) and \( A(G) \) defined by

\[
A_\alpha(G) = \alpha D(G) + (1 - \alpha) A(G),
\]
where $0 \leq \alpha < 1$. The $\alpha$-spectral radius of $G$ is the spectral radius of $A_\alpha(G)$, denoted by $\rho_\alpha(G)$. Note that $\rho_0(G)$ is the spectral radius of $G$, while $2\rho_{1/2}(G)$ is the signless Laplacian spectral radius of $G$.

For $k \geq 2$, let $G$ be a $k$-uniform hypergraph with $V(G) = [n]$, and $x$ a $n$-dimensional column vector. Let $x_V = \prod_{v \in V} x_v$ for $V \subseteq V(G)$. Then
\[
x^\top (A_\alpha(G)x) = \alpha \sum_{u \in V(G)} d_u x_u^k + (1 - \alpha)k \sum_{e \in E(G)} x_e,
\]
or equivalently,
\[
x^\top (A_\alpha(G)x) = \sum_{e \in E(G)} \left( \alpha \sum_{u \in e} x_u^k + (1 - \alpha)k x_e \right).
\]

For a uniform hypergraph $G$, bounds for the spectral radius $\rho_0(G)$ have been given in [1, 12, 13, 29], and bounds for the signless Laplacian spectral radius $2\rho_{1/2}(G)$ may be found in [6, 12, 21]. Recently, Lin et al. [11] gave upper bounds for $\alpha$-spectral radius of connected irregular $k$-uniform hypergraphs, extending some known bounds for ordinary graphs. Some hypergraph transformations have been proposed to investigate the change of the $0$-spectral radius, and the unique hypergraphs that maximize or minimize the $0$-spectral radius have been determined among some classes of uniform hypergraphs (especially for hypertrees), see, e.g., [2, 4, 8, 16, 24, 25, 28, 31].

In this paper, we give upper bounds for the $\alpha$-spectral radius of a uniform hypergraph, propose some hypergraph transformations that increase the $\alpha$-spectral radius, and determine the unique hypergraphs with maximum $\alpha$-spectral radius in some classes of uniform hypergraphs such as the class of $k$-uniform hypercacti with $m$ edges and $r$ cycles for $0 \leq r \leq \lfloor \frac{m}{2} \rfloor$, and the class of $k$-uniform hypertrees with $m$ edges and diameter $d \geq 3$.

2. Preliminaries

A tensor $T$ of order $k \geq 2$ and dimension $n$ is said to be weakly reducible, if there is a nonempty proper subset $J$ of $[n]$ such that for $i_1 \in J$ and $i_j \in [n] \setminus J$ for some $j = 2, \ldots, k$, $T_{i_1 \cdots i_k} = 0$. Otherwise, $T$ is weakly irreducible.

For $k \geq 2$, an $n$-dimensional vector $x$ is said to be $k$-unit if $\sum_{i=1}^n x_i^k = 1$.

Lemma 1 [3, 27]. Let $T$ be a nonnegative tensor of order $k \geq 2$ and dimension $n$. Then $\rho(T)$ is an eigenvalue of $T$ and there is a $k$-unit nonnegative eigenvector corresponding to $\rho(T)$. If furthermore $T$ is weakly irreducible, then there is a unique $k$-unit positive eigenvector corresponding to $\rho(T)$.
If $G$ is a $k$-uniform hypergraph with $k \geq 2$, then $\mathcal{A}_\alpha(G)$ is weakly irreducible if and only if $G$ is connected (see [17, 20] for the treatment of $\mathcal{A}_0(G)$ and $2A_{1/2}(G)$, respectively). Thus, if $G$ is connected, then by Lemma 1, there is a unique $k$-unit positive $H$-eigenvector $x$ corresponding to $\rho_\alpha(G)$, which is called the $\alpha$-Perron vector of $G$.

For a nonnegative tensor $T$ of order $k \geq 2$ and dimension $n$, let $r_i(T) = \sum_{i_2 \cdots i_k=1}^n T_{i_2 \cdots i_k}$ for $i = 1, \ldots, n$.

**Lemma 2** [7, 27]. Let $T$ be a nonnegative tensor of order $k \geq 2$ and dimension $n$. Then

$$\rho(T) \leq \max_{1 \leq i \leq n} r_i(T)$$

with equality when $T$ is weakly irreducible if and only if $r_1(T) = \cdots = r_n(T)$.

For two tensors $\mathcal{M}$ and $\mathcal{N}$ of order $k \geq 2$ and dimension $n$, if there is an $n \times n$ nonsingular diagonal matrix $U$ such that $\mathcal{N} = U^{-(k-1)} \mathcal{M} U$, then we say that $\mathcal{M}$ and $\mathcal{N}$ are diagonal similar.

**Lemma 3** [22]. Let $\mathcal{M}$ and $\mathcal{N}$ be two diagonal similar tensors of order $k \geq 2$ and dimension $n$. Then $\mathcal{M}$ and $\mathcal{N}$ have the same real eigenvalues.

Let $G$ be a connected $k$-uniform hypergraph on $n$ vertices, where $k \geq 2$. Let $0 \leq \alpha < 1$. For an $n$-dimensional $k$-unit nonnegative vector $x$, by [19, Theorem 2] (and its proof) and Lemma 1, we have $\rho_\alpha(G) \geq x^\top (\mathcal{A}_\alpha(G)x)$ with equality if and only if $x$ is the $\alpha$-Perron vector of $G$. If $x$ is the $\alpha$-Perron vector of $G$, then for any $v \in V(G)$,

$$\rho_\alpha(G)x_v^{k-1} = \alpha d_v x_v^{k-1} + (1 - \alpha) \sum_{e \in E_v(G)} x_{e \setminus \{v\}},$$

which is called the eigenequation of $G$ at $v$.

For a hypergraph $G$ with $\emptyset \neq X \subseteq V(G)$, let $G[X]$ be the subhypergraph induced by $X$, i.e., $G[X]$ has vertex set $X$ and edge set $\{e \subseteq X : e \in E(G)\}$. If $E' \subseteq E(G)$, then $G - E'$ is the hypergraph obtained from $G$ by deleting the edges in $E'$. If $E' \subseteq E(G)$, then $G \cup E'$ is the hypergraph obtained from $G$ by adding elements of $E'$ as edges.

