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DETERMINING GRAPHS BY THE COMPLEMENTARY SPECTRUM

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Abstract

The complementary spectrum of a connected graph G is the set of the complementary eigenvalues of the adjacency matrix of G. In this note, we discuss the possibility of representing G using this spectrum. On one hand, we give evidence that this spectrum distinguishes more graphs than other standard graph spectra. On the other hand, we show that it is hard to compute the complementary spectrum. In particular, we see that computing the complementary spectrum is equivalent to finding all connected induced subgraphs.

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1. Introduction

One of the main goals of spectral graph theory consists in giving information about a graph G by just looking at its spectrum. For this, we associate a graph

G to a matrix M and analyze the eigenvalues or eigenvectors of M. These eigenvalues are called the spectrum of G with respect to the matrix M and its multiset will be denoted by M-spectrum(G).

We say that two graphs are cospectral with respect to the matrix M, or that are M-cospectral, if their M-spectra are equal. A graph G is *determined by its M-spectrum*, M-DS for short, if only isomorphic graphs are cospectral with G.

Is there a matrix M for which all graphs are M-DS? The answer is no. For all known matrices M associated to graphs, there exist many examples of non isomorphic graphs that share the same M-spectrum. The smallest example known for the adjacency matrix is given in Figure 1.

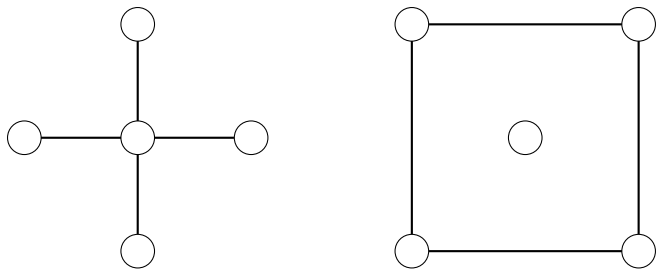

Figure 1. Nonisomorphic graphs with the same spectrum, but distinct complementary spectrum.

This missing feature is unfortunate for the area of spectral graph theory because it means the spectrum of a graph G is not enough to determine all properties of G. Important motivation for our question comes from complexity theory. It is still undecided whether graph isomorphism is an NP-hard problem. Since checking whether two graphs are M-cospectral can be done in polynomial time, the isomorphism problem concentrates on checking isomorphism between cospectral graphs.

A main research topic in this direction is to study if there is a matrix M (say adjacency, Laplacian, signless Laplacian, etc.) that distinguishes more graphs. We will make this concept more precise in the next section, but in general, the idea is to verify whether the portion of graphs that have an M-cospectral mate is smaller for a particular matrix M.

The main purpose of this note is to underscore and to understand a recent new proposal for representing a graph using complementary eigenvalues. Instead of changing the matrix associated with G, the suggestion is to modify the concept of eigenvalue. In order to explain the new proposal and its consequences, we will recall here a few facts.

Definition 1. Let A be a real matrix of order n. A real number λ is called a *complementary eigenvalue* of A if there exists a nonzero vector $x \in \mathbb{R}^n$ satisfying the complementarity system

$$0 \le x \perp (Ax - \lambda x) \ge 0$$

where $\perp$ stands for orthogonality and $x \ge 0$ means that every entry of vector x is non-negative.

Fernandes *et al.* [6], studied the complementary eigenvalues of matrices associated to graphs (Laplacian, adjacency, etc.) and we say that the *complementary spectrum of a graph G* is the set of complementary eigenvalues of its adjacency matrix. Seeger [13] proposes to represent a graph by its complementary spectrum. As an example, the smallest pair of two nonisomorphic A-cospectral graphs of Figure 1 have different complementary spectrum.

Indeed, the spectrum of both graphs is the multiset $\{-2, 0, 0, 0, 2\}$, whereas the complementary spectrum of the graph on the left is $\{0, 1, \sqrt{2}, \sqrt{3}, 2\}$ and the complementary spectrum of the graph on right is $\{0, 1, \sqrt{2}, 2\}$. (See Section 3 how to compute the complementary spectrum of graphs).

