

## ON THE $\alpha$ -SPECTRAL RADIUS OF UNIFORM HYPERGRAPHS

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### Abstract

For  $0 \leq \alpha < 1$  and a uniform hypergraph  $G$ , the  $\alpha$ -spectral radius of  $G$  is the largest  $H$ -eigenvalue of  $\alpha\mathcal{D}(G) + (1 - \alpha)\mathcal{A}(G)$ , where  $\mathcal{D}(G)$  and  $\mathcal{A}(G)$  are the diagonal tensor of degrees and the adjacency tensor of  $G$ , respectively. We give upper bounds for the  $\alpha$ -spectral radius of a uniform hypergraph, propose some transformations that increase the  $\alpha$ -spectral radius, and determine the unique hypergraphs with maximum  $\alpha$ -spectral radius in some classes of uniform hypergraphs.

**Keywords:**  $\alpha$ -spectral radius,  $\alpha$ -Perron vector, adjacency tensor, uniform hypergraph, extremal hypergraph.

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### 1. INTRODUCTION

Let  $G$  be a hypergraph on  $n$  vertices with vertex set  $V(G)$  and edge set  $E(G)$ . If  $|e| = k$  for each  $e \in E(G)$ , then  $G$  is said to be a  $k$ -uniform hypergraph. For a vertex  $v \in V(G)$ , the set of the edges containing  $v$  in  $G$  is denoted by  $E_G(v)$ , and the degree of  $v$  in  $G$ , denoted by  $d_G(v)$  or  $d_v$ , is the size of  $E_G(v)$ . We say that  $G$  is regular if all vertices of  $G$  have equal degrees. Otherwise,  $G$  is irregular.

For  $u, v \in V(G)$ , a walk from  $u$  to  $v$  in  $G$  is defined to be an alternating sequence of vertices and edges  $(v_0, e_1, v_1, \dots, v_{s-1}, e_s, v_s)$  with  $v_0 = u$  and  $v_s = v$  such that edge  $e_i$  contains vertices  $v_{i-1}$  and  $v_i$ , and  $v_{i-1} \neq v_i$  for  $i = 1, \dots, s$ . The value  $s$  is the length of this walk. A path is a walk with all  $v_i$  distinct and all  $e_i$

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distinct. A cycle is a walk containing at least two edges, all  $e_i$  are distinct and all  $v_i$  are distinct except  $v_0 = v_s$ . If there is a path from  $u$  to  $v$  for any  $u, v \in V(G)$ , then we say that  $G$  is connected. A hypertree is a connected hypergraph with no cycles. For  $k \geq 2$ , the number of vertices of a  $k$ -uniform hypertree with  $m$  edges is  $1 + (k - 1)m$ .

The distance between vertices  $u$  and  $v$  in a connected hypergraph  $G$  is the length of a shortest path from  $u$  to  $v$  in  $G$ . The diameter of connected hypergraph  $G$  is the maximum distance between any two vertices of  $G$ .

For positive integers  $k$  and  $n$ , a tensor  $\mathcal{T} = (\mathcal{T}_{i_1 \dots i_k})$  of order  $k$  and dimension  $n$  is a multidimensional array with entries  $\mathcal{T}_{i_1 \dots i_k} \in \mathbb{C}$  for  $i_j \in [n] = \{1, \dots, n\}$  and  $j \in [k]$ , where  $\mathbb{C}$  is the complex field.

Let  $\mathcal{M}$  be a tensor of order  $k \geq 2$  and dimension  $n$ , and  $\mathcal{N}$  a tensor of order  $\ell \geq 1$  and dimension  $n$ . The product  $\mathcal{MN}$  is the tensor of order  $(k - 1)(\ell - 1) + 1$  and dimension  $n$  with entries [22]

$$(\mathcal{MN})_{ij_1 \dots j_{k-1}} = \sum_{i_2, \dots, i_k \in [n]} \mathcal{M}_{ii_2 \dots i_k} \mathcal{N}_{i_2 j_1} \dots \mathcal{N}_{i_k j_{k-1}},$$

with  $i \in [n]$  and  $j_1, \dots, j_{k-1} \in [n]^{\ell-1}$ . Then for a tensor  $\mathcal{T}$  of order  $k$  and dimension  $n$  and an  $n$ -dimensional vector  $x = (x_1, \dots, x_n)^\top$ ,  $\mathcal{T}x$  is an  $n$ -dimensional vector whose  $i$ -th entry is

$$(\mathcal{T}x)_i = \sum_{i_2, \dots, i_k=1}^n \mathcal{T}_{ii_2 \dots i_k} x_{i_2} \dots x_{i_k},$$

where  $i \in [n]$ . For some complex  $\lambda$ , if there is a nonzero vector  $x$  such that

$$\mathcal{T}x = \lambda \left( x_1^{k-1}, \dots, x_n^{k-1} \right)^\top,$$

then  $\lambda$  is called an eigenvalue of  $\mathcal{T}$ , and  $x$  is called an eigenvector of  $\mathcal{T}$  corresponding to  $\lambda$ . Moreover, if both  $\lambda$  and  $x$  are real, then we call  $\lambda$  an  $H$ -eigenvalue and  $x$  an  $H$ -eigenvector of  $\mathcal{T}$ . See [10, 18, 20] for more details. The spectral radius of  $\mathcal{T}$  is the largest modulus of its eigenvalues, denoted by  $\rho(\mathcal{T})$ .

