BALANCEDNESS AND THE LEAST LAPLACIAN EIGENVALUE OF SOME COMPLEX UNIT GAIN GRAPHS

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Abstract

Let $\mathbb{T}_4 = \{\pm 1, \pm i\}$ be the subgroup of 4-th roots of unity inside $\mathbb{T}$, the multiplicative group of complex units. A complex unit gain graph $\Gamma$ is a simple graph $\Gamma = (V(\Gamma) = \{v_1, \ldots, v_n\}, E(\Gamma))$ equipped with a map $\varphi : \overrightarrow{E}(\Gamma) \to \mathbb{T}$ defined on the set of oriented edges such that $\varphi(v_i v_j) = \varphi(v_j v_i)^{-1}$.

The gain graph $\Phi$ is said to be balanced if and only if $\varphi(v_i v_j) \varphi(v_j v_i) \cdots \varphi(v_k v_1) = 1$ for every cycle $C = v_1 v_2 \cdots v_k v_1$.

It is known that $\Phi$ is balanced if and only if the least Laplacian eigenvalue $\lambda_n(\Phi)$ is 0. Here we show that, if $\Phi$ is unbalanced and $\varphi(\Phi) \subseteq \mathbb{T}_4$, the eigenvalue $\lambda_n(\Phi)$ measures how far is $\Phi$ from being balanced. More precisely, let $\nu(\Phi)$ (respectively, $\epsilon(\Phi)$) be the number of vertices (respectively, edges) to cancel in order to get a balanced gain subgraph. We show that

$$\lambda_n(\Phi) \leq \nu(\Phi) \leq \epsilon(\Phi).$$

We also analyze the case when $\lambda_n(\Phi) = \nu(\Phi)$. In fact, we identify the structural conditions on $\Phi$ that lead to such equality.

Keywords: gain graph, Laplacian eigenvalues, balanced graph, algebraic frustration.

2010 Mathematics Subject Classification: 05C50, 05C22.
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doi:10.1007/s10114-007-0002-2

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doi:10.1016/0095-8956(89)90063-4


Received 11 June 2019
Revised 15 October 2019
Accepted 24 October 2019