A $k$-uniform hypertree with $m$ edges is a hyperstar, denoted by $S_{m,k}$, if all edges share a common vertex. A $k$-uniform loose path with $m \geq 1$ edges, denoted by $P_{m,k}$, is the $k$-uniform hypertree whose vertices and edges may be labelled as $(v_0, e_1, v_1, \ldots, v_{m-1}, e_m, v_m)$ such that the vertices $v_1, \ldots, v_{m-1}$ are of degree 2, and all the other vertices of $G$ are of degree 1.

If $P$ is a path or a cycle of a hypergraph $G$, $V(P)$ denotes the vertex set of the hypergraph $P$. 

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3. Upper Bounds for $\alpha$-Spectral Radius

For a connected irregular $k$-uniform hypergraph $G$ with $n$ vertices, maximum degree $\Delta$ and diameter $D$, where $2 \leq k < n$, it was shown in [11] that for $0 \leq \alpha < 1$,

$$\rho_\alpha(G) < \Delta - \frac{4(1-\alpha)}{((4D-1-2\alpha)(k-1)+1)n}.$$  

For a $k$-uniform hypergraph $G$, upper bounds on $\rho_0(G)$ and $2\rho_{1/2}(G)$ have been given in [12, 29].

**Theorem 4.** Let $G$ be a $k$-uniform hypergraph on $n$ vertices with maximum degree $\Delta$ and second maximum degree $\Delta'$, where $k \geq 2$. For $\alpha = 0$, let $\delta = (\frac{\Delta}{\Delta'})^\frac{1}{k}$, and for $0 < \alpha < 1$, let $\delta = 1$ if $\Delta = \Delta'$ and $\delta$ be a root of $h(t) = 0$ in $(\frac{\Delta}{\Delta'})^{\frac{1}{k}}, +\infty)$ if $\Delta > \Delta'$, where $h(t) = (1-\alpha)\Delta' t + \alpha(\Delta' - \Delta)t^{k-1} - (1-\alpha)\Delta$ for $0 \leq \alpha < 1$. Then

$$(1) \quad \rho_\alpha(G) \leq \alpha \Delta + (1-\alpha)\Delta \delta^{-(k-1)}.$$  

Moreover, if $G$ is connected, then equality holds in (1) if and only if $G$ is a regular hypergraph or $G \cong G'$, where $V(G') = V(H) \cup \{v\}$, $E(G') = \{e \cup \{v\}: e \in E(H)\}$, and $H$ is a regular $(k-1)$-uniform hypergraph on $n-1$ vertices with $v \notin V(H)$.

**Proof.** By Theorem 2.1 and Lemma 2.2 in [22], we may assume that $d_1 \geq \cdots \geq d_n$. Then $\Delta = d_1$ and $\Delta' = d_2$.

If $d_1 = d_2$, then $\delta = 1$, and by Lemma 2, we have

$$\rho_\alpha(G) \leq \max_{1 \leq i \leq n} r_i(A_\alpha(G)) = \max_{1 \leq i \leq n} d_i = d_1 = \alpha d_1 + (1-\alpha) d_2 \delta^{-(k-1)},$$

and when $G$ is connected, $A_\alpha(G)$ is weakly irreducible, thus by Lemma 2, equality (1) holds if and only if $r_1(A_\alpha(G)) = \cdots = r_n(A_\alpha(G))$, i.e., $G$ is a regular hypergraph.

Suppose in the following that $d_1 > d_2$. Let $U = \text{diag}(t, 1, \ldots, 1)$ be an $n \times n$ diagonal matrix, where $t > 1$ is a variable to be determined later. Let $T = U^{-(k-1)}A_\alpha(G)U$. By Lemma 3, $A_\alpha(G)$ and $T$ have the same real eigenvalues. Obviously, both $A_\alpha(G)$ and $T$ are nonnegative tensors of order $k$ and dimension $n$. By Lemma 1, $\rho(A_\alpha(G))$ is an eigenvalue of $A_\alpha(G)$ and $\rho(T)$ is an eigenvalue of $T$. Therefore $\rho_\alpha(G) = \rho(A_\alpha(G)) = \rho(T)$. For $i \in [n] \setminus \{1\}$, let $d_{1,i} = |\{e : 1, i \in e \in E(G)\}|$. Obviously, $d_{1,i} \leq d_i$. Note that
\[ r_1(T) = \sum_{i_2, \ldots, i_k \in [n]} T_{i_2 \cdots i_k} \]
\[ = \alpha \mathcal{D}_{1 \cdots 1} + (1 - \alpha) \sum_{i_2, \ldots, i_k \in [n]} U_{i_1}^{-1} \mathcal{A}_{i_1 i_2 \cdots i_k} U_{i_2 i_3} \cdots U_{i_k i_k} \]
\[ = \alpha d_1 + (1 - \alpha) \sum_{i_2, \ldots, i_k \in [n]} \frac{1}{t_{k-1}} \mathcal{A}_{i_1 i_2 \cdots i_k} = \alpha d_1 + \frac{(1 - \alpha)d_1}{t_{k-1}}, \]
and for \( 2 \leq i \leq n, \)
\[ r_i(T) = \sum_{i_2, \ldots, i_k \in [n]} T_{i_2 \cdots i_k} = \alpha \mathcal{D}_{i \cdots i} + (1 - \alpha) \sum_{i_2, \ldots, i_k \in [n]} U_{ii}^{-1} \mathcal{A}_{ii i_2 \cdots i_k} U_{i_2 i_3} \cdots U_{i_k i_k} \]
\[ = \alpha d_i + (1 - \alpha) \sum_{i_2, \ldots, i_k \in [n]} U_{ii}^{-1} \mathcal{A}_{ii i_2 \cdots i_k} U_{i_2 i_3} \cdots U_{i_k i_k} \]
\[ + (1 - \alpha) \sum_{i_2, \ldots, i_k \in [n]} U_{ii}^{-1} \mathcal{A}_{ii i_2 \cdots i_k} U_{i_2 i_3} \cdots U_{i_k i_k} \]
\[ = \alpha d_i + (1 - \alpha) \sum_{i_2, \ldots, i_k \in [n]} \mathcal{A}_{ii i_2 \cdots i_k} t + (1 - \alpha) \sum_{i_2, \ldots, i_k \in [n]} \mathcal{A}_{ii i_2 \cdots i_k} \]
\[ = \alpha d_i + (1 - \alpha)(t - 1)d_{1,i} + (1 - \alpha)(d_i - d_{1,i}) \]
\[ = d_i + (1 - \alpha)(t - 1)d_{1,i} \leq (1 + (1 - \alpha)(t - 1))d_i \leq (1 + (1 - \alpha)(t - 1))d_2 \]
with equality if and only if \( d_{1,i} = d_i = d_2. \)