Let  $G$  be a  $k$ -uniform hypergraph with vertex set  $V(G) = \{v_1, \dots, v_n\}$ , where  $k \geq 2$ . The adjacency tensor of  $G$  is defined in [1] as the tensor  $\mathcal{A}(G)$  of order  $k$  and dimension  $n$  whose  $(i_1, \dots, i_k)$ -entry is  $\frac{1}{(k-1)!}$  if  $\{v_{i_1}, \dots, v_{i_k}\} \in E(G)$ , and 0 otherwise. The degree tensor of  $G$  is the diagonal tensor  $\mathcal{D}(G)$  of order  $k$  and dimension  $n$  with  $(i, \dots, i)$ -entry to be the degree of vertex  $v_i \in [n]$ . Then  $\mathcal{Q}(G) = \mathcal{D}(G) + \mathcal{A}(G)$  is the signless Laplacian tensor of  $G$  [20]. Motivated by work of Nikiforov [14] (see also [5, 15]), Lin *et al.* [11] proposed to study the convex linear combinations  $\mathcal{A}_\alpha(G)$  of  $\mathcal{D}(G)$  and  $\mathcal{A}(G)$  defined by

$$\mathcal{A}_\alpha(G) = \alpha \mathcal{D}(G) + (1 - \alpha) \mathcal{A}(G),$$

where  $0 \leq \alpha < 1$ . The  $\alpha$ -spectral radius of  $G$  is the spectral radius of  $\mathcal{A}_\alpha(G)$ , denoted by  $\rho_\alpha(G)$ . Note that  $\rho_0(G)$  is the spectral radius of  $G$ , while  $2\rho_{1/2}(G)$  is the signless Laplacian spectral radius of  $G$ .

For  $k \geq 2$ , let  $G$  be a  $k$ -uniform hypergraph with  $V(G) = [n]$ , and  $x$  a  $n$ -dimensional column vector. Let  $x_V = \prod_{v \in V} x_v$  for  $V \subseteq V(G)$ . Then

$$x^\top (\mathcal{A}_\alpha(G)x) = \alpha \sum_{u \in V(G)} d_u x_u^k + (1 - \alpha)k \sum_{e \in E(G)} x_e,$$

or equivalently,

$$x^\top (\mathcal{A}_\alpha(G)x) = \sum_{e \in E(G)} \left( \alpha \sum_{u \in e} x_u^k + (1 - \alpha)k x_e \right).$$

For a uniform hypergraph  $G$ , bounds for the spectral radius  $\rho_0(G)$  have been given in [1, 12, 13, 29], and bounds for the signless Laplacian spectral radius  $2\rho_{1/2}(G)$  may be found in [6, 12, 21]. Recently, Lin *et al.* [11] gave upper bounds for  $\alpha$ -spectral radius of connected irregular  $k$ -uniform hypergraphs, extending some known bounds for ordinary graphs. Some hypergraph transformations have been proposed to investigate the change of the 0-spectral radius, and the unique hypergraphs that maximize or minimize the 0-spectral radius have been determined among some classes of uniform hypergraphs (especially for hypertrees), see, e.g., [2, 4, 8, 16, 24, 25, 28, 31].

In this paper, we give upper bounds for the  $\alpha$ -spectral radius of a uniform hypergraph, propose some hypergraph transformations that increase the  $\alpha$ -spectral radius, and determine the unique hypergraphs with maximum  $\alpha$ -spectral radius in some classes of uniform hypergraphs such as the class of  $k$ -uniform hypercacti with  $m$  edges and  $r$  cycles for  $0 \leq r \leq \lfloor \frac{m}{2} \rfloor$ , and the class of  $k$ -uniform hypertrees with  $m$  edges and diameter  $d \geq 3$ .

## 2. PRELIMINARIES

A tensor  $\mathcal{T}$  of order  $k \geq 2$  and dimension  $n$  is said to be weakly reducible, if there is a nonempty proper subset  $J$  of  $[n]$  such that for  $i_1 \in J$  and  $i_j \in [n] \setminus J$  for some  $j = 2, \dots, k$ ,  $\mathcal{T}_{i_1 \dots i_k} = 0$ . Otherwise,  $\mathcal{T}$  is weakly irreducible.

For  $k \geq 2$ , an  $n$ -dimensional vector  $x$  is said to be  $k$ -unit if  $\sum_{i=1}^n x_i^k = 1$ .

**Lemma 1** [3, 27]. *Let  $\mathcal{T}$  be a nonnegative tensor of order  $k \geq 2$  and dimension  $n$ . Then  $\rho(\mathcal{T})$  is an eigenvalue of  $\mathcal{T}$  and there is a  $k$ -unit nonnegative eigenvector corresponding to  $\rho(\mathcal{T})$ . If furthermore  $\mathcal{T}$  is weakly irreducible, then there is a unique  $k$ -unit positive eigenvector corresponding to  $\rho(\mathcal{T})$ .*

If  $G$  is a  $k$ -uniform hypergraph with  $k \geq 2$ , then  $\mathcal{A}_\alpha(G)$  is weakly irreducible if and only if  $G$  is connected (see [17, 20] for the treatment of  $\mathcal{A}_0(G)$  and  $2\mathcal{A}_{1/2}(G)$ , respectively). Thus, if  $G$  is connected, then by Lemma 1, there is a unique  $k$ -unit positive  $H$ -eigenvector  $x$  corresponding to  $\rho_\alpha(G)$ , which is called the  $\alpha$ -Perron vector of  $G$ .

For a nonnegative tensor  $\mathcal{T}$  of order  $k \geq 2$  and dimension  $n$ , let  $r_i(\mathcal{T}) = \sum_{i_2 \dots i_k=1}^n \mathcal{T}_{i_2 \dots i_k}$  for  $i = 1, \dots, n$ .