In this paper, we shall reason that this new spectral way of representing a graph may do a better job in distinguishing them. For this, we formalize the proper definitions. Two graphs are said to be *complementary cospectral* if they have the same complementary spectrum.

Definition 2. We say that a connected graph G is *determined by its complementary spectrum*—DCS for short—if any cospectral graph H is either isomorphic to G or the number of vertices of H and G are distinct.

Throughout the paper, while reviewing some well known facts about spectra of graphs, we pose research questions that seem to be relevant in light of this new look on the spectra of graphs. In particular, we address the question of whether there exist pairs of non isomorphic graphs with the same complementary spectrum. We advance this by saying that we have found no examples of non isomorphic graphs with the same complementary spectrum.

The remainder of the paper is organized as follows. In the next Section 2 we review and discuss the issue of distinguishing graphs by their spectra. In Section 3 we show how to compute the complementary spectrum of a graph G—an interesting interplay between algebraic and combinatorial problems. We also explain Seeger's proposal for determining graphs from their complementary spectra. In Section 4 we show that some graphs are DCS. The path, the cycle, the complete and the star are DCS. We also show that all graph with less than 8 vertices are DCS. In section 5 we find several classes of graphs $\mathcal{G}$ whose elements have unique complementary spectrum in $\mathcal{G}$. Finally, in Section 6, we discuss the advantages and disadvantages of the proposal representation of the graphs. Particularly, we address the question of the cardinality of the complementary spectrum of a graph.

2. Distinguishing Graphs by Their Spectra

When a graph G is M-DS? This means that G has a unique M-spectrum over all the graphs having the same number of vertices of G. As van Dam and Haemers [14] point out, it is very hard to prove that a graph G is M-DS, for any matrix M. In fact it seems easier to find families of graphs that have M-cospectral mates.

The milestone work of Schwenk [12] shows that almost all trees have an A-spectral mate, meaning that hardly any tree can be characterized by its A-spectrum. This result has been extended to Laplacian, signless Laplacian and distance matrix by McKay in [10]. In 1982, Godsil and Mckay [7] introduced an operation, now called the GM-switching, that has been used to construct families of cospectral graphs with respect to the adjacency and other matrices associated to a graph.

These developments seem to go against the conjecture that almost all graphs are DS. This conjecture has been forged by van Dam and Haemers in the papers [8, 14, 15]. The conjecture means, if true, that among all non-isomorphic graphs on at most n vertices, the fraction that is DS goes to 1 when n goes to infinity. We observe that since the number of trees compared to the number of all graphs is negligible, the fact that almost all trees have a cospectral mate does not interfere with the general conjecture. This conjecture appears to be formulated for any matrix M associated to graphs (Laplacian, Adjacency, signless Laplacian, etc.). However, it is our understanding that the conjecture is far from being settled. We noticed that there even exist a few arguments against the validity of the conjecture [8].

Before the conjecture was firmly stated, there was a debate whether any particular matrix would distinguish more graphs than other matrices. More precisely, it was discussed whether the portion of DS graphs among all graphs on at most n vertices is larger for a particular matrix M associated to a graph. In 2009, Cvetković and Simić, in the beautiful series of papers [3, 4, 5], introduced many properties of the signless Laplacian matrix for graphs and, in particular, advocate that the signless Laplacian matrix would distinguish more graphs. From the table

n	4	5	6	7	8	9	10	11
r_n	0	0.059	0.064	0.105	0.139	0.186	0.213	0.211
s_n	0	0	0.026	0.125	0.143	0.155	0.118	0.090
q_n	0.182	0.118	0.103	0.098	0.097	0.069	0.053	0.038

where r_n, s_n and q_n are the *spectral uncertainty* associated with the adjacency, Laplacian and signless Laplacian, respectively, that is the portion of graphs on n vertices that have a cospectral mate among graphs on n vertices. Quoting the authors: "We see that numbers q_n are smaller than the numbers r_n and s_n for

$n \geq 7$. In addition, the sequence q_n is decreasing for $n \leq 11$ while the sequence r_n is increasing for $n \leq 10$. This is a strong basis for believing that studying graphs by Q-spectra is more efficient than studying them by their (adjacency) spectra."

