A SPECTRAL CHARACTERIZATION OF THE $s$-CLIQUE EXTENSION OF THE TRIANGULAR GRAPHS

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This paper is dedicated to the memory of Prof. Slobodan Simić.

Abstract

A regular graph is co-edge regular if there exists a constant $\mu$ such that any two distinct and non-adjacent vertices have exactly $\mu$ common neighbors. In this paper, we show that for integers $s \geq 2$ and $n$ large enough, any co-edge-regular graph which is cospectral with the $s$-clique extension of the triangular graph $T(n)$ is exactly the $s$-clique extension of the triangular graph $T(n)$.

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1. Introduction

All graphs in this paper are simple and undirected. For definitions related to distance-regular graphs, see [1, 11]. Before we state the main result, we give more definitions.

Let \( G \) be a simple connected graph on vertex set \( V(G) \), edge set \( E(G) \) and adjacency matrix \( A \). The eigenvalues of \( G \) are the eigenvalues of \( A \). Let \( \lambda_0, \lambda_1, \ldots, \lambda_t \) be the distinct eigenvalues of \( G \) and \( m_i \) be the multiplicity of \( \lambda_i \) \((i = 0, 1, \ldots, t)\). Then the multiset \( \{\lambda_{m_0}, \lambda_{m_1}, \ldots, \lambda_{m_t}\} \) is called the spectrum of \( G \). Two graphs are called cospectral if they have the same spectrum. Note that a graph \( H \) cospectral with a \( k \)-regular graph \( G \) is also \( k \)-regular.

Recall that a regular graph is called co-edge-regular, if there exists a constant \( \mu \) such that any two distinct and non-adjacent vertices have exactly \( \mu \) common neighbors. Our main result in this paper is as follows.

**Theorem 1.** Let \( \Gamma \) be a co-edge-regular graph with spectrum

\[
\{(2sn - 3s - 1)^1, (sn - 3s - 1)^{n-1}, (-s - 1)^{n^2 - 3n - 1} \bigg(\frac{n^2 - 3n - 1}{2}\bigg), (-1)^{\frac{(s-1)n(n-1)}{2}}\} \]

where \( s \geq 2 \) and \( n \geq 1 \) are integers. If \( n \geq 48s \), then \( \Gamma \) is the \( s \)-clique extension of the triangular graph \( T(n) \).

This paper is a follow-up paper of Hayat, Koolen and Riaz [4]. They showed a similar result for the square grid graphs. In that paper, they gave the following conjecture.

**Conjecture 2 [4].** Let \( \Gamma \) be a connected co-edge-regular graph with four distinct eigenvalues. Let \( t \geq 2 \) be an integer and \( |V(\Gamma)| = n(\Gamma) \). Then there exists a constant \( n_t \) such that, if \( \theta_{\min}(\Gamma) \geq -t \) and \( n(\Gamma) \geq n_t \) both hold, then \( \Gamma \) is the \( s \)-clique extension of a strongly regular graph for some \( 2 \leq s \leq t - 1 \).

This conjecture is wrong as the \( p \times q \)-grids \((p > q \geq 2)\) show. So we would like to modify this conjecture as follows.

**Conjecture 3.** Let \( \Gamma \) be a connected co-edge-regular graph with parameters \((n, k, \mu)\) having four distinct eigenvalues. Let \( t \geq 2 \) be an integer. Then there exists a constant \( n_t \) such that, if \( \theta_{\min}(\Gamma) \geq -t \), \( n \geq n_t \) and \( k < n - 2 - \frac{(t-1)^2}{4} \), then either \( \Gamma \) is the \( s \)-clique extension of a strongly regular graph for \( 2 \leq s \leq t - 1 \) or \( \Gamma \) is a \( p \times q \)-grid with \( p > q \geq 2 \).

The reason for the valency condition is, that in [12], it was shown that for \( \lambda \geq 2 \), there exist constants \( C(\lambda) \) such that a connected \( k \)-regular co-edge-regular graph with order \( v \) and smallest eigenvalue at least \(-\lambda\) satisfies one of the following conditions.
(i) $v - k - 1 \leq \frac{(\lambda - 1)^2}{4} + 1$, or;
(ii) Every pair of distinct non-adjacent vertices has at most $C(\lambda)$ common neighbours.