**Lemma 2** [7, 27]. *Let  $\mathcal{T}$  be a nonnegative tensor of order  $k \geq 2$  and dimension  $n$ . Then*

$$\rho(\mathcal{T}) \leq \max_{1 \leq i \leq n} r_i(\mathcal{T})$$

with equality when  $\mathcal{T}$  is weakly irreducible if and only if  $r_1(\mathcal{T}) = \dots = r_n(\mathcal{T})$ .

For two tensors  $\mathcal{M}$  and  $\mathcal{N}$  of order  $k \geq 2$  and dimension  $n$ , if there is an  $n \times n$  nonsingular diagonal matrix  $U$  such that  $\mathcal{N} = U^{-(k-1)}\mathcal{M}U$ , then we say that  $\mathcal{M}$  and  $\mathcal{N}$  are diagonal similar.

**Lemma 3** [22]. *Let  $\mathcal{M}$  and  $\mathcal{N}$  be two diagonal similar tensors of order  $k \geq 2$  and dimension  $n$ . Then  $\mathcal{M}$  and  $\mathcal{N}$  have the same real eigenvalues.*

Let  $G$  be a connected  $k$ -uniform hypergraph on  $n$  vertices, where  $k \geq 2$ . Let  $0 \leq \alpha < 1$ . For an  $n$ -dimensional  $k$ -unit nonnegative vector  $x$ , by [19, Theorem 2] (and its proof) and Lemma 1, we have  $\rho_\alpha(G) \geq x^\top (\mathcal{A}_\alpha(G)x)$  with equality if and only if  $x$  is the  $\alpha$ -Perron vector of  $G$ . If  $x$  is the  $\alpha$ -Perron vector of  $G$ , then for any  $v \in V(G)$ ,

$$\rho_\alpha(G)x_v^{k-1} = \alpha d_v x_v^{k-1} + (1 - \alpha) \sum_{e \in E_v(G)} x_{e \setminus \{v\}},$$

which is called the eigenequation of  $G$  at  $v$ .

For a hypergraph  $G$  with  $\emptyset \neq X \subseteq V(G)$ , let  $G[X]$  be the subhypergraph induced by  $X$ , i.e.,  $G[X]$  has vertex set  $X$  and edge set  $\{e \subseteq X : e \in E(G)\}$ . If  $E' \subseteq E(G)$ , then  $G - E'$  is the hypergraph obtained from  $G$  by deleting the edges in  $E'$ . If  $E'$  is set of subsets of  $V(G)$  and no element of  $E'$  is an edge of  $G$ , then  $G + E'$  is the hypergraph obtained from  $G$  by adding elements of  $E'$  as edges.

A  $k$ -uniform hypertree with  $m$  edges is a hyperstar, denoted by  $S_{m,k}$ , if all edges share a common vertex. A  $k$ -uniform loose path with  $m \geq 1$  edges, denoted by  $P_{m,k}$ , is the  $k$ -uniform hypertree whose vertices and edges may be labelled as  $(v_0, e_1, v_1, \dots, v_{m-1}, e_m, v_m)$  such that the vertices  $v_1, \dots, v_{m-1}$  are of degree 2, and all the other vertices of  $G$  are of degree 1.

If  $P$  is a path or a cycle of a hypergraph  $G$ ,  $V(P)$  denotes the vertex set of the hypergraph  $P$ .

3. UPPER BOUNDS FOR  $\alpha$ -SPECTRAL RADIUS

For a connected irregular  $k$ -uniform hypergraph  $G$  with  $n$  vertices, maximum degree  $\Delta$  and diameter  $D$ , where  $2 \leq k < n$ , it was shown in [11] that for  $0 \leq \alpha < 1$ ,

$$\rho_\alpha(G) < \Delta - \frac{4(1-\alpha)}{((4D-1-2\alpha)(k-1)+1)n}.$$

For a  $k$ -uniform hypergraph  $G$ , upper bounds on  $\rho_0(G)$  and  $2\rho_{1/2}(G)$  have been given in [12, 29].

**Theorem 4.** *Let  $G$  be a  $k$ -uniform hypergraph on  $n$  vertices with maximum degree  $\Delta$  and second maximum degree  $\Delta'$ , where  $k \geq 2$ . For  $\alpha = 0$ , let  $\delta = (\frac{\Delta}{\Delta'})^{\frac{1}{k}}$ , and for  $0 < \alpha < 1$ , let  $\delta = 1$  if  $\Delta = \Delta'$  and  $\delta$  be a root of  $h(t) = 0$  in  $(\frac{\Delta}{\Delta'})^{\frac{1}{k}}, +\infty)$  if  $\Delta > \Delta'$ , where  $h(t) = (1-\alpha)\Delta't^k + \alpha(\Delta' - \Delta)t^{k-1} - (1-\alpha)\Delta$  for  $0 \leq \alpha < 1$ . Then*

$$(1) \quad \rho_\alpha(G) \leq \alpha\Delta + (1-\alpha)\Delta\delta^{-(k-1)}.$$

Moreover, if  $G$  is connected, then equality holds in (1) if and only if  $G$  is a regular hypergraph or  $G \cong G'$ , where  $V(G') = V(H) \cup \{v\}$ ,  $E(G') = \{e \cup \{v\} : e \in E(H)\}$ , and  $H$  is a regular  $(k-1)$ -uniform hypergraph on  $n-1$  vertices with  $v \notin V(H)$ .

**Proof.** By Theorem 2.1 and Lemma 2.2 in [22], we may assume that  $d_1 \geq \dots \geq d_n$ . Then  $\Delta = d_1$  and  $\Delta' = d_2$ .

