

## GRAPHS WITH CLUSTERS PERTURBED BY REGULAR GRAPHS — $A_\alpha$ -SPECTRUM AND APPLICATIONS

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### Abstract

Given a graph  $G$ , its adjacency matrix  $A(G)$  and its diagonal matrix of vertex degrees  $D(G)$ , consider the matrix  $A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$ , where  $\alpha \in [0, 1)$ . The  $A_\alpha$ -spectrum of  $G$  is the multiset of eigenvalues of  $A_\alpha(G)$  and these eigenvalues are the  $\alpha$ -eigenvalues of  $G$ . A cluster in  $G$  is a pair of vertex subsets  $(C, S)$ , where  $C$  is a set of cardinality  $|C| \geq 2$  of pairwise co-neighbor vertices sharing the same set  $S$  of  $|S|$  neighbors. Assuming that  $G$  is connected and it has a cluster  $(C, S)$ ,  $G(H)$  is obtained from  $G$  and an  $r$ -regular graph  $H$  of order  $|C|$  by identifying its vertices with the vertices in  $C$ , eigenvalues of  $A_\alpha(G)$  and  $A_\alpha(G(H))$  are deduced and if  $A_\alpha(H)$  is positive semidefinite, then the  $i$ -th eigenvalue of  $A_\alpha(G(H))$  is greater than or equal to  $i$ -th eigenvalue of  $A_\alpha(G)$ . These results are extended to graphs with several pairwise disjoint clusters  $(C_1, S_1), \dots, (C_k, S_k)$ . As an application, the effect on the energy,  $\alpha$ -Estrada index and  $\alpha$ -index of a graph  $G$  with clusters when the edges of regular graphs are added to  $G$  are analyzed. Finally, the  $A_\alpha$ -spectrum of the corona product  $G \circ H$  of a connected graph  $G$  and a regular graph  $H$  is determined.

**Keywords:** cluster, convex combination of matrices,  $A_\alpha$ -spectrum, corona product of graphs.

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## 1. INTRODUCTION AND PRELIMINARIES

We deal with simple undirected graphs  $G = (V(G), E(G))$  on  $n$  vertices with vertex set  $V(G)$  and edge set  $E(G)$ . The complement of  $G$  is the graph  $\overline{G}$  with the same vertex set as  $G$  in which any two distinct vertices are adjacent if and only if they are non-adjacent in  $G$ . The complete graph on  $n$  vertices is denoted by  $K_n$  (therefore,  $\overline{K_n}$  has no edges, that is, all its vertices are isolated). The complete bipartite graph on  $p + q$  vertices is denoted by  $K_{p,q}$  (in particular,  $K_{1,s}$  is a star on  $s + 1$  vertices).

Throughout the text,  $N_k$  denotes the set of positive integers not greater than  $k$ , the identity matrix of order  $m$  and the transpose of a matrix  $A$  are denoted by  $I_m$  and  $A^T$ , respectively. Furthermore,  $0$  is the zero matrix of appropriate order,  $\mathbf{1}_n$  is the all-one column vector of size  $n$  and  $J_{p,q}$  is the all-one matrix of order  $p \times q$ . The remainder notation is standard. However for the reader's convenience, as it follows, the fundamental concepts and their notation is briefly recalled.

Let  $D(G)$  be the diagonal matrix of order  $n$  whose  $(i, i)$ -entry is the degree of the  $i$ -th vertex of  $G$  and let  $A(G)$  be the adjacency matrix of  $G$ . The matrices  $L(G) = D(G) - A(G)$  and  $Q(G) = D(G) + A(G)$  are the Laplacian and signless Laplacian matrix of  $G$ , respectively. The matrices  $L(G)$  and  $Q(G)$  are both positive semidefinite and  $(0, \mathbf{1})$  is an eigenpair of  $L(G)$ . Fiedler [7] proved that  $G$  is a connected graph if and only if the second smallest eigenvalue of  $L(G)$  is positive. This eigenvalue is called the algebraic connectivity of  $G$ . Moreover, it is known that for any bipartite graph  $G$ , the characteristic polynomials of  $L(G)$  and  $Q(G)$  coincide [6, Prop. 2.3]. For a connected graph  $G$ , the least eigenvalue of  $Q(G)$  is positive if and only if  $G$  is non-bipartite [6, Proposition 2.1].

In [13] Nikiforov introduced the family of matrices

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$$

where  $\alpha \in [0, 1]$ . We see that  $A_\alpha(G)$  is a convex combination of the matrices  $A(G)$  and  $D(G)$ . The multiset of eigenvalues of  $A_\alpha(G)$  is called the  $A_\alpha$ -spectrum of  $G$ .

Since  $A_\alpha(G)$  is a real symmetric matrix, its eigenvalues are real numbers. Observe that  $A_0(G) = A(G)$  and  $2A_{1/2}(G) = Q(G)$ . Thus, the family  $A_\alpha(G)$  extends both  $A(G)$  and  $Q(G)$ . Since  $A_1(G) = D(G)$ , from now on, we take  $\alpha \in [0, 1)$ .

If  $G$  is a graph of order  $n$ , we denote by

$$\nu_1(G) \leq \nu_2(G) \leq \dots \leq \nu_n(G)$$

the eigenvalues of  $A_\alpha(G)$ . If necessary, these eigenvalues are also denoted by  $\nu_1(A_\alpha(G)), \nu_2(A_\alpha(G)), \dots, \nu_n(A_\alpha(G))$ .