Even though it is no longer clear that the Q matrix distinguishes more graphs than other matrices, it is a fact that these computational results were used for a long time by many authors to justify the use of this matrix. It is worth mentioning a somewhat unexpected result by Carvalho *et al.* [2] which shows the existence of exponentially many Q-cospectral threshold graphs.

Nevertheless, the series of papers by Cvetković and Simić presented the Q-theory for graphs. The signless Laplacian matrix is now considered an important matrix that determines many structural properties of graphs. The following question still remains.

Problem 3. Is there a matrix M associated to graphs such that the M-spectra distinguish more graphs than other matrices?

3. Computing the Complementary Spectrum of a Graph

Let G be a connected graph with n vertices. The largest eigenvalue of the adjacency matrix $A(G)$ of G, denoted by $\lambda(A(G))$, is called the *spectral radius* or the *index* of G. The most important information about computing the complementary spectrum is the following result [6].

Theorem 4. *Let G be a connected graph with n vertices. The complementary spectrum $\mathcal{CS}(G)$ of G is the set composed by the spectral radius of all induced connected subgraphs of G.*

We observe that the complementary spectrum has only nonnegative and no repeated values. Moreover, because the complementary spectrum of a disconnected graph is the union of the complementary spectrum of its components, we may consider, without loss of generality, only connected graphs.

We refer to [13] for several important properties of the complementary spectrum $\mathcal{CS}(G)$ of G, but recall here a few facts that are relevant to this paper. Let us denote by $\varrho = \varrho(G)$ the largest complementary eigenvalue and by $\varrho_2 = \varrho_2(G)$ its second largest complementary eigenvalue.

Fact 1. 0 *is the smallest complementary eigenvalue of any graph G.*

Fact 2. $\varrho = \lambda(A(G))$ *is the spectral radius of $A(G)$.*

Fact 3. $\varrho_2 = \max\{\lambda(G - v) : v \in V(G)$ *and v is not a cut vertex of G*$\}$.

Theorem 4 is an interplay between a combinatorial problem—the determination of connected induced subgraphs—an algebraic problem—the computation of spectral radii of principal submatrices of the adjacency matrix—and an optimization problem—the computation of complementary eigenvalues.

We will see further in this note that the cardinality of the set $\mathcal{CS}(G)$ plays an important role. Only as an observation, we point out that two graphs with the same number of vertices may have a different number of complementary eigenvalues. As an example the cycle C_4 on 4 vertices has $\mathcal{CS}(C_4) = \{0, 1, \sqrt{2}, 2\}$, while the graph H composed by a triangle with a pendent vertex has $\mathcal{CS}(H) = \{0, 1, \sqrt{2}, 2, \lambda(H)\}$, where $\lambda(H) \approx 2.17009$. Additionally, there may exist nonisomorphic subgraphs of G having the same index, hence the following holds [6].

Corollary 5. *Let G be a connected graph with n vertices and $\mathfrak{b}(G)$ be the number of induced nonisomorphic connected subgraphs of G. Then*

$$|\mathcal{CS}(G)| \leq \mathfrak{b}(G) \quad \text{and} \quad n \leq \mathfrak{b}(G) \leq 2^n - 1.$$

The lower bound of Theorem 4 is nicely settled by Seeger [13] as follows. For a given n, we say that the complete graph K_n, the star S_n, the path P_n and the cycle C_n, all with n vertices, are *elementary graphs*.

Theorem 6. *Let G be a connected graph with n vertices. Then*

$$n \leq |\mathcal{CS}(G)|.$$

Equality holds if and only if G is an elementary graph.