If  $d_1 = d_2$ , then  $\delta = 1$ , and by Lemma 2, we have

$$\rho_\alpha(G) \leq \max_{1 \leq i \leq n} r_i(\mathcal{A}_\alpha(G)) = \max_{1 \leq i \leq n} d_i = d_1 = \alpha d_1 + (1-\alpha)d_1\delta^{-(k-1)},$$

and when  $G$  is connected,  $\mathcal{A}_\alpha(G)$  is weakly irreducible, thus by Lemma 2, equality (1) holds if and only if  $r_1(\mathcal{A}_\alpha(G)) = \dots = r_n(\mathcal{A}_\alpha(G))$ , i.e.,  $G$  is a regular hypergraph.

Suppose in the following that  $d_1 > d_2$ . Let  $U = \text{diag}(t, 1, \dots, 1)$  be an  $n \times n$  diagonal matrix, where  $t > 1$  is a variable to be determined later. Let  $\mathcal{T} = U^{-(k-1)}\mathcal{A}_\alpha(G)U$ . By Lemma 3,  $\mathcal{A}_\alpha(G)$  and  $\mathcal{T}$  have the same real eigenvalues. Obviously, both  $\mathcal{A}_\alpha(G)$  and  $\mathcal{T}$  are nonnegative tensors of order  $k$  and dimension  $n$ . By Lemma 1,  $\rho(\mathcal{A}_\alpha(G))$  is an eigenvalue of  $\mathcal{A}_\alpha(G)$  and  $\rho(\mathcal{T})$  is an eigenvalue of  $\mathcal{T}$ . Therefore  $\rho_\alpha(G) = \rho(\mathcal{A}_\alpha(G)) = \rho(\mathcal{T})$ . For  $i \in [n] \setminus \{1\}$ , let  $d_{1,i} = |\{e : 1, i \in e \in E(G)\}|$ . Obviously,  $d_{1,i} \leq d_i$ . Note that

$$\begin{aligned}
r_1(\mathcal{T}) &= \sum_{i_2, \dots, i_k \in [n]} \mathcal{T}_{1i_2 \dots i_k} \\
&= \alpha \mathcal{D}_{1 \dots 1} + (1 - \alpha) \sum_{i_2, \dots, i_k \in [n]} U_{11}^{-(k-1)} \mathcal{A}_{1i_2 \dots i_k} U_{i_2 i_2} \cdots U_{i_k i_k} \\
&= \alpha d_1 + (1 - \alpha) \sum_{i_2, \dots, i_k \in [n] \setminus \{1\}} \frac{1}{t^{k-1}} \mathcal{A}_{1i_2 \dots i_k} = \alpha d_1 + \frac{(1 - \alpha)d_1}{t^{k-1}},
\end{aligned}$$

and for  $2 \leq i \leq n$ ,

$$\begin{aligned}
r_i(\mathcal{T}) &= \sum_{i_2, \dots, i_k \in [n]} \mathcal{T}_{ii_2 \dots i_k} = \alpha \mathcal{D}_{i \dots i} + (1 - \alpha) \sum_{i_2, \dots, i_k \in [n]} U_{ii}^{-(k-1)} \mathcal{A}_{ii_2 \dots i_k} U_{i_2 i_2} \cdots U_{i_k i_k} \\
&= \alpha d_i + (1 - \alpha) \sum_{\substack{i_2, \dots, i_k \in [n] \\ 1 \in \{i_2, \dots, i_k\}}} U_{ii}^{-(k-1)} \mathcal{A}_{ii_2 \dots i_k} U_{i_2 i_2} \cdots U_{i_k i_k} \\
&\quad + (1 - \alpha) \sum_{\substack{i_2, \dots, i_k \in [n] \\ 1 \notin \{i_2, \dots, i_k\}}} U_{ii}^{-(k-1)} \mathcal{A}_{ii_2 \dots i_k} U_{i_2 i_2} \cdots U_{i_k i_k} \\
&= \alpha d_i + (1 - \alpha) \sum_{\substack{i_2, \dots, i_k \in [n] \\ 1 \in \{i_2, \dots, i_k\}}} \mathcal{A}_{ii_2 \dots i_k} t + (1 - \alpha) \sum_{\substack{i_2, \dots, i_k \in [n] \\ 1 \notin \{i_2, \dots, i_k\}}} \mathcal{A}_{ii_2 \dots i_k} \\
&= \alpha d_i + (1 - \alpha) t d_{1,i} + (1 - \alpha) (d_i - d_{1,i}) \\
&= d_i + (1 - \alpha) (t - 1) d_{1,i} \leq (1 + (1 - \alpha) (t - 1)) d_i \leq (1 + (1 - \alpha) (t - 1)) d_2
\end{aligned}$$

with equality if and only if  $d_{1,i} = d_i = d_2$ .

Note that  $h\left(\left(\frac{d_1}{d_2}\right)^{\frac{1}{k}}\right) = \alpha(d_2 - d_1) \left(\frac{d_1}{d_2}\right)^{\frac{k-1}{k}} \leq 0$  with equality if and only if  $\alpha = 0$ , and that  $h(+\infty) > 0$ . Thus  $h(t) = 0$  does have a root  $\delta$ , as required. Let  $t = \delta$ . Then  $t > 1$ ,

$$\alpha d_1 + \frac{(1 - \alpha)d_1}{t^{k-1}} = (1 + (1 - \alpha)(t - 1))d_2,$$

and thus for  $1 \leq i \leq n$ ,

$$r_i(\mathcal{T}) \leq \alpha d_1 + (1 - \alpha) d_1 \delta^{-(k-1)}.$$

Now by Lemma 2,

$$\rho_\alpha(G) = \rho(\mathcal{T}) \leq \max_{1 \leq i \leq n} r_i(\mathcal{T}) \leq \alpha d_1 + (1 - \alpha) d_1 \delta^{-(k-1)}.$$

This proves (1).

Suppose that  $G$  is connected. Then  $\mathcal{A}_\alpha$  is weakly irreducible, and so is  $\mathcal{T}$ .