In particular,  $\nu_n(G)$  is called the  $\alpha$ -index of  $G$ . From the Perron-Frobenius Theory for nonnegative matrices, it follows that

- for a connected graph  $G$ , the  $\alpha$ -index of  $G$  (Perron root) is a simple eigenvalue of  $A_\alpha(G)$  that has a positive eigenvector (Perron vector),
- for a connected graph  $G$ , the  $\alpha$ -index of  $G$  increases if any entry of  $A_\alpha(G)$  increases,
- if  $G$  is a proper subgraph of a connected graph  $H$ , then  $\nu_n(G) < \nu_n(H)$ , and
- if  $G$  is an  $r$ -regular graph of order  $n$ , then  $A_\alpha(G) = r\alpha I_n + (1 - \alpha)A(G)$  and  $\nu_n(G) = r$  with eigenvector  $\mathbf{1}_n$ .

Now, we recall the concept of cluster which appears first in [11] and more recently in [5].

**Definition 1.1.** A cluster of order  $c$  and degree  $s$  in a graph  $G$  is a pair of vertex subsets  $(C, S)$ , where  $C$  is a set of cardinality  $|C| = c \geq 2$  of pairwise co-neighbor vertices sharing the same set  $S$  of  $s$  neighbors.

A pendent vertex is a vertex of degree 1 and a quasi-pendent vertex is a vertex adjacent to at least one pendent vertex. For the star  $K_{1,s}$ ,  $C$  is the set of the pendent vertices and  $S = \{v\}$  where  $v$  is the root vertex and a complete bipartite graph  $K_{p,q}$  has the clusters  $(\overline{K}_p, \overline{K}_q)$  and  $(\overline{K}_q, \overline{K}_p)$ . Also, note that each quasi-pendent vertex adjacent with more than one pendent vertex define a cluster  $(C, S)$  in which  $|S| = 1$ . In [4], among other results, it was proved that  $\alpha$  is an eigenvalue of  $A_\alpha(G)$  with multiplicity at least  $p(G) - q(G)$ , when  $G$  has  $p(G) > 0$  pendent vertices and  $q(G)$  quasi-pendent vertices. It is easy to prove that any set of pairwise co-neighbor vertices is an independent set.

**Definition 1.2.** Let  $G$  be a connected graph of order  $n$  with a cluster  $(C, S)$  and let  $H$  be a graph of order  $|C|$ . Assuming that  $V(H) = C$ , then  $G(H)$  is the graph with vertex set  $V(G(H)) = V(G)$  and edge set  $E(G(H)) = E(G) \cup E(H)$ .

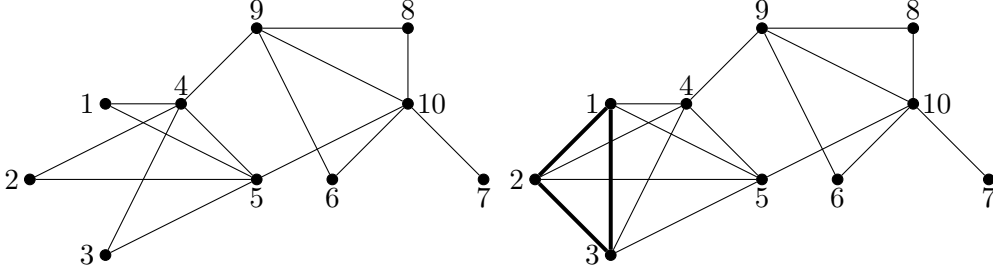
From Definition 1.2,  $G(H)$  is the graph obtained from  $G$  and  $H$  adding the edges of  $H$  to the edges of  $G$  by identifying the vertices of  $H$  with the vertices in  $C$ .

**Example 1.3.** Let  $G$  be the graph below depicted which has the cluster  $(C, S)$ , where  $C = \{1, 2, 3\}$  and  $S = \{4, 5\}$ . Let  $H$  be the cycle on 3 vertices,  $V(H) = \{1, 2, 3\}$ . Then the graphs  $G$  and  $G(H)$  are displayed, respectively, below.

**Definition 1.4.** Let  $(C_1, S_1)$  and  $(C_2, S_2)$  be clusters in a graph  $G$ . We say that  $(C_1, S_1)$  and  $(C_2, S_2)$  are disjoint if  $C_1 \cap C_2 = \emptyset$  and  $S_1 \cap S_2 = \emptyset$ .

The Laplacian and signless Laplacian spectra of a graph  $G$  with a cluster  $(C, S)$  are studied in [1]. The effects on the Laplacian spectral radius and algebraic

connectivity of a graph perturbed by adding edges between its pendent vertices are considered in [9] and [17], respectively. Moreover, the effects on others spectral invariants are determined in [15] and [16].



**Definition 1.5.** Let  $G$  be a connected graph with pairwise disjoint clusters  $(C_1, S_1), \dots, (C_k, S_k)$ . For  $i = 1, \dots, k$ , let  $H_i$  be a graph of order  $|C_i|$ . Let  $G(H_i : i \in N_k)$  be the graph obtained from  $G$  and the graphs  $H_i$  when the edges of  $H_i$  are added to the edges of  $G$  by identifying the vertices of  $H_i$  with the vertices in  $C_i$  for  $i = 1, \dots, k$ .

From this definition, we have  $V(H_i) = C_i$ , for  $i = 1, \dots, k$ ,

$$V(G(H_i : i \in N_k)) = V(G)$$

and

$$E(G(H_i : i \in N_k)) = E(G) \cup E(H_1) \cup \dots \cup E(H_k).$$

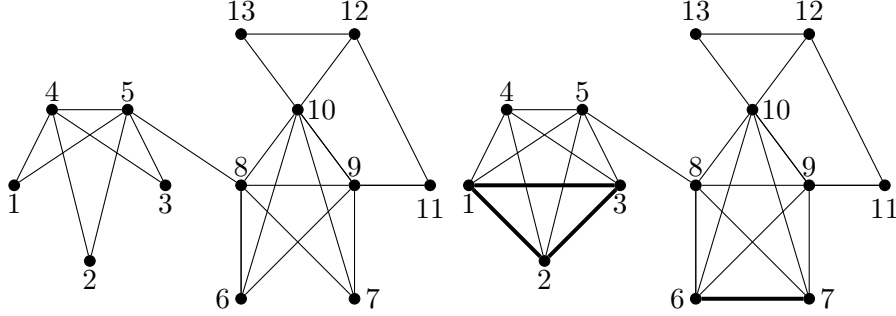
Observe that the graph  $G(H_i : i \in N_k)$  can be constructed as follows.