Note that \( h \left( \left( \frac{d_i}{d_2} \right)^\frac{1}{\delta} \right) = \alpha (d_2 - d_1) \left( \frac{d_i}{d_2} \right)^\frac{k-1}{\delta} \leq 0 \) with equality if and only if \( \alpha = 0, \) and that \( h(+\infty) > 0. \) Thus \( h(t) = 0 \) does have a root \( \delta, \) as required. Let \( t = \delta. \) Then \( t > 1, \)
\[ \alpha d_1 + \frac{(1 - \alpha)d_1}{t_{k-1}} = (1 + (1 - \alpha)(t - 1))d_2, \]
and thus for \( 1 \leq i \leq n, \)
\[ r_i(T) \leq \alpha d_1 + (1 - \alpha)d_1 \delta^{-(k-1)}. \]

Now by Lemma 2,
\[ \rho_\alpha(G) = \rho(T) \leq \max_{1 \leq i \leq n} r_i(T) \leq \alpha d_1 + (1 - \alpha)d_1 \delta^{-(k-1)}. \]

This proves (1).
Suppose that $G$ is connected. Then $A_\alpha$ is weakly irreducible, and so is $T$.

Suppose that equality holds in (1). From the above arguments and by Lemma 2, we have $r_1(T) = \cdots = r_n(T) = \alpha d_1 + (1 - \alpha) d_1 \delta^{-(k-1)}$, and $d_{1,i} = d_i = d_2$ for $i = 2, \ldots, n$. Then vertex 1 is contained in each edge of $G$. Let $H$ be the hypergraph with $V(H) = V(G) \setminus \{1\} = \{2, \ldots, n\}$ and $E(H) = \{e \setminus \{1\} : e \in E(G)\}$. Then $H$ is a regular $(k-1)$-uniform hypergraph on vertices $2, \ldots, n$, of degree $d_2$.

Therefore $G \cong G'$, where $V(G') = V(H) \cup \{1\}$, $E(G') = \{e \cup \{1\} : e \in E(H)\}$, and $H$ is a regular $(k-1)$-uniform hypergraph on vertices $2, \ldots, n$ of degree $d_2$.

Conversely, if $G \cong G'$, where $V(G') = V(H) \cup \{1\}$, $E(G') = \{e \cup \{1\} : e \in E(H)\}$, and $H$ is a regular $(k-1)$-uniform hypergraph on vertices $2, \ldots, n$ of degree $d_2$, then by the above arguments, we have $r_i(T) = \alpha d_1 + (1 - \alpha) d_1 \delta^{-(k-1)}$ for $1 \leq i \leq n$, and thus by Lemma 3, $\rho(A_\alpha(G)) = \rho(T) = \alpha d_1 + (1 - \alpha) d_1 \delta^{-(k-1)}$, i.e., (1) is an equality.

As $\delta \geq \left( \frac{d_2}{d_1} \right)^{\frac{1}{k}}$ with equality if and only if $d_1 = d_2$, we have by Theorem 4 that $\rho_\alpha(G) \leq \alpha d_1 + (1 - \alpha) d_1^{\frac{1}{k}} d_2^{1 - \frac{1}{k}}$ with equality if and only if $G$ is regular.

Letting $\alpha = 0$ in Theorem 4, we have $\delta = \left( \frac{d_2}{d_1} \right)^{\frac{1}{k}}$ and thus (1) becomes $\rho_0(G) \leq d_1^{\frac{1}{k}} d_2^{1 - \frac{1}{k}}$, see [29]. Letting $\alpha = \frac{1}{2}$ in Theorem 4, $\delta$ is the root of $d_2 t^k + (d_2 - d_1) t^{k-1} - d_1 = 0$, and (1) becomes $2 \rho_{1/2}(G) \leq d_1 + d_1 \delta^{-(k-1)}$, see [12].

Let $G$ be a connected $k$-uniform hypergraph with $n$ vertices, $m$ edges, maximum degree $\Delta$ and diameter $D$, where $k \geq 2$. For $0 \leq \alpha < 1$, let $\bar{\pi}$ be the maximum entry of the $\alpha$-Perron vector of $G$. From [11], we have

$$\rho_\alpha(G) \leq \Delta - \frac{(1 - \alpha) k(n\Delta - km)}{2(n\Delta - km)(k-1)D + (1 - \alpha) k} \bar{\pi}^k,$$

and if $D = 1$ and $k \geq 3$, then

$$\rho_\alpha(G) \leq \Delta - \frac{(1 - \alpha) (n\Delta - km)n}{2(n\Delta - km)(k-1) + (1 - \alpha) n} \bar{\pi}^k.$$

Theorem 5. Let $G$ be a connected $k$-uniform hypergraph on $n$ vertices with $m$ edges and maximum degree $\Delta$, where $k \geq 2$. Let $x$ be the $\alpha$-Perron vector of $G$ with maximum entry $\bar{x}$. For $0 \leq \alpha < 1$, we have

$$\rho_\alpha(G) \leq \alpha \Delta + (1 - \alpha) km \bar{x}^k$$

$$\rho_\alpha(G) \leq \alpha \Delta + (1 - \alpha) \left( \sum_{i \in V(G)} d_i^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}} \bar{x}^{k-1}$$

with either equality if and only if $G$ is regular.
Proof. From the eigenequation of $G$ at $i \in V(G)$, we have
\[(\rho_\alpha - \alpha \Delta)x_i^{k-1} \leq (\rho_\alpha - \alpha d_i)x_i^{k-1} = (1 - \alpha) \sum_{e \in E_i(G) \setminus \{i\}} \prod_{v \in e \setminus \{i\}} x_v \leq (1 - \alpha)d_i x_i^{k-1}\]
with equality if and only if for $v \in e \setminus \{i\}$ with $e \in E_i(G)$, $x_v = x$. Then
\[(\rho_\alpha - \alpha \Delta)x_i^k \leq (1 - \alpha)d_i x_i^k,
\]
and thus
\[\rho_\alpha - \alpha \Delta \leq (1 - \alpha)(1 - \alpha)x_i^k \sum_{i \in V(G)} d_i = (1 - \alpha)k \max x^k \]
with equality if and only if all entries of $x$ are equal, or equivalently, $G$ is regular.