Suppose that equality holds in (1). From the above arguments and by Lemma 2, we have  $r_1(\mathcal{T}) = \dots = r_n(\mathcal{T}) = \alpha d_1 + (1 - \alpha)d_1\delta^{-(k-1)}$ , and  $d_{1,i} = d_i = d_2$  for  $i = 2, \dots, n$ . Then vertex 1 is contained in each edge of  $G$ . Let  $H$  be the hypergraph with  $V(H) = V(G) \setminus \{1\} = \{2, \dots, n\}$  and  $E(H) = \{e \setminus \{1\} : e \in E(G)\}$ . Then  $H$  is a regular  $(k-1)$ -uniform hypergraph on vertices  $2, \dots, n$  of degree  $d_2$ . Therefore  $G \cong G'$ , where  $V(G') = V(H) \cup \{1\}$ ,  $E(G') = \{e \cup \{v\} : e \in E(H)\}$ , and  $H$  is a regular  $(k-1)$ -uniform hypergraph on vertices  $2, \dots, n$  of degree  $d_2$ .

Conversely, if  $G \cong G'$ , where  $V(G') = V(H) \cup \{1\}$ ,  $E(G') = \{e \cup \{v\} : e \in E(H)\}$ , and  $H$  is a regular  $(k-1)$ -uniform hypergraph on vertices  $2, \dots, n$  of degree  $d_2$ , then by the above arguments, we have  $r_i(\mathcal{T}) = \alpha d_1 + (1 - \alpha)d_1(\frac{1}{\delta})^{k-1}$  for  $1 \leq i \leq n$ , and thus by Lemma 3,  $\rho(\mathcal{A}_\alpha(G)) = \rho(\mathcal{T}) = \alpha d_1 + (1 - \alpha)d_1\delta^{-(k-1)}$ , i.e., (1) is an equality.  $\blacksquare$

As  $\delta \geq \left(\frac{d_1}{d_2}\right)^{\frac{1}{k}}$  with equality if and only if  $d_1 = d_2$ , we have by Theorem 4 that  $\rho_\alpha(G) \leq \alpha d_1 + (1 - \alpha)d_1^{\frac{1}{k}}d_2^{1-\frac{1}{k}}$  with equality if and only if  $G$  is regular.

Letting  $\alpha = 0$  in Theorem 4, we have  $\delta = \left(\frac{d_1}{d_2}\right)^{\frac{1}{k}}$  and thus (1) becomes  $\rho_0(G) \leq d_1^{\frac{1}{k}}d_2^{1-\frac{1}{k}}$ , see [29]. Letting  $\alpha = \frac{1}{2}$  in Theorem 4,  $\delta$  is the root of  $d_2t^k + (d_2 - d_1)t^{k-1} - d_1 = 0$ , and (1) becomes  $2\rho_{1/2}(G) \leq d_1 + d_1\delta^{-(k-1)}$ , see [12].

Let  $G$  be a connected  $k$ -uniform hypergraph with  $n$  vertices,  $m$  edges, maximum degree  $\Delta$  and diameter  $D$ , where  $k \geq 2$ . For  $0 \leq \alpha < 1$ , let  $\bar{x}$  be the maximum entry of the  $\alpha$ -Perron vector of  $G$ . From [11], we have

$$\rho_\alpha(G) \leq \Delta - \frac{(1 - \alpha)k(n\Delta - km)}{2(n\Delta - km)(k-1)D + (1 - \alpha)k} \bar{x}^k,$$

and if  $D = 1$  and  $k \geq 3$ , then

$$\rho_\alpha(G) \leq \Delta - \frac{(1 - \alpha)(n\Delta - km)n}{2(n\Delta - km)(k-1) + (1 - \alpha)n} \bar{x}^k.$$

**Theorem 5.** *Let  $G$  be a connected  $k$ -uniform hypergraph on  $n$  vertices with  $m$  edges and maximum degree  $\Delta$ , where  $k \geq 2$ . Let  $x$  be the  $\alpha$ -Perron vector of  $G$  with maximum entry  $\bar{x}$ . For  $0 \leq \alpha < 1$ , we have*

$$\rho_\alpha(G) \leq \alpha\Delta + (1 - \alpha)km\bar{x}^k$$

$$\rho_\alpha(G) \leq \alpha\Delta + (1 - \alpha) \left( \sum_{i \in V(G)} d_i^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}} \bar{x}^{k-1}$$

with either equality if and only if  $G$  is regular.

**Proof.** From the eigenequation of  $G$  at  $i \in V(G)$ , we have

$$(\rho_\alpha - \alpha\Delta)x_i^{k-1} \leq (\rho_\alpha - \alpha d_i)x_i^{k-1} = (1 - \alpha) \sum_{e \in E_i(G)} \prod_{v \in e \setminus \{i\}} x_v \leq (1 - \alpha)d_i \bar{x}^{k-1}$$

with equality if and only if for  $v \in e \setminus \{i\}$  with  $e \in E_i(G)$ ,  $x_v = \bar{x}$ . Then

$$(\rho_\alpha - \alpha\Delta)x_i^k \leq (1 - \alpha)d_i \bar{x}^k,$$

and thus

$$\rho_\alpha - \alpha\Delta \leq (1 - \alpha)\bar{x}^k \sum_{i \in V(G)} d_i = (1 - \alpha)km\bar{x}^k$$

with equality if and only if all entries of  $x$  are equal, or equivalently,  $G$  is regular.