Koolen et al. [8] improved this result by showing that one can take $C(\lambda) = (\lambda - 1)\lambda^2$ if $k$ is much larger than $\lambda$. This paper is part of the project to show the conjecture for $t = 3$.

Another motivation comes from the lecture notes [9]. In these notes, Terwilliger shows that any local graph of a thin $Q$-polynomial distance-regular graph is co-edge-regular and has at most five distinct eigenvalues. So it is interesting to study co-edge-regular graphs with a few distinct eigenvalues.

We mainly follow the method of Hayat et al. [4]. The main difference is that we simplify the method of Hayat et al. when we show that every vertex lies on exactly two lines. This leads to a better bound for which we can show this. This will also improve the bound given in the result of Hayat et al.

2. Preliminaries

2.1. Definitions

For two distinct vertices $x$ and $y$, we write $x \sim y$ (respectively, $x \nsim y$) if they are adjacent (respectively, nonadjacent) to each other. For a vertex $x$ of $G$, we define $N_G(x) = \{y \in V(G) \mid y \sim x\}$, and $N_G(x)$ is called the neighborhood of $x$. The graph induced by $N_G(x)$ is called the local graph of $G$ with respect to $x$ and is denoted by $G(x)$. We denote the number of common neighbors between two distinct vertices $x$ and $y$ by $\lambda_{xy}$ (respectively, $\mu_{xy}$) if $x \sim y$ (respectively, $x \nsim y$).

A graph is called regular if every vertex has the same valency. A regular graph $G$ with $n$ vertices and valency $k$ is called co-edge-regular with parameters $(n, k, \mu)$ if any two nonadjacent vertices have exactly $\mu = \mu(G)$ common neighbors. In addition, if any two adjacent vertices have precisely $\lambda = \lambda(G)$ common neighbors, then $G$ is called strongly regular with parameters $(n, k, \lambda, \mu)$. A graph $G$ is called walk-regular if the number of closed walks of length $r$ from a given vertex $x$ is independent of the choice of $x$ for all $r$, that is to say, for any $x$, $A_{xx}^r$ is constant for all $r$, where $A$ is the adjacency matrix of $G$.

Let $X$ be a set of size $t$. The Johnson graph $J(t, d)$ ($t \geq 2d$) is a graph with vertex set $\binom{X}{d}$, the set of $d$-subsets of $X$, where two $d$-subsets are adjacent whenever they have $d - 1$ elements in common. $J(t, 2)$ is the triangular graph $T(t)$. Recall that a clique (or a complete graph) is a graph in which every pair of vertices is adjacent. A co-clique is a graph that any two distinct vertices are nonadjacent. A $t$-clique is a clique with $t$-vertices and is denoted by $K_t$. The line graph of $K_t$ is also the triangular graph $T(t)$ which is strongly regular with
parameters \((t^3, 2t - 4, t - 2, 4)\) and spectrum \(\{(2t - 4)^1, (t - 4)^{t-1}, (-2)^{2t-2}\}\).

The Kronecker product \(M_1 \otimes M_2\) of two matrices \(M_1\) and \(M_2\) is obtained by replacing the \(ij\)-entry of \(M_1\) by \((M_1)_{ij}M_2\) for all \(i\) and \(j\). Note that if \(\tau\) and \(\eta\) are eigenvalues of \(M_1\) and \(M_2\), respectively, then \(\tau\eta\) is an eigenvalue of \(M_1 \otimes M_2\).

2.2. Interlacing

**Lemma 4** ([6], Interlacing). Let \(N\) be a real symmetric \(n \times n\) matrix with eigenvalues \(\theta_1 \geq \cdots \geq \theta_n\) and \(R\) be a real \(n \times m\) \((m < n)\) matrix with \(R^TR = I\). Set \(M = R^TNR\) with eigenvalues \(\mu_1 \geq \cdots \geq \mu_m\). Then

(i) the eigenvalues of \(M\) interlace those of \(N\), i.e.,

\[ \theta_i \geq \mu_i \geq \theta_{n-m+i}, \quad i = 1, 2, \ldots, m, \]

(ii) if the interlacing is tight, that is, there exists an integer \(j \in \{1, 2, \ldots, m\}\) such that \(\theta_i = \mu_i\) for \(1 \leq i \leq j\) and \(\theta_{n-m+i} = \mu_i\) for \(j+1 \leq i \leq m\), then \(RM = NR\).