On the other hand, we have
\[(\rho_\alpha - \alpha \Delta) x_i^k \leq (1 - \alpha) \sum_{i \in V(G)} d_i = (1 - \alpha)k \max x^k \]
and thus
\[\rho_\alpha - \alpha \Delta \leq (1 - \alpha)(1 - \alpha)x_i^k \sum_{i \in V(G)} d_i^k = (1 - \alpha)k \max x^k \]
implying that
\[\rho_\alpha(G) \leq \alpha \Delta + (1 - \alpha) \left( \sum_{i \in V(G)} d_i^{k-1} \right)^{k-1} \]
with equality if and only if $G$ is regular.

Let $\alpha = 0$ in Theorem 5, we have $x \geq \frac{\rho_0^{1/k}}{\sum_{i \in V(G)} d_i^{k-1} x_i}$, which has been reported in [9].

4. Transformations Increasing $\alpha$-Spectral Radius

In the following, we propose several types of hypergraph transformations that increase the $\alpha$-spectral radius.

**Theorem 6.** For $k \geq 2$, let $G$ be a $k$-uniform hypergraph with $u, v_1, \ldots, v_r \in V(G)$ and $e_1, \ldots, e_r \in E(G)$ for $r \geq 1$ such that $u \notin e_i$ and $v_i \in e_i$ for $i = 1, \ldots, r$, where $v_1, \ldots, v_r$ are not necessarily distinct. Let $e'_i = (e_i \setminus \{v_i\}) \cup \{u\}$ for $i = 1, \ldots, r$. Suppose that $e'_i \notin E(G)$ for $i = 1, \ldots, r$. Let $G' = G - \{e_1, \ldots, e_r\} + \{e'_1, \ldots, e'_r\}$. Let $x$ the $\alpha$-Perron vector of $G$. If $x_u \geq \max \{x_{v_1}, \ldots, x_{v_r}\}$, then $\rho_\alpha(G') > \rho_\alpha(G)$ for $0 \leq \alpha < 1$. 

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Proof. From the eigenequation of $G$ at $i \in V(G)$, we have
\[(\rho_\alpha - \alpha \Delta)x_i^{k-1} \leq (\rho_\alpha - \alpha d_i)x_i^{k-1} = (1 - \alpha) \sum_{e \in E_i(G) \setminus \{i\}} \prod_{v \in e \setminus \{i\}} x_v \leq (1 - \alpha)d_i x_i^{k-1}\]
with equality if and only if for $v \in e \setminus \{i\}$ with $e \in E_i(G)$, $x_v = x$. Then
\[(\rho_\alpha - \alpha \Delta)x_i^k \leq (1 - \alpha)d_i x_i^k,
\]
and thus
\[\rho_\alpha - \alpha \Delta \leq (1 - \alpha)(1 - \alpha)x_i^k \sum_{i \in V(G)} d_i = (1 - \alpha)k \max x^k \]
with equality if and only if all entries of $x$ are equal, or equivalently, $G$ is regular.

On the other hand, we have
\[(\rho_\alpha - \alpha \Delta) x_i^k \leq (1 - \alpha) \sum_{i \in V(G)} d_i = (1 - \alpha)k \max x^k \]
and thus
\[\rho_\alpha - \alpha \Delta \leq (1 - \alpha)(1 - \alpha)x_i^k \sum_{i \in V(G)} d_i^k = (1 - \alpha)k \max x^k \]
implying that
\[\rho_\alpha(G) \leq \alpha \Delta + (1 - \alpha) \left( \sum_{i \in V(G)} d_i^{k-1} \right)^{k-1} \]
with equality if and only if $G$ is regular.

Let $\alpha = 0$ in Theorem 5, we have $x \geq \frac{\rho_0^{1/k}}{\sum_{i \in V(G)} d_i^{k-1} x_i}$, which has been reported in [9].
Proof. Note that \( \rho_{\alpha}(G) = x^\top(A_{\alpha}(G)x) \) and \( \rho_{\alpha}(G') \geq x^\top(A_{\alpha}(G')x) \) with equality if and only if \( x \) is also the \( \alpha \)-Perron vector of \( G' \). Thus

\[
\rho_{\alpha}(G') - \rho_{\alpha}(G) \geq x^\top(A_{\alpha}(G')x) - x^\top(A_{\alpha}(G)x) = \alpha \left( rx_u - \sum_{i=1}^r x_{e_i}^k \right) + (1 - \alpha)k \sum_{i=1}^r (x_u - x_{v_i})x_{e_i \setminus \{v_i\}} \geq 0,
\]

and thus \( \rho_{\alpha}(G') \geq \rho_{\alpha}(G) \). Suppose that \( \rho_{\alpha}(G') = \rho_{\alpha}(G) \). Then \( \rho_{\alpha}(G') = x^\top(A_{\alpha}(G')x) \), and thus \( x \) is the \( \alpha \)-Perron vector of \( G' \). From the eigenequations of \( G' \) and \( G \) at \( u \) and noting that \( E_u(G') = E_u(G) \cup \{e', \ldots, e'_i\} \), we have

\[
\rho_{\alpha}(G')x_u^{k-1} = \alpha(d_u + r)x_u^{k-1} + (1 - \alpha) \sum_{e \in E_u(G')} x_{e \setminus \{u\}} > \alpha d_u x_u^{k-1} + (1 - \alpha) \sum_{e \in E_u(G)} x_{e \setminus \{u\}} = \rho_{\alpha}(G)x_u^{k-1},
\]

a contradiction. It follows that \( \rho_{\alpha}(G') > \rho_{\alpha}(G) \). \( \blacksquare \)

We say that the hypergraph \( G' \) in Theorem 6 is obtained from \( G \) by moving edges \( e_1, \ldots, e_r \) from \( v_1, \ldots, v_r \) to \( u \). Theorem 6 has been established in [8] for \( \alpha \in \{0, \frac{1}{2}\} \).

Theorem 7. Let \( G \) be a connected \( k \)-uniform hypergraph with \( k \geq 2 \), and \( e \) and \( f \) be two edges of \( G \) with \( e \cap f = \emptyset \). Let \( x \) be the \( \alpha \)-Perron vector of \( G \). Let \( U \subset e \) and \( V \subset f \) with \( 1 \leq |U| = |V| \leq k - 1 \). Let \( e' = U \cup (f \setminus V) \) and \( f' = V \cup (e \setminus U) \). Suppose that \( e', f' \notin E(G) \). Let \( G' = G - \{e, f\} + \{e', f'\} \). If \( x_U \geq x_V, x_{e \setminus U} \leq x_{f \setminus V} \) and one is strict, then \( \rho_{\alpha}(G) < \rho_{\alpha}(G') \) for \( 0 \leq \alpha < 1 \).