On the other hand, we have

$$(\rho_\alpha - \alpha\Delta)^{\frac{k}{k-1}} x_i^k \leq (1 - \alpha)^{\frac{k}{k-1}} d_i^{\frac{k}{k-1}} \bar{x}^k,$$

and thus

$$(\rho_\alpha - \alpha\Delta)^{\frac{k}{k-1}} \leq (1 - \alpha)^{\frac{k}{k-1}} \bar{x}^k \sum_{i \in V(G)} d_i^{\frac{k}{k-1}},$$

implying that

$$\rho_\alpha(G) \leq \alpha\Delta + (1 - \alpha) \left( \sum_{i \in V(G)} d_i^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}} \bar{x}^{k-1}$$

with equality if and only if  $G$  is regular. ■

Let  $\alpha = 0$  in Theorem 5, we have  $\bar{x} \geq \frac{\rho_0^{\frac{1}{k-1}}}{\left( \sum_{i \in V(G)} d_i^{\frac{k}{k-1}} \right)^{\frac{1}{k}}}$ , which has been reported in [9].

#### 4. TRANSFORMATIONS INCREASING $\alpha$ -SPECTRAL RADIUS

In the following, we propose several types of hypergraph transformations that increase the  $\alpha$ -spectral radius.

**Theorem 6.** For  $k \geq 2$ , let  $G$  be a  $k$ -uniform hypergraph with  $u, v_1, \dots, v_r \in V(G)$  and  $e_1, \dots, e_r \in E(G)$  for  $r \geq 1$  such that  $u \notin e_i$  and  $v_i \in e_i$  for  $i = 1, \dots, r$ , where  $v_1, \dots, v_r$  are not necessarily distinct. Let  $e'_i = (e_i \setminus \{v_i\}) \cup \{u\}$  for  $i = 1, \dots, r$ . Suppose that  $e'_i \notin E(G)$  for  $i = 1, \dots, r$ . Let  $G' = G - \{e_1, \dots, e_r\} + \{e'_1, \dots, e'_r\}$ . Let  $x$  the  $\alpha$ -Perron vector of  $G$ . If  $x_u \geq \max\{x_{v_1}, \dots, x_{v_r}\}$ , then  $\rho_\alpha(G') > \rho_\alpha(G)$  for  $0 \leq \alpha < 1$ .

**Proof.** Note that  $\rho_\alpha(G) = x^\top(\mathcal{A}_\alpha(G)x)$  and  $\rho_\alpha(G') \geq x^\top(\mathcal{A}_\alpha(G')x)$  with equality if and only if  $x$  is also the  $\alpha$ -Perron vector of  $G'$ . Thus

$$\begin{aligned} \rho_\alpha(G') - \rho_\alpha(G) &\geq x^\top(\mathcal{A}_\alpha(G')x) - x^\top(\mathcal{A}_\alpha(G)x) \\ &= \alpha \left( rx_u^k - \sum_{i=1}^r x_{v_i}^k \right) + (1-\alpha)k \sum_{i=1}^r (x_u - x_{v_i})x_{e_i \setminus \{v_i\}} \geq 0, \end{aligned}$$

and thus  $\rho_\alpha(G') \geq \rho_\alpha(G)$ . Suppose that  $\rho_\alpha(G') = \rho_\alpha(G)$ . Then  $\rho_\alpha(G') = x^\top(\mathcal{A}_\alpha(G')x)$ , and thus  $x$  is the  $\alpha$ -Perron vector of  $G'$ . From the eigenequations of  $G'$  and  $G$  at  $u$  and noting that  $E_u(G') = E_u(G) \cup \{e'_1, \dots, e'_r\}$ , we have

$$\begin{aligned} \rho_\alpha(G')x_u^{k-1} &= \alpha(d_u + r)x_u^{k-1} + (1-\alpha) \sum_{e \in E_u(G')} x_{e \setminus \{u\}} \\ &> \alpha d_u x_u^{k-1} + (1-\alpha) \sum_{e \in E_u(G)} x_{e \setminus \{u\}} = \rho_\alpha(G)x_u^{k-1}, \end{aligned}$$

a contradiction. It follows that  $\rho_\alpha(G') > \rho_\alpha(G)$ .  $\blacksquare$

We say that the hypergraph  $G'$  in Theorem 6 is obtained from  $G$  by moving edges  $e_1, \dots, e_r$  from  $v_1, \dots, v_r$  to  $u$ . Theorem 6 has been established in [8] for  $\alpha \in \{0, \frac{1}{2}\}$ .

**Theorem 7.** *Let  $G$  be a connected  $k$ -uniform hypergraph with  $k \geq 2$ , and  $e$  and  $f$  be two edges of  $G$  with  $e \cap f = \emptyset$ . Let  $x$  be the  $\alpha$ -Perron vector of  $G$ . Let  $U \subset e$  and  $V \subset f$  with  $1 \leq |U| = |V| \leq k-1$ . Let  $e' = U \cup (f \setminus V)$  and  $f' = V \cup (e \setminus U)$ . Suppose that  $e', f' \notin E(G)$ . Let  $G' = G - \{e, f\} + \{e', f'\}$ . If  $x_U \geq x_V$ ,  $x_{e \setminus U} \leq x_{f \setminus V}$  and one is strict, then  $\rho_\alpha(G) < \rho_\alpha(G')$  for  $0 \leq \alpha < 1$ .*

**Proof.** Note that

$$\begin{aligned} \rho_\alpha(G') - \rho_\alpha(G) &\geq x^\top(\mathcal{A}_\alpha(G')x) - x^\top(\mathcal{A}_\alpha(G)x) \\ &= (1-\alpha)k \sum_{g \in E(G')} x_g - (1-\alpha)k \sum_{g \in E(G)} x_g \\ &= (1-\alpha)k (x_U x_{f \setminus V} + x_V x_{e \setminus U} - x_U x_{e \setminus U} - x_V x_{f \setminus V}) \\ &= (1-\alpha)k (x_U - x_V) (x_{f \setminus V} - x_{e \setminus U}) \geq 0. \end{aligned}$$