- The graph  $G_1 = G(H_1)$  is obtained from  $G$  and  $H_1$  identifying the vertices of  $H_1$  with  $C_1$ , and
- for  $i = 2, \dots, k$ , the graph  $G_i = G(H_1, \dots, H_i)$  is obtained from  $G_{i-1} = G(H_1, \dots, H_{i-1})$  and  $H_i$  identifying the vertices of  $H_i$  with  $C_i$ .

**Example 1.6.** Let  $G$  be the graph below depicted which has two disjoint clusters  $(C_1, S_1)$  and  $(C_2, S_2)$  where  $C_1 = \{1, 2, 3\}$ ,  $S_1 = \{4, 5\}$  and  $C_2 = \{6, 7\}$ ,  $S_2 = \{8, 9, 10\}$ . Let  $H_1$  be the cycle on 3 vertices,  $V(H_1) = \{1, 2, 3\}$ , and  $H_2$  be the path on 2 vertices,  $V(H_2) = \{6, 7\}$ . Then the graphs  $G$  and  $G(H_1, H_2)$  are displayed, respectively, below.

A unified approach to the determination of the spectra of adjacency, Laplacian and signless Laplacian matrices of graphs with edge perturbation on their clusters was presented in [5]. Moreover, the invariance of algebraic connectivity and Laplacian index under those perturbation was proved.

In this article, using a methodology similar to the one followed in [5], new results about the spectra of  $A_\alpha(G)$  and  $A_\alpha(G(H))$  are deduced. Namely, in



Section 2, assuming that  $G$  is a connected graph of order  $n$  with a cluster  $(C, S)$  and  $G(H)$  is obtained according to Definition 1.2, the following results about the spectra of  $A_\alpha(G)$  and  $A_\alpha(G(H))$  are proven.

1.  $|S|\alpha + \nu_j(H)$ ,  $1 \leq j \leq |C| - 1$ , are eigenvalues of  $A_\alpha(G(H))$ , where

$$\nu_1(H) \leq \cdots \leq \nu_{|C|-1}(H) \leq \nu_{|C|}(H) = r$$

are the eigenvalues of  $A_\alpha(H)$ . As direct consequence,  $|S|\alpha$  is an eigenvalue of  $A_\alpha(G)$  with multiplicity at least  $|C| - 1$ . In both cases, the remaining eigenvalues can be computed from a special matrix, (5) and (8), respectively (Theorem 2.1 and Corollary 2.2).

2. If  $A_\alpha(H)$  is a positive semidefinite matrix, then

$$\nu_i(G) \leq \nu_i(G(H)),$$

for  $i = 1, \dots, n$ , where  $\{\nu_i(G) : 1 \leq i \leq n\}$  and  $\{\nu_i(G(H)) : 1 \leq i \leq n\}$  are the  $A_\alpha$ -spectra of  $G$  and  $G(H)$ , respectively (Theorem 2.6).

3. Assuming that  $G$  has  $k \geq 2$  pairwise disjoint clusters  $(C_1, S_1), \dots, (C_k, S_k)$ , the above results are extended to the graph  $G(H_i : i = 1, \dots, k)$  (Theorem 2.7).

Finally, in Section 3, the obtained results are applied to study the effect on the energy (Theorems 3.1 and 3.2),  $\alpha$ -Estrada index (Theorems 3.3 and 3.4) and  $\alpha$ -index (Theorem 3.5) of a graph  $G$  with clusters when the edges of regular graphs are added to  $G$ . Additionally, the  $A_\alpha$ -spectrum of the corona product  $G \circ H$  of a connected graph  $G$  and a regular graph  $H$  is determined (Theorem 3.7).

## 2. EFFECTS BY ADDING THE EDGES OF A REGULAR GRAPH

Consider  $G(H)$  as in Definition 1.2. Let  $|C| = c$  and  $|S| = s$ . We assume that  $H$  is a connected  $r$ -regular graph of order  $|C| = c$  and that

$$\nu_1(H) \leq \cdots \leq \nu_{c-1}(H) < \nu_c(H) = r$$

are the eigenvalues of  $A_\alpha(H)$  with an orthogonal basis of eigenvectors

$$\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{c-1}, \mathbf{x}_c = \frac{1}{\sqrt{c}} \mathbf{1}_c$$

in which, for  $1 \leq i \leq c$ ,  $A_\alpha(H)\mathbf{x}_i = \nu_i(H)\mathbf{x}_i$ . In particular

$$(1) \quad A_\alpha(H)\mathbf{1}_c = r\mathbf{1}_c.$$

Let

$$X = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_{c-1} & \frac{1}{\sqrt{c}}\mathbf{1}_c \end{bmatrix}$$

and

$$(2) \quad U = \begin{bmatrix} X & \\ & I_{n-c} \end{bmatrix}.$$

Clearly  $X$  and  $U$  are both orthonormal matrices.

Through this paper  $\beta = 1 - \alpha$  and  $d_i$  is the degree of the vertex  $i$  of the graph  $G$ .