In the case that \(R\) is permutation-similar to \(\begin{pmatrix} I & O \\ O & O \end{pmatrix}\), then \(M\) is just a principal submatrix of \(N\).

Let \(\pi = \{V_1, \ldots, V_m\}\) be the partition of the index set of the columns of \(N\) and let \(N\) be partitioned according to \(\pi\) as

\[
\begin{pmatrix}
N_{1,1} & \ldots & N_{1,m} \\
\vdots & \ddots & \vdots \\
N_{m,1} & \ldots & N_{m,m}
\end{pmatrix},
\]

where \(N_{i,j}\) denotes the block matrix of \(N\) formed by rows in \(V_i\) and columns in \(V_j\). The **characteristic matrix** \(P\) is the \(n \times m\) matrix whose \(j\)th column is the characteristic vector of \(V_j\) \((j = 1, \ldots, m)\). The **quotient matrix** of \(N\) with respect to \(\pi\) is the \(m \times m\) matrix \(Q\) whose entries are the average row sum of the blocks \(N_{ij}\) of \(N\), i.e.,

\[
Q_{i,j} = \frac{1}{V_i} (P^TNP)_{i,j}.
\]

The partition \(\pi\) is called **equitable** if each block \(N_{i,j}\) of \(N\) has constant row (and column) sum, i.e., \(PQ = NP\). The following lemma can be shown by using Lemma 4.

**Lemma 5** [5]. Let \(N\) be a real symmetric matrix with \(\pi\) as a partition of the index set of its columns. Suppose \(Q\) is the quotient matrix of \(N\) with respect to \(\pi\), then the following hold.
A Spectral Characterization of the $s$-Clique Extension of ... 5

(i) The eigenvalue of $Q$ interlace the eigenvalues of $N$.

(ii) If the interlacing is tight (as defined in Lemma 4(ii)), then the partition $\pi$ is equitable.

By an equitable partition of a graph, we always mean an equitable partition of its adjacency matrix $A$.

2.3. Clique extensions of $T(n)$

In this subsection, we define $s$-clique extensions of graphs and we will give some specific results for the $s$-clique extension of triangular graphs.

Recall an $s$-clique is a clique with $s$ vertices, where $s$ is a positive integer. The $s$-clique extension of a graph $G$ with $|V(G)|$ vertices is the graph $\tilde{G}$ obtained from $G$ by replacing each vertex $x \in V(G)$ by a clique $\tilde{X}$ with $s$ vertices, satisfying $\tilde{x} \sim \tilde{y}$ in $\tilde{G}$ if and only if $x \sim y$ in $G$, where $\tilde{x} \in \tilde{X}, \tilde{y} \in \tilde{Y}$. If $\tilde{G}$ is an $s$-clique extension of $G$, then the adjacency matrix of $\tilde{G}$ is $(A + I_{|V(G)|}) \otimes J_s - I_s |V(G)|$, where $J_s$ is the all-ones matrix of size $s$ and $I_s |V(G)|$ is the identity matrix of size $|V(G)|$. In particular, if $G$ has $t + 1$ distinct eigenvalues and its spectrum is

\begin{equation}
\{\theta_{m_0}, \theta_{m_1}, \ldots, \theta_{m_t}\},
\end{equation}

then the spectrum of $\tilde{G}$ is

\begin{equation}
\left\{ (s(\theta_0 + 1) - 1)^{m_0}, (s(\theta_1 + 1) - 1)^{m_1}, \ldots, (s(\theta_t + 1) - 1)^{m_t}, (-1)^{(s-1)(m_0 + m_1 + \cdots + m_t)} \right\}.
\end{equation}

Note that if the adjacency matrix $A$ of a connected regular graph $G$ with $|V(G)|$ vertices and valency $k$ has four distinct eigenvalues $\{\theta_0 = k, \theta_1, \theta_2, \theta_3\}$, then $A$ satisfies the following equation (see [7]):

\begin{equation}
A^3 - \left( \sum_{i=1}^{3} \theta_i \right) A^2 + \left( \sum_{1 \leq i < j \leq 3} \theta_i \theta_j \right) A - \theta_1 \theta_2 \theta_3 I = \frac{\prod_{i=1}^{3} (k - \theta_i)}{|V(G)|} J.
\end{equation}

This implies that $G$ is walk-regular, see [10].