Proof. Note that

\[
\rho_{\alpha}(G') - \rho_{\alpha}(G) \geq x^\top(A_{\alpha}(G')x) - x^\top(A_{\alpha}(G)x) = (1 - \alpha)k \sum_{g \in E(G')} x_g - (1 - \alpha)k \sum_{g \in E(G)} x_g = (1 - \alpha)k \left( x_U x_{f \setminus V} + x_V x_{e \setminus U} - x_U x_{e \setminus U} - x_V x_{f \setminus V} \right) = (1 - \alpha)k(x_U - x_V)(x_{f \setminus V} - x_{e \setminus U}) \geq 0.
\]

Thus \( \rho_{\alpha}(G') \geq \rho_{\alpha}(G) \). Suppose that \( \rho_{\alpha}(G') = \rho_{\alpha}(G) \). Then \( \rho_{\alpha}(G') = x^\top(A_{\alpha}(G')x) \) and thus \( x \) is the \( \alpha \)-Perron vector of \( G' \). Suppose without loss of generality that \( x_{e \setminus U} < x_{f \setminus V} \). Then for \( u \in U \)

\[
-x_{e \setminus \{u\}} + x_{e' \setminus \{u\}} = -x_{U \setminus \{u\}} \left( x_{e \setminus U} - x_{f \setminus V} \right) > 0.
\]
From the eigenequations of $G'$ and $G$ at a vertex $u \in U$, we have
\[
\rho_\alpha(G')x_u^{k-1} = \alpha d_u x_u^{k-1} + (1 - \alpha) \sum_{g \in E_u(G')} x_g \setminus \{u\} \\
= \alpha d_u x_u^{k-1} + (1 - \alpha) \left( \sum_{g \in E_u(G)} x_g \setminus \{u\} - x_e \setminus \{u\} + x_{e'} \setminus \{u\} \right) \\
> \alpha d_u x_u^{k-1} + (1 - \alpha) \sum_{g \in E_u(G)} x_g \setminus \{u\} = \rho_\alpha(G)x_u^{k-1},
\]
a contradiction. It follows that $\rho_\alpha(G') > \rho_\alpha(G)$. \hfill \Box

The above result has been known for $k = 2$ in [5] and $\alpha = 0$ [25].

A path $P = (v_0, e_1, v_1, \ldots, v_{s-1}, e_s, v_s)$ in a $k$-uniform hypergraph $G$ is called a pendant path at $v_0$, if $d_G(v_0) \geq 2$, $d_G(v_i) = 2$ for $1 \leq i \leq s - 1$, $d_G(v_s) = 1$ for $v \in e_i \setminus \{v_{i-1}, v_i\}$ with $1 \leq i \leq s$, and $d_G(v_s) = 1$. If $s = 1$, then we call $P$ or $e_1$ a pendant edge of $G$ (at $v_0$). A pendant path of length 0 at $v_0$ is understood as the trivial path consisting of a single vertex $v_0$.

If $P$ is a pendant path at $u$ in a $k$-uniform hypergraph $G$, we say $G$ is obtained from $H$ by attaching a pendant path $P$ at $u$ with $H = G[V(G) \setminus (V(P) \setminus \{u\})]$. In this case, we write $G = H_u(s)$ if the length of $P$ is $s$. Let $H_u(0) = H$.

For a $k$-uniform hypergraph $G$ with $u \in V(G)$, and $p \geq q \geq 0$, let $G_u(p, q) = (G_u(p))_u(q)$.

**Theorem 8.** For $k \geq 2$, let $G$ be a connected $k$-uniform hypergraph with $|E(G)| \geq 1$ and $u \in V(G)$. For $p \geq q \geq 1$ and $0 \leq \alpha < 1$, we have $\rho_\alpha(G_u(p, q)) > \rho_\alpha(G_u(p + 1, q - 1))$.

**Proof.** Let $(u, e_1, v_1, \ldots, e_p, v_{p+1})$ and $(u, f_1, v_1, \ldots, v_{q-1}, f_{q-1}, v_q)$ be the pendant paths of $G_u(p + 1, q - 1)$ at $u$ of lengths $p + 1$ and $q - 1$, respectively. Let $v_0 = u$. Let $x$ be the $\alpha$-Perron vector of $G_u(p, q)$.

Suppose that $\rho_\alpha(G_u(p, q)) < \rho_\alpha(G_u(p + 1, q - 1))$. We prove that $x_{u_{q-i}} > x_{v_{q-i-1}}$ for $i = 0, \ldots, q - 1$.

Suppose that $x_{v_{q-1}} \geq x_{u_p}$. Let $H$ be the $k$-uniform hypergraph obtained from $G_u(p + 1, q - 1)$ by moving $e_{p+1}$ from $u_p$ to $v_{q-1}$. By Theorem 6 and noting that $H \cong G_u(p, q)$, we have $\rho_\alpha(G_u(p, q)) = \rho_\alpha(H) > \rho_\alpha(G_u(p + 1, q - 1))$, a contradiction. Thus $x_{u_p} > x_{v_{q-1}}$.

Suppose that $q \geq 2$ and $x_{u_{q-i}} > x_{v_{q-i-1}}$, where $0 \leq i \leq q - 2$. We want to show that $x_{u_{p-(i+1)}} > x_{v_{q-(i+1)-1}}$. Suppose that this is not true, i.e., $x_{v_{q-i}} \geq x_{u_{p-i}}$. Suppose that $x_{v_{p-i}} \setminus \{u_{p-i}, u_{p-i-1}\} \leq x_{v_{q-i}} \setminus \{v_{q-i-2}, v_{q-i-1}\}$. Then $x_{v_{q-i}} \setminus \{v_{p-i}\} \leq x_{v_{p-i}} \setminus \{u_{p-i}\}$. Let $H' = G_u(p + 1, q - 1) - \{e_{p-i}, f_{q-i-1}\} + \{e', f'\}$, where $e' = \{u_{p-i}\} \cup (f_{q-i-1} \setminus \{v_{q-i-1}\})$ and $f' = \{v_{q-i-1}\} \cup (e_{p-i} \setminus \{v_{p-i}\})$. Therefore, $H' \cong G_u(p + 1, q - 1)$ and $\rho_\alpha(H') = \rho_\alpha(G_u(p + 1, q - 1))$, a contradiction. Thus $x_{u_{p-(i+1)}} > x_{v_{q-(i+1)-1}}$.