We recall that  $G$  is a graph that has a cluster  $(C, S)$ . The graphs  $G$  and  $G(H)$  have the same set of vertices. We label the vertices of  $G$  as follows. The labels  $1, 2, \dots, c$  are for the vertices of  $C$ , the labels  $c + 1, c + 2, \dots, c + s$  are for the vertices in  $S$  and the labels  $c + s + 1, \dots, n$  are for the remaining vertices of  $G$ . This labeling is illustrated in Example 1.3. For this labeling,  $A_\alpha(G)$  and  $A_\alpha(G(H))$  become as follows

$$(3) \quad A_\alpha(G) = \begin{bmatrix} s\alpha I_c & [\beta \mathbf{1}_c \mathbf{1}_s^T \ 0] \\ \begin{bmatrix} \beta \mathbf{1}_s \mathbf{1}_c^T \\ 0 \end{bmatrix} & R(\alpha) \end{bmatrix}$$

and

$$(4) \quad A_\alpha(G(H)) = \begin{bmatrix} s\alpha I_c + A_\alpha(H) & [\beta \mathbf{1}_c \mathbf{1}_s^T \ 0] \\ \begin{bmatrix} \beta \mathbf{1}_s \mathbf{1}_c^T \\ 0 \end{bmatrix} & R(\alpha) \end{bmatrix}$$

where  $R(\alpha) = \begin{bmatrix} A & B \\ B^T & Z \end{bmatrix}$  with submatrices  $A$ ,  $B$  and  $Z$  of size  $s \times s$ ,  $s \times (n - c - s)$  and  $(n - c - s) \times (n - c - s)$ , respectively. The diagonal entries of the matrices  $A$  and  $Z$  are  $\alpha d_i$ ,  $c + 1 \leq i \leq n$  and the off-diagonal entries of  $A$  and  $Z$  as well as the entries of  $B$  are  $\beta$  if the corresponding vertices of  $G$  are adjacent and 0 otherwise.

**Theorem 2.1.** *Let  $G$  be a graph with a cluster  $(C, S)$  of order  $|C| = c$  and degree  $|S| = s$ . If  $H$  is an  $r$ -regular graph of order  $c$  and  $G(H)$  is obtained according to*

Definition 1.2, then

$$s\alpha + \nu_j(H), \quad 1 \leq j \leq c-1,$$

are eigenvalues of  $A_\alpha(G(H))$ , where  $\nu_1(H) \leq \dots \leq \nu_{c-1}(H) \leq \nu_c(H) = r$  are the eigenvalues of  $A_\alpha(H)$  and the remaining eigenvalues of  $A_\alpha(G(H))$  are the eigenvalues of the matrix

$$(5) \quad X = \begin{bmatrix} s\alpha + r & [\beta\sqrt{c}\mathbf{1}_s^T \ 0] \\ \left[ \begin{array}{c} \beta\sqrt{c}\mathbf{1}_s \\ 0 \end{array} \right] & R(\alpha) \end{bmatrix}.$$

**Proof.** We use (4) and the orthogonal matrix  $U$  defined in (2) obtaining

$$\begin{aligned} & U^T A_\alpha(G(H))U \\ &= \begin{bmatrix} X^T & \\ & I_{n-c} \end{bmatrix} \begin{bmatrix} s\alpha I_c + A_\alpha(H) & [\beta\mathbf{1}_c\mathbf{1}_s^T \ 0] \\ \left[ \begin{array}{c} \beta\mathbf{1}_s\mathbf{1}_c^T \\ 0 \end{array} \right] & R(\alpha) \end{bmatrix} \begin{bmatrix} X & \\ & I_{n-c} \end{bmatrix} \\ &= \begin{bmatrix} s\alpha I_c + X^T A_\alpha(H) X & [\beta X^T \mathbf{1}_c \mathbf{1}_s^T \ 0] \\ \left[ \begin{array}{c} \beta\mathbf{1}_s\mathbf{1}_c^T X \\ 0 \end{array} \right] & R(\alpha) \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} s\alpha + \nu_1(H) & & & \\ & \ddots & & \\ & & s\alpha + \nu_{c-1}(H) & \\ & & & s\alpha + r \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \beta\sqrt{c}\mathbf{1}_s^T & 0 \end{bmatrix} \\ \left[ \begin{array}{ccc} 0 & \dots & 0 \\ \dots & & 0 \end{array} \right] & \begin{bmatrix} \beta\sqrt{c}\mathbf{1}_s \\ 0 \end{bmatrix} \\ & R(\alpha) \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} s\alpha + \nu_1(H) & & \\ & \ddots & \\ & & s\alpha + \nu_{c-1}(H) \end{bmatrix} & \\ & \begin{bmatrix} s\alpha + r & [\beta\sqrt{c}\mathbf{1}_s^T \ 0] \\ \left[ \begin{array}{c} \beta\sqrt{c}\mathbf{1}_s \\ 0 \end{array} \right] & R(\alpha) \end{bmatrix} \end{bmatrix}. \end{aligned}$$

Then  $U^T A_\alpha(G(H))U =$

$$(6) \quad \begin{bmatrix} s\alpha + \nu_1(H) & & \\ & \ddots & \\ & & s\alpha + \nu_{c-1}(H) \end{bmatrix} \oplus \begin{bmatrix} s\alpha + r & [\beta\sqrt{c}\mathbf{1}_s^T \ 0] \\ \left[ \begin{array}{c} \beta\sqrt{c}\mathbf{1}_s \\ 0 \end{array} \right] & R(\alpha) \end{bmatrix}.$$

Therefore, the conclusion follows from (6).  $\blacksquare$

Applying Theorem 2.1 to the particular case of  $H = \overline{K}_c$ , it follows that  $G(H) = G$  and

$$(7) \quad U^T A_\alpha(G)U = \begin{bmatrix} s\alpha & & \\ & \ddots & \\ & & s\alpha \end{bmatrix} \oplus \begin{bmatrix} s\alpha & [\beta\sqrt{c}\mathbf{1}_s^T \ 0] \\ \left[ \begin{array}{c} \beta\sqrt{c}\mathbf{1}_s \\ 0 \end{array} \right] & R(\alpha) \end{bmatrix}.$$

Thus the next corollary is immediate.