Now we assume $\Gamma$ is a cospectral graph with the $s$-clique extension of the triangular graph $T(n)$, where $s \geq 2$ and $n \geq 4$ are integers. Then by (2.1) and (2.2), the graph $\Gamma$ has spectrum

\begin{equation}
\left\{ \theta_{m_0}, \theta_{m_1}, \theta_{m_2}, \theta_{m_3} \right\}

= \left\{ (s(2n - 3) - 1)^1, (s(n - 3) - 1)^{n-1}, (-s - 1)^{\frac{s^2 - 3n}{2}}, (-1)^{(s-1)(n-1)} \right\}.
\end{equation}
Note that $\Gamma$ is regular with valency $k$, where $k = (s-1) + 2(n-2)s = s(2n-3) - 1$. Using (2.3), we obtain
\[
A^3 + (3 + 4s - sn)A^2 + ((3 - n)s^2 + (8 - 2n)s + 3)A
+ (1 - (n - 4)s - (n - 3)s^2)I = 4s^2(2n - 3)J.
\]
Therefore,
\[
A^3_{xy} = \begin{cases} 
2s^2n^2 - 2s^2n - 6sn - 3s^2 + 9s + 2, & \text{if } x = y, \\
9s^2n + 2sn - 15s^2 - 8s - 3 - (3 + 4s - sn)\lambda_{xy}, & \text{if } x \sim y, \\
8s^2n - 12s^2 - (3 + 4s - sn)\mu_{xy}, & \text{if } x \nsim y.
\end{cases}
\]

The following result is known as the Hoffman bound.

**Lemma 6** (Cf. [2], Theorem 3.5.2). Let $X$ be a $k$-regular graph with least eigenvalue $\tau$. Let $\alpha(X)$ be the size of maximum coclique in $X$. Then
\[
\alpha(X) \leq \frac{|X|}{k - \tau} (-\tau).
\]
If equality holds, then each vertex not in a coclique of size $\alpha(X)$ has exactly $-\tau$ neighbours in it.

Applying Lemma 6 to the complement of $\Gamma$, we obtain the following lemma.

**Lemma 7.** For any clique $C$ of $\Gamma$ with order $c$, we have
\[
c \leq s(n - 1).
\]
If equality holds, then every vertex $x \in V(\Gamma) \setminus V(C)$ has exactly $2s$ neighbours in $C$.

3. **Lines in $\Gamma$**

Recall that $\Gamma$ is a graph that is cospectral with the $s$-clique extension of the triangular graph $T(n)$, where $s \geq 2$ and $n \geq 1$ are integers. This implies that $\Gamma$ is walk-regular. Now we assume that $\Gamma$ is also co-edge-regular, i.e., there exist precisely $\mu = \mu(\Gamma)$ common neighbors between any two distinct nonadjacent vertices of $\Gamma$. Note that for $\Gamma$, we have $\mu = 4s$ from the spectrum of the $s$-clique extension of $T(n)$.

Fix a vertex, denoted by $\infty$ and let $\Gamma(\infty)$ be the local graph of $\Gamma$ at vertex $\infty$. Let $V(\Gamma(\infty)) = \{x_1, x_2, \ldots, x_k\}$, where $k = s(2n - 3) - 1$. Let $x_i$ have valency $d_i$ inside $\Gamma(\infty)$ for $i = 1, 2, \ldots, k$. Because $\Gamma$ is walk-regular, the number of closed walks through a fixed vertex $\infty$ of length 3 and 4 only depends on the spectrum.
This means that the number of edges in $\Gamma(\infty)$ is determined by the spectrum and as $\Gamma$ is co-edge-regular, we also see that the number of walks of length 2 in $\Gamma(\infty)$ is determined by the spectrum of $\Gamma$. This implies these numbers are the same as in a local graph of the $s$-clique extension of $T(n)$.

Let $\Delta$ be the $s$-clique extension of $T(n)$. Fix a vertex $u$ of $\Delta$. Then there are $s - 1$ vertices with valency $(s - 2) + 2s(n - 2)$ and $2s(n - 2)$ vertices with valency $s(n - 2) + 2(s - 1)$ in the local graph of $T(n)$ with respect to a fixed vertex. Using (2.5), this implies that the sum of valencies and the sum of square of valencies of vertices in $\Gamma(\infty)$ are constant, and are given by the following equations.

\[(3.1) \sum_{i=1}^{k} d_i = 2\varepsilon = 2s^2n^2 - 2s^2n - 6sn - 3s^2 + 9s + 2,\]

\[(3.2) \sum_{i=1}^{k} (d_i)^2 = 2sn(s^2n^2 - 6sn - 6s^2 + 10s + 8) + 9s^3 + 3s^2 - 24s - 4,\]

where $\varepsilon$ is the number of edges inside $\Gamma(\infty)$. By (3.1) and (3.2), we obtain

\[(3.3) \sum_{i=1}^{k} (d_i - (sn - 2))^2 = (s - 1)s^2(n - 3)^2.\]

It turns out that (3.3) is of crucial importance in proving our main result. Now we show the following lemma that will be used later.