Thus  $\rho_\alpha(G') \geq \rho_\alpha(G)$ . Suppose that  $\rho_\alpha(G') = \rho_\alpha(G)$ . Then  $\rho_\alpha(G') = x^\top(\mathcal{A}_\alpha(G')x)$  and thus  $x$  is the  $\alpha$ -Perron vector of  $G'$ . Suppose without loss of generality that  $x_{e \setminus U} < x_{f \setminus V}$ . Then for  $u \in U$

$$-x_{e \setminus \{u\}} + x_{e' \setminus \{u\}} = -x_{U \setminus \{u\}} (x_{e \setminus U} - x_{f \setminus V}) > 0.$$

From the eigenequations of  $G'$  and  $G$  at a vertex  $u \in U$ , we have

$$\begin{aligned} \rho_\alpha(G')x_u^{k-1} &= \alpha d_u x_u^{k-1} + (1-\alpha) \sum_{g \in E_u(G')} x_{g \setminus \{u\}} \\ &= \alpha d_u x_u^{k-1} + (1-\alpha) \left( \sum_{g \in E_u(G)} x_{g \setminus \{u\}} - x_{e \setminus \{u\}} + x_{e' \setminus \{u\}} \right) \\ &> \alpha d_u x_u^{k-1} + (1-\alpha) \sum_{g \in E_u(G)} x_{g \setminus \{u\}} = \rho_\alpha(G)x_u^{k-1}, \end{aligned}$$

a contradiction. It follows that  $\rho_\alpha(G') > \rho_\alpha(G)$ .  $\blacksquare$

The above result has been known for  $k = 2$  in [5] and  $\alpha = 0$  [25].

A path  $P = (v_0, e_1, v_1, \dots, v_{s-1}, e_s, v_s)$  in a  $k$ -uniform hypergraph  $G$  is called a pendant path at  $v_0$ , if  $d_G(v_0) \geq 2$ ,  $d_G(v_i) = 2$  for  $1 \leq i \leq s-1$ ,  $d_G(v) = 1$  for  $v \in e_i \setminus \{v_{i-1}, v_i\}$  with  $1 \leq i \leq s$ , and  $d_G(v_s) = 1$ . If  $s = 1$ , then we call  $P$  or  $e_1$  a pendant edge of  $G$  (at  $v_0$ ). A pendant path of length 0 at  $v_0$  is understood as the trivial path consisting of a single vertex  $v_0$ .

If  $P$  is a pendant path at  $u$  in a  $k$ -uniform hypergraph  $G$ , we say  $G$  is obtained from  $H$  by attaching a pendant path  $P$  at  $u$  with  $H = G[V(G) \setminus (V(P) \setminus \{u\})]$ . In this case, we write  $G = H_u(s)$  if the length of  $P$  is  $s$ . Let  $H_u(0) = H$ .

For a  $k$ -uniform hypergraph  $G$  with  $u \in V(G)$ , and  $p \geq q \geq 0$ , let  $G_u(p, q) = (G_u(p))_u(q)$ .

**Theorem 8.** *For  $k \geq 2$ , let  $G$  be a connected  $k$ -uniform hypergraph with  $|E(G)| \geq 1$  and  $u \in V(G)$ . For  $p \geq q \geq 1$  and  $0 \leq \alpha < 1$ , we have  $\rho_\alpha(G_u(p, q)) > \rho_\alpha(G_u(p+1, q-1))$ .*

**Proof.** Let  $(u, e_1, u_1, \dots, u_p, e_{p+1}, u_{p+1})$  and  $(u, f_1, v_1, \dots, v_{q-2}, f_{q-1}, v_{q-1})$  be the pendant paths of  $G_u(p+1, q-1)$  at  $u$  of lengths  $p+1$  and  $q-1$ , respectively. Let  $v_0 = u$ . Let  $x$  be the  $\alpha$ -Perron vector of  $G_u(p+1, q-1)$ .

Suppose that  $\rho_\alpha(G_u(p, q)) < \rho_\alpha(G_u(p+1, q-1))$ . We prove that  $x_{u_{p-i}} > x_{v_{q-i-1}}$  for  $i = 0, \dots, q-1$ .

Suppose that  $x_{v_{q-1}} \geq x_{u_p}$ . Let  $H$  be the  $k$ -uniform hypergraph obtained from  $G_u(p+1, q-1)$  by moving  $e_{p+1}$  from  $u_p$  to  $v_{q-1}$ . By Theorem 6 and noting that  $H \cong G_u(p, q)$ , we have  $\rho_\alpha(G_u(p, q)) = \rho_\alpha(H) > \rho_\alpha(G_u(p+1, q-1))$ , a contradiction. Thus  $x_{u_p} > x_{v_{q-1}}$ .

Suppose that  $q \geq 2$  and  $x_{u_{p-i}} > x_{v_{q-i-1}}$ , where  $0 \leq i \leq q-2$ . We want to show that  $x_{u_{p-(i+1)}} > x_{v_{q-(i+1)-1}}$ . Suppose that this is not true, i.e.,  $x_{v_{q-i-2}} \geq x_{u_{p-i-1}}$ . Suppose that  $x_{e_{p-i} \setminus \{u_{p-i-1}, u_{p-i}\}} \leq x_{f_{q-i-1} \setminus \{v_{q-i-2}, v_{q-i-1}\}}$ . Then  $x_{e_{p-i} \setminus \{u_{p-i}\}} \leq x_{f_{q-i-1} \setminus \{v_{q-i-1}\}}$ . Let  $H' = G_u(p+1, q-1) - \{e_{p-i}, f_{q-i-1}\} + \{e', f'\}$ , where  $e' = \{u_{p-i}\} \cup (f_{q-i-1} \setminus \{v_{q-i-1}\})$  and  $f' = \{v_{q-i-1}\} \cup (e_{p-i} \setminus$