**Corollary 2.2.** *Let  $G$  be a graph with a cluster  $(C, S)$  of order  $|C| = c$  and degree  $|S| = s$ . Then  $s\alpha$  is an eigenvalue of  $A_\alpha(G)$  with multiplicity at least  $c - 1$  and the remaining eigenvalues are the eigenvalues of the matrix*

$$(8) \quad Y = \begin{bmatrix} s\alpha & [\beta\sqrt{c}\mathbf{1}_s^T \ 0] \\ \left[ \begin{array}{c} \beta\sqrt{c}\mathbf{1}_s \\ 0 \end{array} \right] & R(\alpha) \end{bmatrix}.$$

Taking into account that  $A_0(G) = A(G)$  and  $2A_{\frac{1}{2}}(G) = Q(G)$ , another immediate corollary is the following.

**Corollary 2.3.** *Let  $G$  be a graph with a cluster  $(C, S)$  of order  $|C| = c$  and degree  $|S| = s$ . If  $H$  is an  $r$ -regular graph of order  $c$  and  $G(H)$  is obtained according to Definition 1.2, then*

- (i)  $0$  is an eigenvalue of  $A(G)$  with multiplicity at least  $c - 1$ ,
- (ii) if  $\lambda_j(H) \neq r$  is an eigenvalue of  $A(H)$ , then it is also an eigenvalue of  $A(G(H))$ ,
- (iii)  $s$  is an eigenvalue of  $Q(G)$  with multiplicity at least  $c - 1$ , and
- (iv) if  $q_j(H) \neq 2r$  is an eigenvalue of  $Q(H)$ , then  $q_j(H) + s$  is an eigenvalue of  $Q(G(H))$ .

## 2.1. The nonnegative $A_\alpha$ -spectrum case

In this subsection we study the  $A_\alpha$ -spectrum of  $G(H)$  when  $A_\alpha(H)$  is a positive semidefinite matrix.

Among the basic results on  $A_\alpha(G)$  obtained in [13] we recall the following theorem.

**Theorem 2.4** [13, Proposition 4]. *Let  $1 \geq \alpha > \beta \geq 0$ . Then*

$$(9) \quad \nu_j(A_\alpha(G)) \geq \nu_j(A_\beta(G))$$

for  $j = 1, 2, \dots, n$ . If  $G$  is connected, then inequality (9) is strict, unless  $j = n$  and  $G$  is regular.

The function  $f_G(\alpha) = \nu_1(A_\alpha(G))$  is continuous and, from (9) with  $j = 1$ , it is nondecreasing in  $\alpha$ . Moreover,  $f_G(0) = \nu_1(A_0(G)) < 0$ . Therefore, there is a smallest value  $\alpha \in (0, \frac{1}{2}]$  such that  $\nu_1(A_\alpha(G)) = 0$ . Hence, denoting this value by  $\alpha_0(G)$ ,  $A_\alpha(G)$  is a positive semidefinite matrix if and only if  $\alpha_0(G) \leq \alpha \leq 1$ .

Now, we restate a problem proposed in [13, Problem 8] as follows: *given a graph  $G$ , find  $\alpha_0(G)$ .*

Some advances on this problem obtained in [14] are presented in the next proposition.



**Proposition 2.5** [14, Proposition 5]. *If  $H$  is an  $r$ -regular graph, then*

$$(10) \quad \alpha_0(H) = \frac{-\nu_{\min}(A(H))}{r - \nu_{\min}(A(H))}$$

where  $\nu_{\min}(A(H))$  is the least eigenvalue of  $A(H)$ .

**Theorem 2.6.** *Let  $G$  be a graph with a cluster  $(C, S)$  of order  $|C| = c$  and degree  $|S| = s$ . If  $H$  is an  $r$ -regular graph of order  $c$ ,  $\alpha \geq \alpha_0(H)$ , where  $\alpha_0(H)$  is given by (10), and  $G(H)$  is obtained according to Definition 1.2, then*

$$\nu_i(G) \leq \nu_i(G(H)),$$

for  $i = 1, \dots, n$ , where  $\{\nu_i(G) : 1 \leq i \leq n\}$  and  $\{\nu_i(G(H)) : 1 \leq i \leq n\}$  are the  $A_\alpha$ -spectra of  $G$  and  $G(H)$ , respectively.

**Proof.** Since  $\alpha \geq \alpha_0(H)$  with  $\alpha_0(H)$  given by (10),  $A_\alpha(H)$  is a positive semidefinite matrix and then its eigenvalues are nonnegative. Thus the result follows from (6) and (7) applying the Weyl's inequalities for eigenvalues of Hermitian matrices ([10], p. 181). ■

## 2.2. The multiple pairwise disjoint clusters case

In this subsection the graphs with more than one cluster are analyzed.

**Theorem 2.7.** *Let  $G$  be a graph with a set of pairwise disjoint clusters  $\{(C_i, S_i) : i \in N_k\}$ , with  $k \geq 2$ , and let  $|C_i| = c_i$  and  $|S_i| = s_i$ , for  $i \in N_k$ . Assuming that each  $H_i$  is an  $r_i$ -regular graph of order  $c_i$  and  $G(H_i : i \in N_k)$  is obtained according to Definition 1.5, it follows, for each  $p \in N_k$ , that*

- (i)  $s_p \alpha$  is an eigenvalue of  $A_\alpha(G)$  with multiplicity at least  $c_p - 1$ ,
- (ii)  $s_p \alpha + \nu_j(H_p)$ ,  $1 \leq j \leq c_p - 1$ , is an eigenvalue of  $A_\alpha(G(H_i : i \in N_k))$ , where

$$\nu_1(H_p) \leq \dots \leq \nu_{c_p-1}(H_p) \leq \nu_{c_p}(H_p) = r_p$$

are the eigenvalues of  $A_\alpha(H_p)$ ,

- (iii) if

$$\alpha \geq \frac{-\alpha_{\min}(A(H_p))}{r_p - \alpha_{\min}(A(H_p))}$$

where  $\alpha_{\min}(A(H_p))$  is the least eigenvalue of  $A(H_p)$ , then the  $j$ -th eigenvalue of  $A_\alpha(G(H_i : i \in N_k))$  is greater or equal to the  $j$ -th eigenvalue of  $A_\alpha(G)$ .