4. Graphs Determined by the Complementary Spectrum

The number of complementary eigenvalues of a graph G is not determined by the number of vertices of G, instead it depends on the number of different spectral radii of the induced subgraphs of G. Hence, an interesting strategy to characterize graphs or classes of graphs by this spectral property is to study the cardinality of the complementary spectrum. For example, if we show that a graph G with n vertices is the only graph among all graphs on n vertices that has k complementary eigenvalues, we will have shown that this graph is DCS.

Let K_n, C_n, P_n and S_n be, respectively, the complete graph, the cycle, the path and the star on n vertices. Following Seeger, we will call them elementary graphs.

Theorem 7. *Elementary graphs are DCS.*

Proof. Let $\mathcal{S}(G)$ denote the set of all induced connected subgraphs of G and $\mathcal{CS}(G)$ denote the set of complementary spectrum of G. We know that

$$\mathcal{S}(K_n) = \{K_1, K_2, \ldots, K_{n-1}, K_n\},$$
$$\mathcal{S}(C_n) = \{P_1, P_2, \ldots, P_{n-1}, C_n\},$$
$$\mathcal{S}(P_n) = \{P_1, P_2, \ldots, P_{n-1}, P_n\},$$
$$\mathcal{S}(S_n) = \{S_1, S_2, \ldots, S_{n-1}, S_n\}.$$

So, we have

$$\mathcal{CS}(K_n) = \{0, 1, 2, \ldots, n-1\},$$
$$\mathcal{CS}(C_n) = \{\omega_1, \omega_2, \ldots, \omega_{n-1}, 2\},$$
$$\mathcal{CS}(P_n) = \{\omega_1, \omega_2, \ldots, \omega_{n-1}, \omega_n\},$$
$$\mathcal{CS}(S_n) = \{0, 1, \sqrt{2}, \ldots, \sqrt{n-1}\},$$

where $\omega_i = 2\cos\left(\frac{\pi}{i+1}\right)$. As we know the set of all induced subgraphs of the elementary graphs, we also know that their complementary spectra are different from each other. Actually, except for C_5 and S_5, we just need to compute the spectral radius of the elementary graphs to see this. And for C_5 and S_5, we just need to compute the second largest complementary eigenvalue to see that $\varrho_2(C_5) \neq \varrho_2(S_5)$, in spite of $\varrho(C_5) = \varrho(S_5)$.

Moreover, by Theorem 6, these are the only graphs G having $|\mathcal{CS}(G)| = n$, hence their complementary spectrum is different from any other graph with n vertices. This proves the result. ■

Notice that we know all the induced connected subgraphs of K_n, C_n, P_n and S_n. Hence, we not only determine these graphs by their complementary spectrum, but we can also compute the whole complementary spectra of the elementary graphs K_n, C_n, P_n and S_n.

4.1. Ordering of graphs

In this subsection, we give further details of the spectral representation of graphs proposed by Seeger [13].

As a motivation, consider the set $\mathcal{C}_6$ of all graphs with $n \leq 6$ vertices. Denote by $|G|$ the number of vertices of G. Define in $\mathcal{C}_6$ the following order

$$H \preceq G \leftrightarrow (|H|, \varrho(H), \varrho_2(H)) \preceq_{\text{lex}} (|G|, \varrho(G), \varrho_2(G)), \tag{1}$$

where $\preceq_{\text{lex}}$ is the lexicographic order in $\mathbf{R}^3$.

Seeger has shown, by computing numerically the complementary spectrum, that this lexicographic rule is a total ordering in $\mathcal{C}_6$, that is, all graphs with $n \leq 6$ vertices can be distinguished either by the largest or the second largest complementary eigenvalue. According to our definition, this means that all graphs up to 6 vertices are DCS.

For graphs with 7 vertices, we report the following experiment. For the first step, we computed the index of all 853 graphs on 7 vertices. Notice that when the spectral radii of these graphs are different, it means that these graphs are determined by their complementary spectrum once that the largest complementary eigenvalue of them are all different.