**Lemma 8.** Fix a vertex $\infty$ of $\Gamma$ and let $\Gamma(\infty)$ be the local graph of $\Gamma$ at $\infty$. Define $E = \{y \sim \infty \mid d_y > \frac{3}{4}s(n-1)\}$ and let $e = |E|$. Let $F = \{y \sim \infty \mid d_y \leq \frac{3}{4}s(n-1)\}$ and $f = |F|$. If $n \geq 55$, then the following hold.

1. $f \leq 16(s - 1)$.
2. The subgraph of $\Gamma$ induced on $E$ is not complete.
3. The subgraph of $\Gamma$ induced on $E$ does not contain a coclique of order three.

**Proof.** Note that $f = k - e$. As $\frac{3}{4}s(n-1) + 1 \leq \frac{3}{4}(sn - 2)$, by (3.3), we obtain

\[(s - 1)s^2(n - 3)^2 = \sum_{y \sim \infty} (d_y - (sn - 2))^2 \geq \sum_{y \in F} (d_y - (sn - 2))^2 \geq \sum_{y \in F} \left(\frac{1}{4}(sn - 2)\right)^2 = \frac{1}{16} f(sn - 2)^2 \geq \frac{1}{16} f(sn - s)^2.

So

\[f \leq 16(s - 1),\]
which implies \( f < \frac{1}{2}(sn - 2) \) if \( n \geq 55 \) (and \( s \geq 2 \)). This means
\[
e = k - f > sn.
\]
By Lemma 7, we obtain that \( e \) is greater than the order of a maximum size clique and hence the subgraph induced on \( E \) is not complete.

Now we show that \( E \) does not contain a coclique of order three. Suppose \( X \subset E \) is a coclique in \( \Gamma(\infty) \) with vertices \( \{x_1, x_2, x_3\} \). Define \( A_i \) (\( i = 1, 2, 3 \)) such that
\[
A_i = \{y \sim \infty \mid y \sim x_i, y \sim x_j \text{ for all } x_j \in X, j \neq i\} \cup \{x_i\}.
\]
Since \( \Gamma \) is co-edge-regular, the vertices \( x_i \) and \( x_j \) (\( i \neq j \)) have at most \( 4s - 1 \) common neighbours. By the inclusion-exclusion principle, we have
\[
3 \times \left(\frac{3}{4}s(n - 1) + 1\right) - k \leq 4s - 1.
\]
This gives \( n < 54 \). This shows the lemma.

A maximal clique of \( \Gamma \) is called a line if it contains more than \( \frac{3}{4}s(n - 1) \) vertices. We show the existence of lines of \( \Gamma \) in the following.

**Proposition 9.** If \( n \geq 48s \geq 96 \), then for every vertex \( \infty \), there are exactly two lines through \( \infty \), say \( C_1 \) and \( C_2 \). Denote \( m = |V(C_1) \cap V(C_2) \setminus \{\infty\}| \) and \( \ell = k + 1 - |V(C_1) \cup V(C_2)| \). Then \( m \leq 4s - 1 \) and \( \ell \leq 16(s - 1) \).

**Proof.** Fix a vertex \( \infty \) of \( \Gamma \), let \( E = \{y \sim \infty \mid d_y > \frac{3}{4}s(n - 1)\} \). By Lemma 8, a maximum coclique in \( E \) has order two as \( n \geq 48s \geq 55 \). Let \( x_1, x_2 \) be distinct nonadjacent vertices in \( E \) and let \( y \in E \). Then \( y \) has at least one neighbour in \( \{x_1, x_2\} \).