**Proof.** Considering  $p \in N_k$ , since

$$G(H_i : i \in N_k \setminus \{p\})(H_p) = G(H_i : i \in N_k),$$

the results are immediate from Theorems 2.1 and 2.6. ■

As a consequence, we have the following corollary.

**Corollary 2.8.** *Let  $G$  be a graph with a set of pairwise disjoint clusters  $\{(C_i, S_i) : i \in N_k\}$ , with  $k \geq 2$ , and let  $|C_i| = c_i$  and  $|S_i| = s_i$ , for  $i \in N_k$ . Assuming that each  $H_i$  is an  $r_i$ -regular graph of order  $c_i$  and  $G(H_i : i \in N_k)$  is obtained according to Definition 1.5, then 0 is an eigenvalue of  $A(G)$  with multiplicity at least  $\sum_{i=1}^k c_i - k$ . Moreover, for each  $p \in N_k$ ,*

- (i) *if  $\lambda_j(H_p) \neq r_p$  is an eigenvalue of  $A(H_p)$ , then it is also an eigenvalue of  $A(G(H_i : i \in N_k))$ ,*
- (ii)  *$s_p$  is an eigenvalue of  $Q(G)$  with multiplicity at least  $c_p - 1$ ,*
- (iii) *if  $q_j(H_p) \neq 2r_p$  is an eigenvalue of  $Q(H_p)$ , then  $q_j(H_p) + s_p$  is an eigenvalue of  $Q(G(H_i : i \in N_k))$ .*

### 3. SOME APPLICATIONS

In this section, the energy,  $\alpha$ -Estrada index, and  $\alpha$ -index of graphs with clusters are considered, and the  $A_\alpha$ -spectrum of the corona of a connected graph  $G$  and a regular graph  $H$  is determined.

We recall that the energy of a graph  $G$  is  $\mathcal{E}(G) = \sum_{i=1}^n |\lambda_i(G)|$  and the Estrada index of  $G$  is  $E\mathcal{E}(G) = \sum_{i=1}^n e^{\lambda_i(G)}$ , where

$$\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_{n-1}(G) \leq \lambda_n(G)$$

are the eigenvalues of  $A(G)$ . Similarly, the signless Laplacian Estrada index of  $G$  is defined as  $SLE\mathcal{E}(G) = \sum_{i=1}^n e^{q_i(G)}$ , where

$$q_1(G) \leq q_2(G) \leq \cdots \leq q_{n-1}(G) \leq q_n(G)$$

are the eigenvalues of  $Q(G)$ .

The corona  $G \circ H$  of two graphs  $G$  and  $H$  (where  $|V(G)| = n$  and  $|V(H)| = m$ ) introduced by Frucht and Harary [8] is defined as the graph obtained by taking one copy of  $G$  and  $n$  copies of  $H$  and then joining by an edge the  $i$ -th vertex of  $G$  to every vertex of the  $i$ -th copy of  $H$ . It is immediate that the corona graph operation is not commutative, that is, in general  $G \circ H \neq H \circ G$ .

#### 3.1. The energy of graphs with clusters

Let  $M$  be an  $m \times n$  complex matrix,  $q = \min\{m, n\}$  and

$$\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_q(M)$$

be the singular values of  $M$ . Nikiforov [12] defines the energy of  $M$  as  $\mathcal{E}(M) = \sum_{j=1}^q \sigma_j(M)$ . Since  $A(G)$  is symmetric, its singular values are the modulus of its eigenvalues. Then  $\mathcal{E}(G) = \mathcal{E}(A(G))$ .

Given a natural number  $k$  such that  $1 \leq k \leq n$ , the Ky Fan  $k$ -norm of a matrix  $M$  of order  $n \times n$  is the sum of the  $k$  largest singular values of  $M$ , that is, assuming that  $\sigma_1(M), \dots, \sigma_k(M)$  are the  $k$  largest singular values of  $M$ ,  $\|M\|_k = \sum_{i=1}^k \sigma_i(M)$ . In particular,  $\|M\|_n = \mathcal{E}(M)$ .

**Theorem 3.1.** *Let  $G$  be a graph with a cluster  $(C, S)$  of order  $|C| = c$  and degree  $|S| = s$ . Let  $H$  be an  $r$ -regular graph of order  $c$ . Let  $G(H)$  as in Definition 1.2. Then*

$$\mathcal{E}(G(H)) - \mathcal{E}(G) \leq \mathcal{E}(H).$$

**Proof.** We apply Theorem 2.1 with  $\alpha = 0$ . From (6) and (7), using the fact that the singular values are invariant under unitary transformations, we have

$$(11) \quad \mathcal{E}(G(H)) = \mathcal{E}(A(G(H))) = \sum_{i=1}^{c-1} |\nu_i(H)| + \mathcal{E}(C),$$

where  $C = \begin{bmatrix} r & [\sqrt{c}\mathbf{1}_s^T \ 0] \\ \begin{bmatrix} \sqrt{c}\mathbf{1}_s \\ 0 \end{bmatrix} & R(0) \end{bmatrix}$  and  $\mathcal{E}(G) = \mathcal{E}(A(G)) = \mathcal{E}(D)$ , where  $D = \begin{bmatrix} 0 & [\sqrt{c}\mathbf{1}_s^T \ 0] \\ \begin{bmatrix} \sqrt{c}\mathbf{1}_s \\ 0 \end{bmatrix} & R(0) \end{bmatrix}$ . Then  $C = D + F$ , where

$$(12) \quad F = \begin{bmatrix} r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence  $\mathcal{E}(C) = \|C\|_{n-c+1} \leq \|D\|_{n-c+1} + \|F\|_{n-c+1} = \mathcal{E}(D) + r = \mathcal{E}(G) + r$ . Using this inequality in (11), we obtain

$$\mathcal{E}(G(H)) - \mathcal{E}(G) \leq \sum_{i=1}^{c-1} |\nu_i(H)| + r = \sum_{i=1}^c |\nu_i(H)| = \mathcal{E}(H). \quad \blacksquare$$