Hence, $\rho_\alpha(G_u(p, q)) > \rho_\alpha(G_u(p + 1, q - 1))$.
Theorem 9. Let $G$ be a $k$-uniform hypergraph with $k \geq 2$, $e = \{v_1, \ldots, v_k\}$ be an edge of $G$ with $d_G(v_i) \geq 2$ for $i = 1, \ldots, r$, and $d_G(v_i) = 1$ for $i = r + 1, \ldots, k$, where $3 \leq r \leq k$. Let $G'$ be the hypergraph obtained from $G$ by moving all edges containing $v_3, \ldots, v_r$ but not containing $v_1$ from $v_3, \ldots, v_r$ to $v_1$. Then $\rho_\alpha(G') > \rho_\alpha(G)$ for $0 \leq \alpha < 1$.

Proof. Let $x$ be the $\alpha$-Perron vector of $G$, and $x_{v_1} = \max\{x_{v_i} : 3 \leq i \leq r\}$. If $x_{v_1} \geq x_{v_t}$, then by Theorem 6, $\rho_\alpha(G') > \rho_\alpha(G)$. Suppose that $x_{v_1} < x_{v_t}$. Let $G''$ be the hypergraph obtained from $G$ by moving all edges containing $v_t$ but not containing $v_1$ from $v_t$ to $v_1$ for all $3 \leq i \leq r$ with $i \neq t$, and moving all edges containing $v_1$ but not containing $v_t$ from $v_1$ to $v_t$. It is obvious that $G'' \cong G'$. By Theorem 6, we have $\rho_\alpha(G') = \rho_\alpha(G'') > \rho_\alpha(G)$.

5. Hypergraphs with Large $\alpha$-Spectral Radius

A hyperactus is a connected $k$-uniform hypergraph in which any two cycles (viewed as two hypergraphs) have at most one vertex in common. Let $H_{m,r,k}$ be a $k$-uniform hypergraph consisting of $r$ cycles of length 2 and $m - 2r$ pendant edges with a vertex in common. If $r = 0$, then $H_{m,r,k} \cong S_{m,k}$. 

Theorem 10. For $k \geq 2$, let $G$ be a $k$-uniform hyperactus with $m$ edges and $r$ cycles, where $0 \leq r \leq \left\lfloor \frac{m}{2} \right\rfloor$ and $m \geq 2$. For $0 \leq \alpha < 1$, we have $\rho_\alpha(G) \leq \rho_\alpha(H_{m,r,k})$ with equality if and only if $G \cong H_{m,r,k}$.

Proof. Let $G$ be a $k$-uniform hyperactus with maximum $\alpha$-spectral radius among $k$-uniform hyperacti with $m$ edges and $r$ cycles.

Let $x$ be the $\alpha$-Perron vector of $G$. 

\{u_{p-i}\}. Obviously, $H' \cong G_u(p, q)$. By Theorem 7, we have $\rho_\alpha(G_u(p, q)) = \rho_\alpha(H') > \rho_\alpha(G_u(p+1, q-1))$, a contradiction. Thus $x_{u_{p-i}} > x_{u_{p-i+1}}$, and then $x_{u_{p-i}} > x_{e_{q-i-1}}$. Let $H'' = G_u(p+1, q-1) - \{e_{p-i}, f_{q-i-1}\} + \{e''_i, f''_i\}$, where $e''_i = (e_{p-i} \cup \{u_{p-i-1}\}) \cup \{v_i, q-i-2\}$ and $f''_i = (f_{q-i-1} \cup \{v_i, q-i-2\}) \cup \{u_{p-i-1}\}$. Obviously, $H'' \cong G_u(p, q)$. By Theorem 7, we have $\rho_\alpha(G_u(p, q)) = \rho_\alpha(H'') > \rho_\alpha(G_u(p+1, q-1))$, also a contradiction.

It follows that $\rho_\alpha(G_u(p, q)) > \rho_\alpha(G_u(p+1, q-1))$. 

The above result has been reported for $k = 2$ in [5] and $\alpha = 0$ in [25].
Suppose first that \( r = 0 \), i.e., \( G \) is a hypertree with \( m \) edges. Let \( d \) be diameter of \( G \). Obviously, \( d \geq 2 \). Suppose that \( d \geq 3 \). Let \( (u_0, e_1, u_1, \ldots, e_k, u_d) \) be a diametral path of \( G \). Choose \( u \in e_{d-1} \) with \( u = \max\{x_v : v \in e_{d-1}\} \). Let \( G_1 \) be the hypertree obtained from \( G \) by moving all edges (except \( e_{d-1} \)) containing a vertex of \( e_{d-1} \) different from \( u \) from these vertices to \( u \). By Theorem 6, we have \( \rho_\alpha(G_1) > \rho_\alpha(G) \), a contradiction. Thus \( d = 2 \), implying that \( G \cong S_{m,k} = H_{m,0,k} \).

Suppose in the following that \( r \geq 1 \).

If there exists an edge \( e \) with at least three vertices of degree at least 2, then let \( e = \{v_1, \ldots, v_k\} \) with \( d_G(v_i) \geq 2 \) for \( i = 1, \ldots, \ell \), and \( d_G(v_1) = 1 \) for \( i = \ell + 1, \ldots, k \), where \( 3 \leq \ell \leq k \). Let \( G' \) be the hypergraph obtained from \( G \) by moving all edges containing \( v_3, \ldots, v_\ell \) except \( e \) from \( v_3, \ldots, v_\ell \) to \( v_1 \). Obviously, \( G' \) is a \( k \)-uniform hypercactus with \( m \) edges and \( r \) cycles. By Theorem 9, \( \rho_\alpha(G') > \rho_\alpha(G) \), a contradiction. Thus, every edge in \( G \) has \( k - 2 \) vertices of degree 1.

Suppose that there exist two vertex-disjoint cycles. We choose two such cycles \( C_1 \) and \( C_2 \) by requiring that \( d_G(C_1, C_2) \) is as small as possible, where \( d_G(C_1, C_2) = \min\{d_G(u, v) : u \in V(C_1), v \in V(C_2)\} \). Let \( u \in V(C_1) \) and \( v \in V(C_2) \) with \( d_G(C_1, C_2) = d_G(u, v) \). We may assume that \( u_x \geq v_x \). Let \( G'' \) be the hypergraph obtained from \( G \) by moving edges containing \( v_x \) from \( C_2 \) to \( v \). Obviously, \( G'' \) is a \( k \)-uniform hypercactus with \( m \) edges and \( r \) cycles. By Theorem 6, \( \rho_\alpha(G'') > \rho_\alpha(G) \), a contradiction. Thus, if \( r \geq 2 \), then all cycles in \( G \) share a common vertex, which we denote by \( w \). If \( r = 1 \), then \( w \) is a vertex of degree 2 of the unique cycle.