Let \( A_i = \{y \in E \mid y \sim x_i, y \sim x_j \text{ for } j = 1, 2, j \neq i\} \) for \( i = 1, 2 \). Then the subgraph induced on \( A_i \) is complete for \( i = 1, 2 \). Let \( C_i \) be a maximal clique containing the vertex set \( \{\infty\} \cup A_i \) for \( i = 1, 2 \). Note that \( C_1 \neq C_2 \) as \( x_1 \sim x_2 \).

Let \( M = V(C_1) \cap V(C_2) \setminus \{\infty\} \) and \( L = V(\Gamma(\infty)) \setminus (V(C_1) \cup V(C_2)) \). Let \( m = |M| \) and \( \ell = |L| \). By the co-edge-regularity of \( \Gamma \), we have \( m \leq 4s - 1 \). Let \( F = \{y \sim \infty \mid d_y \leq \frac{3}{4}s(n - 1)\} \) and \( f = |F| \). We have, by Lemma 8, that \( f \leq 16(s - 1) \).

Suppose \( x \in E \setminus (V(C_1) \cup V(C_2)) \). Then \( x \) has at least \( (\frac{3}{4}s(n - 1) - (4s - 2) - 16(s - 1))/2 \) neighbours in at least one of \( C_1 \) and \( C_2 \). If \( n \geq 48s \geq 96 \), then this number is at least \( 4s \), which is a contradiction. Hence \( E \subseteq V(C_1) \cup V(C_2) \).

So, \( L \subseteq F \) and hence \( \ell \leq f \leq 16(s - 1) \) by Lemma 8. This shows that \( |V(C_1)| + |V(C_2)| \geq k - \ell \geq k - 16(s - 1) \). Assume \( |V(C_1)| \geq |V(C_2)| \), then we see that
\[
|V(C_2)| \geq k - 16(s - 1) - s(n - 1) > \frac{3}{4}s(n - 1),
\]
as \( n \geq 48s \geq 96 \). This gives that there are exactly two lines through \( \infty \).
Now we prove the following property for lines through a vertex.

**Lemma 10.** Fix a vertex $\infty$ of $\Gamma$ and let $C_1$ and $C_2$ be the two lines through $\infty$ with respective orders $c_1$ and $c_2$. Let $L = V(\Gamma(\infty)) \setminus (V(C_1) \cup V(C_2))$ and $M = V(C_1) \cap V(C_2) \setminus \{\infty\}$, and $\ell = |L|$, $m = |M| \geq 0$. If $n \geq 48s \geq 96$, then $\ell + m = s - 1$ and

$$s(n - 3) + 1 \leq c_i \leq s(n - 1)$$

for $i = 1, 2$.

**Proof.** Let $Q = V(C_1) \Delta V(C_2)$, where $\Delta$ means “symmetric difference”. Then, by Lemma 7, $|Q| \leq |V(C_1)| + |V(C_2)| \leq 2s(n - 1)$.

Note that $Q$ is the complement of $L \cup M$ inside $V(\Gamma(\infty))$.

For $y \in M$, we have

$$2sn - 19s \leq k - 1 - \ell \leq dy \leq k - 1 = 2sn - 3s - 2,$$

by Proposition 9.

Now let $y \in L$. Then $y$ has at least $4s - 1$ neighbors in each of $C_1$ and $C_2$.

Hence, by Proposition 9, we obtain

$$dy \leq 2 \times (4s - 1) + \ell - 1 \leq 2(4s - 1) + 16(s - 1) - 1 \leq 24s.$$

By (3.3), we obtain

$$s(n - 3)\sum_{y \sim \infty} (dy - (sn - 2))^2$$

$$\geq \sum_{y \in L} (dy - (sn - 2))^2 + \sum_{y \in M} (dy - (sn - 2))^2$$

$$\geq \ell((sn - s) - 24s)^2 + m((2sn - 19s) - sn)^2$$

$$= \ell s^2(n - 25)^2 + ms^2(n - 19)^2 \geq (\ell + m)s^2(n - 25)^2.$$

So

$$\ell + m \leq \frac{(s - 1)(n - 3)^2}{(n - 25)^2} < s$$

if $n \geq 48s$. Hence

$$\ell + m \leq s - 1.$$

This gives for $y \in L \cup M$, using (3.5), (3.6) and $l \leq s - 1$, that

$$dy - (sn - 2) \leq k - 1 - (sn - 2) = sn - 3s.$$
Note that by (3.8),
\begin{align*}
    s(n-1) & \geq |V(C_j)| \geq 1 + k - s(n-1) - l \\
    & \geq 2sn - 3s - s(n-1) - (s-1) = s(n-3) + 1
\end{align*}
for \(j = 1, 2\).