In case two graphs G and H have $\varrho(G) = \varrho(H)$, we compute the second largest complementary eigenvalue. We removed all non-cut vertices of these graphs obtaining all possible connected induced subgraphs (with 6 vertices), after that we compute the spectral radii of all these connected induced subgraphs and chose the largest one. In this second step we looked for the graphs with the same ϱ_2 and, for this set, it was necessary to compute ϱ_3.

In the third step we determined the candidate subgraphs to be ϱ_3 for these graphs, based on the interlacing of eigenvalues. We then computed the ϱ_3 and, for those where ϱ_3 was the same, we computed ϱ_4 in the same way we did for ϱ_3. Finally, this was the final step. There is no pair of graphs that has the same ϱ, ϱ_2, ϱ_3 and ϱ_4.

This means that the order given by equation 1 is not enough to distinguish all graphs with 7 vertices. However, we only need to compute the first four largest complementary eigenvalues to determine all graphs on 7 vertices. In any event, we can state the following result.

Theorem 8. *All graphs with $n \leq 7$ vertices are DCS.*

This may suggest that this complementary spectrum approach may be an alternative spectral technique that defines a greater portion of graphs.

To finish this section, we give the complete ordering formulation given by Seeger. For the set $\mathcal{C}$ of all connected graphs, define the function

$$G \in \mathcal{C} \longrightarrow \Psi_q(G) = (|G|, \varrho(G), \varrho_2(G), \ldots, \varrho_q(G)),$$

where $\varrho_k(G) = 0$ if $k > |\mathcal{CS}(G)|$. Define the order

$$H \preceq_q G \Leftrightarrow \Psi_q(H) \preceq \Psi_q(G), \tag{2}$$

a natural problem is to determine whether there exists q such that this defines a total ordering in $\mathcal{C}$.

Problem 9. Let $\mathcal{C}$ be the set of all connected graphs. Does there exist q such that (2) defines a total order in $\mathcal{C}$?

A positive answer to this question is equivalent to say that all graphs are DCS.

5. CLASSES WITH UNIQUE COMPLEMENTARY SPECTRUM

Consider a class of graphs $\mathcal{G}$ in which each element $G \in \mathcal{G}$ has a unique complementary spectrum. More precisely, if for $H, G \in \mathcal{G}$ we have $\varrho(H) = \varrho(G) \iff H$

and G are isomorphic. We say in this case that the graphs of $\mathcal{G}$ are determined by their complementary spectrum in $\mathcal{G}$. For short we say that $\mathcal{G}$ is DCS. Notice that the graphs of these classes are DCS just inside the class they belong to, which may be a first step to show they are DCS.

It is well known that the largest complementary eigenvalue of a graph G is the spectral radius of G. If a class $\mathcal{G}$ is such that each element $G \in \mathcal{G}$ has a unique spectral radius, then by the above definition, we say that $\mathcal{G}$ is DCS.

In this section we find a few classes $\mathcal{G}$ which are DCS.

5.1. Complete bipartite graphs

We say that a graph G on n vertices is complete bipartite if the set of vertices of G can be partitioned into two disjoint sets of cardinality r and s such that none of the vertices in each set are adjacent and every vertex in one bipartition is adjacent to every vertex in the other bipartition. We will denote this graph by $K_{r,s}$. It is well known that

$$\varrho(K_{r,s}) = \sqrt{rs}. \tag{3}$$

If we fix the number of vertices, we have $r + s = n$, in order to prove the uniqueness of the spectral radius, we shall take two different partitions of n, say $n = p + q = r + s$, and show that if $K_{p,q}$ and $K_{r,s}$ have the same spectral radius then they are isomorphic.

Notice that $rs = rn - r^2$ and $pq = np - p^2$. Suppose $pq = rs$, so that $K_{p,q}$ and $K_{r,s}$ have the same spectral radius. Then $(r - p)n = (r - p)(r + p)$. If $p = r$, then $q = s$ and $K_{p,q}$ and $K_{r,s}$ are isomorphic. If $p \neq r$, then $n = p + r$. But $n = p + q$, so $q = r$ and, consequently, $p = s$. We conclude once again that $K_{p,q}$ and $K_{r,s}$ are isomorphic.