**Theorem 3.2.** *Let  $G$  be a graph with a set of clusters  $\{(C_i, S_i) : i \in N_k\}$ ,  $k \geq 2$ . For  $i \in N_k$ , let  $|C_i| = c_i$ ,  $|S_i| = s_i$  and  $H_i$  be an  $r_i$ -regular graph of order  $c_i$ . Let  $G(H_i : i \in N_k)$  as in Definition 1.5. Then*

$$\mathcal{E}(G(H_i : i \in N_k)) - \mathcal{E}(G) \leq \sum_{i=1}^k \mathcal{E}(H_i).$$

**Proof.** The result follows easily by a repeated application of Theorem 3.1.  $\blacksquare$

### 3.2. The $\alpha$ -Estrada index of graphs with clusters

In [16], for a graph with pendent vertices, the effects on the energy, Estrada index ( $\alpha = 0$ ) and signless Laplacian Estrada index (essentially,  $\alpha = 0.5$ ) are obtained when the edges of regular graphs are added among the pendent vertices. In this subsection, we extend these results to a graph with clusters, for all  $\alpha \in [0, 1]$ .

Since  $A_0(G) = A(G)$ , it seems natural to define the  $\alpha$ -Estrada index of  $G$ , denoted by  $E\mathcal{E}_\alpha(G)$ , as

$$E\mathcal{E}_\alpha(G) = \sum_{i=1}^n e^{\nu_i(G)}$$

where

$$\nu_1(G) \leq \nu_2(G) \leq \cdots \leq \nu_{n-1}(G) \leq \nu_n(G)$$

are the eigenvalues of  $A_\alpha(G)$ . Hence  $E\mathcal{E}_\alpha(G) = \text{trace}(e^{A_\alpha(G)})$

Next, we study the effect on the  $\alpha$ -Estrada index.

**Theorem 3.3.** *Let  $G$  be a graph with a cluster  $(C, S)$  of order  $|C| = c$  and degree  $|S| = s$ . Let  $H$  be an  $r$ -regular graph of order  $c$ . Let  $G(H)$  be as in Definition 1.2. Then*

$$E\mathcal{E}_\alpha(G(H)) - E\mathcal{E}_\alpha(G) \geq e^{s\alpha} E\mathcal{E}_\alpha(H) - [(c-1)e^{s\alpha} + e^r(e^{s\alpha} - 1)].$$

**Proof.** We use again the fact that the singular values under unitary transformations to obtain, from (6) and (7), that

$$(13) \quad E\mathcal{E}_\alpha(G(H)) = \sum_{i=1}^{c-1} e^{(s\alpha + \nu_i(H))} + \text{trace}(e^X)$$

and

$$(14) \quad E\mathcal{E}_\alpha(G) = \sum_{i=1}^{c-1} e^{s\alpha} + \text{trace}(e^Y)$$

where  $X$  and  $Y$  are as in Theorem 2.1. From the series-expansion of  $e^N$ , we have

$$e^X = \sum_{j=0}^{\infty} \frac{1}{j!} X^j = \sum_{j=0}^{\infty} \frac{1}{j!} (Y + F)^j = \sum_{j=0}^{\infty} \frac{1}{j!} (Y^j + \cdots + F^j),$$

where  $F$  is given in (12). Since  $Y$  and  $F$  are nonnegative matrices, it follows that

$$\text{trace}(e^X) \geq \text{trace}\left(\sum_{j=0}^{\infty} \frac{1}{j!} Y^j\right) + \text{trace}\left(\sum_{j=0}^{\infty} \frac{1}{j!} F^j\right).$$

Hence,

$$\text{trace}(e^X) \geq \text{trace}(e^Y) + \sum_{j=0}^{\infty} \frac{1}{j!} r^j = \text{trace}(e^Y) + e^r.$$

Using this inequality in (13), we get

$$\begin{aligned} E\mathcal{E}_\alpha(G(H)) &\geq \sum_{i=1}^{c-1} e^{(s\alpha + \nu_i(H))} + \text{trace}(e^Y) + e^r \\ &= e^{s\alpha} \sum_{i=1}^{c-1} e^{\nu_i(H)} + E\mathcal{E}_\alpha(G) - \sum_{i=1}^{c-1} e^{s\alpha} + e^r. \end{aligned}$$

Finally,

$$\begin{aligned} E\mathcal{E}_\alpha(G(H)) - E\mathcal{E}_\alpha(G) &\geq e^{s\alpha} \sum_{i=1}^{c-1} e^{\nu_i(H)} - \sum_{i=1}^{c-1} e^{s\alpha} + e^r \\ &= e^{s\alpha} \sum_{i=1}^{c-1} e^{\nu_i(H)} - (c-1)e^{s\alpha} + e^r + e^{s\alpha}e^r - e^{s\alpha}e^r \\ &= e^{s\alpha} E\mathcal{E}_\alpha(H) - (c-1)e^{s\alpha} - e^r(e^{s\alpha} - 1). \end{aligned}$$

Therefore,

$$E\mathcal{E}_\alpha(G(H)) - E\mathcal{E}_\alpha(G) \geq e^{s\alpha} E\mathcal{E}_\alpha(H) - [(c-1)e^{s\alpha} + e^r(e^{s\alpha} - 1)]. \quad \blacksquare$$

A repeated application of Theorem 3.3 yields to the following result.