Let \( (v_0, e_1, v_1, \ldots, v_{\ell-1}, e_\ell, v_0) \) be a cycle of \( G \) of length \( \ell \geq 2 \), where \( v_0 = w \). Suppose that \( \ell \geq 3 \). Assume that \( x_{v_0} \geq x_{v_2} \). Let \( G^* \) be the hypergraph obtained from \( G \) by moving the edge \( e_2 \) from \( v_2 \) to \( v_0 \). Obviously, \( G^* \) is a \( k \)-uniform hypercactus with \( m \) edges and \( r \) cycles. By Theorem 6, \( \rho_\alpha(G^*) > \rho_\alpha(G) \), a contradiction. Thus, every cycle of \( G \) is of length 2, and there are exactly \( m - 2r \) edges that are not on any cycle.

Suppose that \( G \not\cong H_{m,r,k} \). Then there exists a vertex \( z \) such that \( d_G(w, z) = 2 \). Let \( z' \) be the unique vertex such that \( d_G(w, z') = d_G(z', z) = 1 \). There are two cases. First suppose that \( z' \) lies on some cycle. Let \( e_1 \) and \( e_2 \) be the cycle containing \( w \) and \( z' \). Let \( H \) be the hypergraph obtained from \( G \) by moving all edges containing \( z' \) except \( e_1 \) and \( e_2 \) from \( z' \) to \( w \) if \( x_w \geq x_{z'} \), and the hypergraph obtained from \( G \) by moving all edges containing \( w \) except \( e_1 \) and \( e_2 \) from \( w \) to \( z \) otherwise. Now suppose that \( z' \) does not lie on any cycle. Let \( e \) be the edge containing \( w \) and \( z' \). Let \( H \) be the hypergraph obtained from \( G \) by moving all edges containing \( z' \) except \( e \) from \( z' \) to \( w \) if \( x_w \geq x_{z'} \), and the hypergraph obtained from \( G \) by moving all edges containing \( w \) except \( e \) from \( w \) to \( z \) otherwise. In either case, \( H \) is a \( k \)-uniform hypercactus with \( m \) edges and \( r \) cycles. By Theorem 6, \( \rho_\alpha(H) > \rho_\alpha(G) \), a contradiction. It follows that \( G \cong H_{m,r,k} \).
Corollary 11. Suppose that $k \geq 2$.

(i) If $G$ is a $k$-uniform hypertree with $m \geq 1$ edges, then $\rho_\alpha(G) \leq \rho_\alpha(S_{m,k})$ for $0 \leq \alpha < 1$ with equality if and only if $G \cong S_{m,k}$.

(ii) If $G$ is a $k$-uniform unicyclic hypergraphs with $m \geq 2$ edges, then $\rho_\alpha(G) \leq \rho_\alpha(H_{m,1,k})$ for $0 \leq \alpha < 1$ with equality if and only if $G \cong H_{m,1,k}$.

The cases when $\alpha = 0$ in Corollary 11 (i) and (ii) have been known in [8, 2].

For $2 \leq d \leq m$, let $S_{m,d,k}$ be the $k$-uniform hypertree obtained from the $k$-uniform loose path $P_{d,k} = (v_0, e_1, v_1, \ldots, v_{d-1}, e_d, v_d)$ by attaching $m - d$ pendant edges at $v_{\lceil \frac{d}{2} \rceil}$. Obviously, $S_{m,2,k} \cong S_{m,k}$.

Theorem 12. For $k \geq 2$, let $G$ be a $k$-uniform hypertree with $m$ edges and diameter $d \geq 2$. For $0 \leq \alpha < 1$, we have $\rho_\alpha(G) \leq \rho_\alpha(S_{m,d,k})$ with equality if and only if $G \cong S_{m,d,k}$.

Proof. It is trivial for $d = 2$. Suppose that $d \geq 3$.

Let $G$ be a $k$-uniform hypertree with maximum $\alpha$-spectral radius among hypertrees with $m$ edges and diameter $d$.

Let $P = (v_0, e_1, v_1, \ldots, e_d, v_d)$ be a diametral path of $G$. Let $x$ be the $\alpha$-Perron vector of $G$.

Claim 1. Every edge of $G$ has at least $k - 2$ vertices of degree 1.

Proof. Suppose that there is at least one edge with at least three vertices of degree at least 2. Let $f = \{u_1, \ldots, u_k\}$ be such an edge. First suppose that $f$ is not an edge on $P$. We may assume that $d_G(u_i, P) = d_G(u_i, P) - 1$ for $i = 2, \ldots, k$, where $d_G(u_i, P) = \min\{d_G(u_i, v) : v \in V(P)\}$. Then $d_G(u_1) \geq 2$. We may assume that $d_G(u_i) \geq 2$ for $i = 2, \ldots, r$ and $d_G(u_i) = 1$ for $i = r + 1, \ldots, k$, where $3 \leq r \leq k$. Let $G'$ be the hypertree obtained from $G$ by moving all edges containing $u_3, \ldots, u_r$ except $f$ from $u_3, \ldots, u_r$ to $u_1$. Obviously, $G'$ is a hypertree with $m$ edges and diameter $d$. By Theorem 9, $\rho_\alpha(G') > \rho_\alpha(G)$, a contradiction. Thus $f$ is an edge on $P$, i.e., $f = e_i$ for some $i$ with $2 \leq i \leq d - 1$.

Let $e_i \setminus \{v_{i-1}, v_i\} = \{v_{i,1}, \ldots, v_{i,k-2}\}$. We may assume that $v_{i,1}, \ldots, v_{i,s}$ are precisely those vertices with degree at least 2 among $v_{i,1}, \ldots, v_{i,k-2}$, where $1 \leq s \leq k - 2$. Let $G''$ be the hypertree obtained from $G$ by moving all edges containing $v_{i,1}, \ldots, v_{i,s}$ except $e_i$ from $v_{i,1}, \ldots, v_{i,s}$ to $v_i$. Obviously, $G''$ is a hypertree with $m$ edges and diameter $d$. By Theorem 9, $\rho_\alpha(G'') > \rho_\alpha(G)$, also a contradiction. It follows that all edges of $G$ have at most two vertices of degree at least 2. Claim 1 follows.