This means that the class of the complete bipartite graphs is DCS.

In this class of graphs, we can actually compute the spectral radius of the graphs. We notice there are methods of ordering a whole class of graphs by their spectral radius, without computing them. This will also allows one to conclude that this class is DCS without really knowing what the spectral radius is.

5.2. Lollipops

A lollipop with n vertices, denoted by $H_{n,k}$, is a graph obtained by pending in a vertex of the cycle C_k, a terminal vertex of the path P_{n-k}, where $3 \leq k \leq n$. In order to prove that the class of lollipop graphs is DCS, we need the following concept.

An internal path in a graph G, denoted by $v_1 v_2 \cdots v_{r-1} v_r$, is a path beginning at v_1 and ending at v_r, where v_1 and v_r both have degree bigger than two, while all other vertices have degree two. The vertices v_1 and v_r are not necessarily distinct. We denote by W_n the tree with n vertices where two vertices have degree three and the distance between them is $n-5$. In the following we denote by λ the spectral radius of the graph we are considering. The following result, according to Belardo [1], appears in the work by Hoffman and Smith [9].

Lemma 10. *Let G be a graph with n vertices, $G \neq C_n, W_n$. Let G' be the graph with $n+1$ vertices obtained from G by inserting a new vertex of degree two in an edge e. Then*

(i) *if e lies on an internal path, then $\lambda(G') < \lambda(G)$,*

(ii) *if e does not lie on an internal path, then $\lambda(G') > \lambda(G)$.*

Theorem 11. *Given n, if we take $3 \leq k \leq n-1$, then $\lambda(H_{n,k+1}) < \lambda(H_{n,k})$.*

Proof. Consider the graph $H_{n,k}$. By Lemma 10, if we add a vertex in an edge of C_k, we have $\lambda(H_{n+1,k+1}) < \lambda(H_{n,k})$. If we delete the end vertex of the path P_{n-k} in $H_{n+1,k+1}$, we obtain $\lambda(H_{n,k+1}) < \lambda(H_{n+1,k+1})$, because $H_{n,k+1}$ is a proper subgraph of $H_{n+1,k+1}$. This proves the result. ∎

From this result we conclude that lollipops with n vertices are determined by their spectral radius. Hence they the class is DCS.

5.3. Starlike trees

A starlike tree is a tree having a unique vertex of degree greater than 2. In [11], the authors prove that, for a fixed n, all starlike trees with n vertices have distinct spectral radius. In the next paragraph, we explain the result in a more precise way.

A starlike tree with n vertices may be represented as a partition of $n-1$, say $T = [m_1, m_2, \ldots, m_k]$, where $m_i \geq 1$, $n-1 = m_1 + m_2 + \cdots + m_k$, $k \geq 3$ and the paths P_{m_i} are attached to common vertex v. Moreover, without loss of generality we assume that $1 \leq m_1 \leq m_2 \leq \cdots \leq m_k$. In the paper [11], it is proven that the lexicographic order of the k-tuple $[m_1, m_2, \ldots, m_k]$ gives a total ordering of the spectral radii of the starlike trees with n vertices.

This shows that the class of starlike trees is DCS.

5.4. Trees—computational results

A tree T is a connected graph without cycles. With respect to the complementary spectrum of a tree, we have performed some experiments and the following results arise.

Considering all trees up to 14 vertices, we observe that there are no cospectral pairs if we consider the complementary spectrum. The experiment consists in fixing n and computing the spectral radii of all trees on n vertices. For trees with the same spectral radius, we further compute the second largest complementary eigenvalue taking all non-cut vertices and compute the spectral radii of all possible induced subgraphs.