**Theorem 3.4.** *Let  $G$  be a graph with a set of clusters  $\{(C_i, S_i) : i \in N_k\}$ ,  $k \geq 2$ . For  $i \in N_k$ , let  $|C_i| = c_i$ ,  $|S_i| = s_i$  and  $H_i$  be an  $r_i$ -regular graph of order  $c_i$ . Let  $G(H_i : i \in N_k)$  as in Definition 1.5. Then*

$$E\mathcal{E}_\alpha(G(H_i : i \in N_k)) - E\mathcal{E}_\alpha(G) \geq \sum_{i=1}^k (e^{s_i\alpha} E\mathcal{E}_\alpha(H_i) - (c_i - 1)e^{s_i\alpha} - e^{r_i}(e^{s_i\alpha} - 1)).$$

### 3.3. The $\alpha$ -index of graphs with a cluster

Now, we study the effect on the  $\alpha$ -index. We remember that  $\nu_n(G)$  and  $\nu_n(G(H))$  denote the  $\alpha$ -index of  $G$  and  $G(H)$ , respectively. We denote by  $\rho(X)$  and  $\rho(Y)$  the spectral radius of the matrices  $X$  and  $Y$  given in (5) and (8), respectively.

**Theorem 3.5.** *Let  $G$  be a graph with a cluster  $(C, S)$  of order  $|C| = c$  and degree  $|S| = s$ . Let  $H$  be an  $r$ -regular graph of order  $c$ . Let  $G(H)$  be as in Definition 1.2. Then*

$$0 < \nu_n(G(H)) - \nu_n(G) < r.$$

**Proof.** Clearly, from Theorem 2.1,  $\nu_n(G(H)) = \rho(X)$  and  $\nu_n(G) = \rho(Y)$ . We have  $X = Y + F$  with  $F$  as in (12). Since  $X - Y \geq 0$  with strict inequality in the entry (1,1), we get that  $0 < \rho(X) - \rho(Y)$ . Moreover, applying the Weyl's inequalities for eigenvalues of Hermitian matrices ([10], p. 181) and the conditions for the equality [18], we obtain that  $\rho(X) - \rho(Y) < r$ . ■

### 3.4. The corona product

In [2, Theorem 3.1] the authors compute the entire spectrum of the adjacency matrix of  $G \circ H$  ( $\alpha = 0$ ), when  $H$  is regular. In this subsection we extend this result to all  $\alpha \in [0, 1)$ , when  $H$  is regular. Before that, it is worth mention the following lemma which is an immediate consequence of Lemma 2.3.1 in [3].

**Lemma 3.6.** *If  $\{X_1, X_2, \dots, X_m\}$  is a partition of  $X = \{1, 2, \dots, n\}$  which is equitable for the square matrix  $A$  whose rows and columns are indexed by the elements of  $X$ , then each eigenvalue of the corresponding quotient matrix is an eigenvalue of  $A$ .*

Let  $V(G) = \{v_1, \dots, v_n\}$ . Observe that  $G \circ H = (G \circ \overline{K_m})(H_i : 1 \leq i \leq n)$  where  $H_i = H$ . Each pair of vertex subsets  $(C_i, S_i)$ , with  $C_i = V(\overline{K_m})$  and  $S_i = \{v_i\}$  is a cluster, for  $i = 1, \dots, n$ .

**Theorem 3.7.** *If  $G$  is a connected graph of order  $n$  and  $H$  is a  $r$ -regular graph of order  $m$ , then  $G \circ H$  is a graph of order  $n(m+1)$  and its  $A_\alpha$ -spectrum includes the eigenvalues*

$$(15) \quad \alpha + \nu_j(H) \text{ for } 1 \leq j \leq m-1,$$

*each one with multiplicity  $n$ .*

*The remaining  $2n$  eigenvalues of  $A_\alpha(G \circ H)$  are the eigenvalues of the matrix*

$$(16) \quad B = \begin{bmatrix} A_\alpha(G) + m\alpha I_n & m\beta I_n \\ \beta I_n & (\alpha + r)I_n \end{bmatrix}.$$

**Proof.** Let  $V(G) = \{v_1, \dots, v_n\}$ . We recall that  $G \circ H = (G \circ \overline{K_m})(H_i : 1 \leq i \leq n)$  with  $H_i = H$  for all  $i$ . Applying Theorem 2.7(ii) to  $(G \circ \overline{K_m})(H_i : 1 \leq i \leq n)$ , it follows that, for  $1 \leq i \leq n$  and  $1 \leq j \leq m-1$ ,  $\alpha + \nu_j(H)$  is an eigenvalue of  $A_\alpha(G \circ H)$  with multiplicity  $n$ . Therefore, the expression (15) follows. We label the vertices of  $G \circ H$  as follows:  $1, \dots, n$  for the vertices of  $G$  and, for  $1 \leq i \leq n$ , the labels  $n+(i-1)m+1, \dots, n+im$  for the vertices of  $H_i$ . Let  $X = \{1, \dots, n, n+1, \dots, n+mn\}$ . Consider the partition  $\{X_1, \dots, X_n, X_{n+1}, \dots, X_{2n}\}$  of  $X$  where  $X_1 = \{1\}, \dots, X_n = \{n\}$  and, for  $1 \leq i \leq n$ ,  $X_{n+i} = \{n+(i-1)m+1, \dots, n+im\}$ . For this partition  $A_\alpha(G \circ H)$  becomes a  $2n \times 2n$  - block matrix such that the

row sum of each of the blocks is constant. Hence  $\{X_1, \dots, X_{2n}\}$  is an equitable partition. The corresponding quotient matrix is the matrix  $B$  given in (16). Therefore, by Lemma 3.6, the eigenvalues of  $B$  are eigenvalues of  $A_\alpha(G \circ H)$ . ■

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