For \(y \in V(\Gamma(\infty)) \setminus (L \cup M)\), we obtain
\[sn - 4s \leq |V(C_2)| - m - 2 \leq d_y \leq |V(C_2)| - 1 + 4s - 1 + \ell \leq sn + 4s - 3.\]
Hence \(|d_y - (sn - 2)| \leq 4s\).

Now (3.3) gives us
\begin{align*}
    (s-1)s^2(n-3)^2 &= \sum_{y \sim \infty} (d_y - (sn - 2))^2 \\
    &\leq \sum_{y \in L \cup M} (d_y - (sn - 2))^2 + \sum_{y \in Q} (d_y - (sn - 2))^2 \\
    &\leq (\ell + m)s^2n^2 + 2s(n-1)(4s)^2.
\end{align*}
So
\[\ell + m \geq \frac{(s-1)(n-3)^2 - 32s(n-1)}{n^2} > s - 2,\]
if \(n \geq 48s \geq 96\). This implies \(\ell + m = s - 1\). This shows the lemma. \(\blacksquare\)

We obtain the following lemma immediately.

**Lemma 11.** Fix a vertex \(\infty\) of \(\Gamma\) and let \(C_1\) and \(C_2\) be the two lines through \(\infty\) with respective orders \(c_1\) and \(c_2\). Assume \(m = |V(C_1) \cap V(C_2) \setminus \{\infty\}|\). If \(n \geq 48s\), then \(c_1 + c_2 = 2s(n-2) + 2(m+1)\).

**Proof.** Let \(\ell = |V(\Gamma(\infty)) \setminus (V(C_1) \cup V(C_2))|\). Then we have
\[(c_1 - m - 1) + (c_2 - m - 1) + m + \ell = k = 2sn - 3s - 1.\]
If \(n \geq 48s\), then we have \(\ell + m = s - 1\) by Lemma 10, hence \(c_1 + c_2 = 2s(n - 2) + 2(m + 1)\). \(\blacksquare\)

In the next two sections, we will follow the method as used in Hayat et al. [4].

### 4. The Order of Lines

In this section, we will show the following lemma on the order of lines.
Lemma 12. Let $s \geq 2$ and $n \geq 1$ be integers. Let $\Gamma$ be a co-edge-regular graph that is cospectral with the $s$-clique extension of the triangular graph $T(n)$. Let $q_i$ be the number of lines with order $s(n - 3) + i$ for $i = 1, \ldots, 2s$ and $\delta = \sum_{i=1}^{2s} q_i$ be the number of lines in $\Gamma$. Assume $n \geq 48s$. Then

\begin{equation}
\sum_{i=1}^{2s} (s(n - 3) + i)q_i = sn(n - 1)
\end{equation}

holds, and the number $\delta$ satisfies

\begin{equation}
\frac{s}{n - 3}n \leq \delta \leq n + 2.
\end{equation}

If $\delta = n$, then $q_i = 0$ for all $i < 2s$, and $q_{2s} = n$.

Proof. Assume $n \geq 48s$. By Proposition 9, any vertex of $\Gamma$ lies on exactly two lines. Now consider the set 

$$W = \{(x, C) \mid x \in V(C), \text{ where } C \text{ is a line}\}.$$ 

Then, by double counting, the cardinality of the set $W$, we see (4.1). Moreover, we see that 

$$\delta = \sum_{i=1}^{2s} q_i < \sum_{i=1}^{2s} \frac{s(n - 3) + i}{s(n - 3)}q_i = n + 2 + \frac{6}{n - 3}.$$ 

Thus, if $n > 10$, we obtain 

$$\delta \leq n + 2.$$ 

On the other hand, we have 

$$\delta = \sum_{i=1}^{2s} q_i \geq \sum_{i=1}^{2s} \frac{s(n - 3) + i}{s(n - 1)}q_i = n.$$ 

This shows $\delta \geq n$, and $\delta = n$ implies that all lines have order $s(n - 1)$, which means $q_i \neq 0$ if and only if $i = 2s$. This completes the proof.

5. The Neighborhood of a Line

In this section we will show the following proposition.