For n up to 6 vertices, no tree has the same spectral radius. For $n \leq 10$ there are no pairs of cospectral trees with respect to the complementary spectrum. More than that, they are determined just by ϱ and ϱ_2. We summarize the results in Table 1 and notice that we need only ϱ, ϱ_2 and ϱ_3 to distinguish all trees up to 14 vertices. In Seeger notation, it means that $\Psi_3(G) = (|G|, \varrho(G), \varrho_2(G), \varrho_3(G))$ is a total order in the class of trees up to 14 vertices.

n	Trees	Cospectral Trees	ϱ equal	same ϱ, ϱ_2	same ϱ, ϱ_2, ϱ_3
7	11	0	2 graphs 1 pair	0	0
8	23	2 graphs 1 pair	4 graphs 2 pairs	0	0
9	47	10 graphs 5 pairs	18 graphs 7 pairs 1 quartet	0	0
10	106	8 graphs 4 pairs	24 graphs 9 pairs 2 triples	0	0
11	235	60 graphs 27 pairs 2 triples	106 graphs 31 pairs, 9 triples, 3 quartets and 1 quintet	2 graphs 1 pair	0
12	551	119 graphs 49 pairs 7 triples	197 graphs 57 pairs, 48 triples, 2 quartets and 5 quintets	8 graphs 4 pairs	0
13	1301	192 sets	662 graphs	29 graphs 10 pairs 3 triples	0
14	3159	390 sets	1245 graphs	51 graphs 24 pairs 1 triple	0

Table 1. Experiment.

6. Final Remarks

In this note, following the ideas of [6] and [13], we have shown how to use complementary eigenvalues of the adjacency matrix to represent the spectrum of a graph. We have defined the notion of a connected graph being *defined by the complementary spectrum—DCS*—when it has a unique complementary spectrum among all graphs with the same order. We show that the elementary graphs (the path, the cycle, the star and the complete graph) are DCS. We show, by computing the complementary spectra, that all graphs with less than 8 vertices are DCS. Additionally, we have not found two non isomorphic connected graphs with the same complementary spectrum.

As we have seen, the complementary spectrum of a graph G is the set composed by the spectral radii of all connected induced subgraphs of G. This result may be seen as a nice relationship between the algebraic problem of computing the complementary eigenvalues of the adjacency matrix and the combinatorial problem of determining the connected induced subgraphs of G. On the other hand, it also may be seen as an evidence of the difficulty of the problem. Which is harder? Computing the complementary eigenvalues of an adjacency matrix or to find all connected induced subgraphs?

The difficulty of these problems is related to the cardinality $|\mathcal{CS}(G)|$ of the complementary spectrum of a graph G. Moreover, we notice that $|\mathcal{CS}(G)|$ can be related to the isomorphism problem in the following way. If $|\mathcal{CS}(G)|$ were bounded by a polynomial in n, the order of G, then the complementary spectrum could be computed in polynomial time. In this case, if all graphs were DCS, the conclusion would be that the isomorphism problem is polynomial. Clearly, this line of reasoning is very speculative and perhaps the only merit is to show that proving that a graph G is DCS, or computing the complementary spectrum of G or merely bounding its cardinality are very hard problems. Indeed, as we see next, the cardinality $|\mathcal{CS}(G)|$ is not bounded by a polynomial.

Let $\mathcal{G}_n$ be the set of all connected graphs of order n. Corollary 5 shows that for $G \in \mathcal{G}_n$, $|\mathcal{CS}(G)| \leq 2^n - 1$. Can we find a better than exponential upper bound? In [6], the authors determined that $|\mathcal{CS}(G)|$ grows faster than any polynomial in n. More precisely, they showed that, for fixed n, there is a starlike tree T with n vertices whose

$$|\mathcal{CS}(T)| \sim \frac{\exp \pi \sqrt{2\sqrt{n}/3}}{4\sqrt{3n}}.$$

This means that $|\mathcal{CS}(G)|$ cannot be bounded by a polynomial in n. It is still unknown whether there is an upper bound whose growth is smaller than an exponential. We finish this note by posing the following question.

Problem 12. Is there a function of n bounding the cardinality $|\mathcal{CS}(G)|$ for all $G \in \mathcal{G}_n$ whose growth is smaller than exponential?

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