Claim 2. Any edge not on $P$ is a pendant edge.

Proof. Suppose that $e$ is an edge not on $P$ and it is not a pendant edge. Then there are two vertices, say $u$ and $v$, in $e$ such that $d_u \geq 2$ and $d_v \geq 2$. Suppose
without loss of generality that $d_G(u, P) < d_G(v, P)$. Let $w$ be the vertex on $P$ with $d_G(u, P) = d_G(u, w)$. Let $G^*$ be the hypertree obtained from $G$ by moving all edges containing $v$ except $e$ from $v$ to $w$ if $x_w \geq x_v$, and the hypertree obtained from $G$ by moving all edges containing $w$ (except the edge in the path connecting $w$ and $v$) from $w$ to $v$ otherwise. By Theorem 6, $\rho_\alpha(G^*) > \rho_\alpha(G)$, a contradiction. This proves Claim 2.

Claim 3. There is at most one vertex of degree greater than two in $G$.

**Proof.** Suppose that there are two vertices, say $s$ and $t$, on $P$ with degree greater than two. We may assume that $x_s \geq x_t$. Let $H$ be the hypertree obtained from $G$ by moving all pendant edges containing $t$ from $t$ to $s$. By Theorem 6, we have $\rho_\alpha(H) > \rho_\alpha(G)$, a contradiction. Claim 3 follows.

Combining Claims 1–3, $G$ is a hypertree obtained from the path $P$ by attaching $m - d$ pendant edges at some $v_i$ with $1 \leq i \leq d - 1$, and by Theorem 8, we have $G \cong S_{m,d,k}$.

The above result for $\alpha = 0$ has been proved in [25] by a relation between the 0-spectral radius of a power hypergraph and the 0-spectral radius of its graph. Recall that for $\alpha = 0$ and $k = 2$, Simić and one author of this paper [23] determined the tree on $n$ vertices and diameter $d$ with the largest 0-spectral radius for $k = 1, \ldots, \left\lfloor \frac{d}{2} \right\rfloor + 1$ if $4 \leq d \leq n - 4$ and for $k = 1, \ldots, \left\lfloor \frac{d}{2} \right\rfloor$ if $d = n - 3$.

Suppose that $m \geq d \geq 3$. Let $H$ be the hypergraph obtained from $S_{m,d,k}$ by moving edge $e_d$ from $v_{d-1}$ to $v_{\left\lfloor \frac{d}{2} \right\rfloor + 1}$ if $x_{v_{d-1}} \geq x_{v_{\left\lfloor \frac{d}{2} \right\rfloor + 1}}$, and the hypergraph obtained from $S_{m,d,k}$ by moving edges containing $v_{\left\lfloor \frac{d}{2} \right\rfloor + 1}$ from $v_{\left\lfloor \frac{d}{2} \right\rfloor + 1}$ to $v_{d-1}$ otherwise. Obviously, $H \cong S_{m,d-1,k}$. By Theorem 6, $\rho_\alpha(S_{m,d,k}) < \rho_\alpha(S_{m,d-1,k})$.

Now by Theorem 12, Corollary 11(i) follows. Moreover, if $G$ is a $k$-uniform hypertree with $m \geq 3$ edges and $G \not\cong S_{m,k}$, $\rho_\alpha(G) \leq \rho_\alpha(S_{m,3,k})$ with equality if and only if $G \cong S_{m,3,k}$, which has been known for $\alpha = 0$ in [8].

For $2 \leq t \leq m$, let $T_{m,t,k}$ be the $k$-uniform hypertree consisting of $t$ pendant paths of almost equal lengths (i.e., $t - (m - t \left\lceil \frac{m}{t} \right\rceil)$ pendant paths of length $\left\lceil \frac{m}{t} \right\rceil$ and $m - t \left\lceil \frac{m}{t} \right\rceil$ pendant paths of length $\left\lceil \frac{m}{t} \right\rceil + 1$) at a common vertex. Particularly, $T_{m,2,k}$ is just the $k$-uniform loose path $P_{m,k}$.

**Theorem 13.** Let $G$ be a $k$-uniform hypertree with $m$ edges and $t \geq 2$ pendant edges. For $0 \leq \alpha < 1$, we have $\rho_\alpha(G) \leq \rho_\alpha(T_{m,t,k})$ with equality if and only if $G \cong T_{m,t,k}$.

**Proof.** Let $G$ be a $k$-uniform hypertree with maximum $\alpha$-spectral radius among hypertrees with $m$ edges and $t$ pendant edges. Let $x$ be the $\alpha$-Perron vector of $G$.

Suppose that there exists an edge $e = \{u_1, \ldots, u_k\}$ with at least three vertices of degree at least 2. Assume that $d_G(u_i) \geq d_G(u_{i+1})$ for $i = 1, \ldots, k-1$. Let $G'$ be the hypertree obtained from $G$ by moving all edges containing $u_3, \ldots, u_k$ except
e from these vertices to $u_1$. Obviously, $G'$ is a hypertree with $m$ edges and $t$ pendant edges. By Theorem 9, $\rho_\alpha(G') > \rho_\alpha(G)$, a contradiction. It follows that each edge of $G$ has at most two vertices of degree at least 2.

Suppose that there are two vertices, say $u, v$ with degree greater than 2. We may assume that $x_u \geq x_v$. Let $H$ be the hypertree obtained from $G$ by moving an edge not on the path connecting $u$ and $v$ containing $v$ from $v$ to $u$. By Theorem 6, we have $\rho_\alpha(H) > \rho_\alpha(G)$, a contradiction. Thus, there is at most one vertex of degree greater than 2 in $G$.

If there is no vertex of degree greater than 2, then $t = 2$, and $G$ is the $k$-uniform loose path $P_{m,k}$. If there is exactly one vertex of degree greater than 2, then $t \geq 3$, $G$ is a hypertree consisting of $t$ pendant paths at a common vertex, and by Theorem 8, we have $G \cong T_{m,t,k}$.

For $\alpha = 0$, this is known in [26, 30].

Acknowledgements

This work was supported by the National Natural Science Foundation of China (No. 11671156) and the Innovation Project of Graduate School of South China Normal University.

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Received 14 August 2018
Revised 24 April 2019
Accepted 24 April 2019