Proposition 13. Let $\Gamma$ be a co-edge-regular graph that is cospectral with the $s$-clique extension of the triangular graph $T(n)$, where $s \geq 2, n \geq 1$ are integers. If $n \geq 48s$, then $\Gamma$ contains exactly $n$ lines.
Proof. In Lemma 12, we have seen that the number $\delta$ of lines satisfies $n \leq \delta \leq n + 2$. Now we assume that $n + 1 \leq \delta \leq n + 2$, in order to obtain a contradiction. Let $q_i$ be the number of lines of order $s(n - 3) + i$ in $\Gamma$, where $i = 1, \ldots, 2s$. Let $h$ be minimal such that $q_h \neq 0$. Then clearly, $1 \leq h \leq 2s$. Fix a line $C$ with exactly $s(n - 3) + h$ vertices. Let $q'_i$ be the number of lines $C'$ with $s(n - 3) + i$ vertices that intersect $C$ in at least one vertex. So $q_i \geq q'_i$. By Lemma 11, we obtain

\begin{equation}
|V(C) \cap V(C')| = \frac{h + i - 2s}{2}.
\end{equation}

By Proposition 9, every vertex lies on exactly two lines, and hence we obtain

\begin{equation}
\sum_{i=1}^{2s} q_i \left( \frac{h + i - 2s}{2} \right) \geq \sum_{i=1}^{2s} q'_i \left( \frac{h + i - 2s}{2} \right) = s(n - 3) + h.
\end{equation}

Now multiply (5.2) by 2 and subtract (4.1) from obtained equation, we find

\begin{equation}
\delta(h + s(1 - n)) = \sum_{i=1}^{2s} q_i(h + s(1 - n)) \geq s(-n^2 + 3n - 6) + 2h
\end{equation}
as $\delta = \sum_{i=1}^{2s} q_i$. This gives

\begin{equation}
h(\delta - 2) \geq 2s(n - 3) + (\delta - n)s(n - 1).
\end{equation}

As $n + 1 \leq \delta \leq n + 2$, we see

\begin{equation}
hn \geq h(\delta - 2) \geq 2s(n - 3) + (\delta - n)s(n - 1) \geq 2s(n - 3) + s(n - 1) = 3sn - 7s.
\end{equation}

Since $n \geq 48s$, (5.4) implies that $h \geq 3s$. This contradicts to $h \leq 2s$. This completes the proof.

6. Proof of the Main Result

In this section we show our main result, Theorem 1.

Proof of Theorem 1. Assume $n \geq 48s$. By Propositions 9 and 13 and Lemma 12, we find that there are exactly $n$ lines, each of order $s(n - 1)$, and every vertex $x$ in $\Gamma$ lies on exactly two lines. Moreover, by Lemma 11, the two lines through any vertex $x$ have exactly $s$ vertices in common, and every neighbor of $x$ lies in one of the two lines through $x$. Now consider the following equivalence relation $R$ on the vertex set $V(\Gamma)$: $xRx'$ if and only if $\{x\} \cup N_\Gamma(x) = \{x'\} \cup N_\Gamma(x')$, where $x, x' \in V(\Gamma)$. 
Every equivalence class under $R$ contains $s$ vertices and it is the intersection of two lines. Let us define the graph $\hat{\Gamma}$ whose vertices are the equivalent classes and two classes, say $S_1$ and $S_2$, are adjacent in $\hat{\Gamma}$ if and only if any vertex in $S_1$ is adjacent to any vertex in $S_2$. Then $\hat{\Gamma}$ is a regular graph with valency $2n - 4$, and $\Gamma$ is the $s$-clique extension of $\hat{\Gamma}$. Note that the spectrum of $\hat{\Gamma}$ is equal to

$$\left\{(2n - 4)^1, (n - 4)^{n-1}, (-2)^{\frac{n^2-3n}{2}}\right\},$$

by the relation of the spectra of $\Gamma$ and $\hat{\Gamma}$, see (2.1) and (2.2). Since $\hat{\Gamma}$ is a connected regular graph with valency $2n - 4$, and it has exactly three distinct eigenvalues, it follows that $\hat{\Gamma}$ is a strongly regular graph with parameters \((\binom{n}{2}, 2n - 4, n - 2, 4)\).

As proved in [3], the triangular graphs are determined by the spectrum except when $n = 8$. Since we assume that $n$ is large enough, the graph $\hat{\Gamma}$ is the triangular graph $T(n)$. This completes the proof.

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