$\{u_{p-i}\}$ ). Obviously,  $H' \cong G_u(p, q)$ . By Theorem 7, we have  $\rho_\alpha(G_u(p, q)) = \rho_\alpha(H') > \rho_\alpha(G_u(p+1, q-1))$ , a contradiction. Thus  $x_{e_{p-i} \setminus \{u_{p-i-1}, u_{p-i}\}} > x_{f_{q-i-1} \setminus \{v_{q-i-2}, v_{q-i-1}\}}$ , and then  $x_{e_{p-i} \setminus \{u_{p-i-1}\}} > x_{f_{q-i-1} \setminus \{v_{q-i-2}\}}$ . Let  $H'' = G_u(p+1, q-1) - \{e_{p-i}, f_{q-i-1}\} + \{e'', f''\}$ , where  $e'' = (e_{p-i} \setminus \{u_{p-i-1}\}) \cup \{v_{q-i-2}\}$  and  $f'' = (f_{q-i-1} \setminus \{v_{q-i-2}\}) \cup \{u_{p-i-1}\}$ . Obviously,  $H'' \cong G_u(p, q)$ . By Theorem 7, we have  $\rho_\alpha(G_u(p, q)) = \rho_\alpha(H'') > \rho_\alpha(G_u(p+1, q-1))$ , also a contradiction. It follows that  $x_{u_{p-i-1}} > x_{v_{q-i-2}}$ , i.e.,  $x_{u_{p-(i+1)}} > x_{v_{q-(i+1)-1}}$ .

Therefore  $x_{u_{p-i}} > x_{v_{q-i-1}}$  for  $i = 0, \dots, q-1$ . Particularly,  $x_{u_{p-q+1}} > x_{v_0}$ .

Now let  $H^*$  be the  $k$ -uniform hypergraph obtained from  $G_u(p+1, q-1)$  by moving all the edges containing  $u$  except  $e_1$  and  $f_1$  from  $u$  to  $u_{p-q+1}$ . By Theorem 6 and noting that  $H^* \cong G_u(p, q)$ , we have  $\rho_\alpha(G_u(p, q)) > \rho_\alpha(G_u(p+1, q-1))$ , a contradiction. Therefore  $\rho_\alpha(G_u(p, q)) > \rho_\alpha(G_u(p+1, q-1))$ . ■

The above result has been reported for  $k = 2$  in [5] and  $\alpha = 0$  in [25].

**Theorem 9.** *Let  $G$  be a  $k$ -uniform hypergraph with  $k \geq 2$ ,  $e = \{v_1, \dots, v_k\}$  be an edge of  $G$  with  $d_G(v_i) \geq 2$  for  $i = 1, \dots, r$ , and  $d_G(v_i) = 1$  for  $i = r+1, \dots, k$ , where  $3 \leq r \leq k$ . Let  $G'$  be the hypergraph obtained from  $G$  by moving all edges containing  $v_3, \dots, v_r$  but not containing  $v_1$  from  $v_3, \dots, v_r$  to  $v_1$ . Then  $\rho_\alpha(G') > \rho_\alpha(G)$  for  $0 \leq \alpha < 1$ .*

**Proof.** Let  $x$  be the  $\alpha$ -Perron vector of  $G$ , and  $x_{v_t} = \max\{x_{v_i} : 3 \leq i \leq r\}$ . If  $x_{v_1} \geq x_{v_t}$ , then by Theorem 6,  $\rho_\alpha(G') > \rho_\alpha(G)$ . Suppose that  $x_{v_1} < x_{v_t}$ . Let  $G''$  be the hypergraph obtained from  $G$  by moving all edges containing  $v_i$  but not containing  $v_t$  from  $v_i$  to  $v_t$  for all  $3 \leq i \leq r$  with  $i \neq t$ , and moving all edges containing  $v_1$  but not containing  $v_t$  from  $v_1$  to  $v_t$ . It is obvious that  $G'' \cong G'$ . By Theorem 6, we have  $\rho_\alpha(G') = \rho_\alpha(G'') > \rho_\alpha(G)$ . ■

## 5. HYPERGRAPHS WITH LARGE $\alpha$ -SPECTRAL RADIUS

A hypercactus is a connected  $k$ -uniform hypergraph in which any two cycles (viewed as two hypergraphs) have at most one vertex in common. Let  $H_{m,r,k}$  be a  $k$ -uniform hypergraph consisting of  $r$  cycles of length 2 and  $m - 2r$  pendant edges with a vertex in common. If  $r = 0$ , then  $H_{m,r,k} \cong S_{m,k}$ .

**Theorem 10.** *For  $k \geq 2$ , let  $G$  be a  $k$ -uniform hypercactus with  $m$  edges and  $r$  cycles, where  $0 \leq r \leq \lfloor \frac{m}{2} \rfloor$  and  $m \geq 2$ . For  $0 \leq \alpha < 1$ , we have  $\rho_\alpha(G) \leq \rho_\alpha(H_{m,r,k})$  with equality if and only if  $G \cong H_{m,r,k}$ .*

**Proof.** Let  $G$  be a  $k$ -uniform hypercactus with maximum  $\alpha$ -spectral radius among  $k$ -uniform hypercacti with  $m$  edges and  $r$  cycles.

Let  $x$  be the  $\alpha$ -Perron vector of  $G$ .

Suppose first that  $r = 0$ , i.e.,  $G$  is a hypertree with  $m$  edges. Let  $d$  be diameter of  $G$ . Obviously,  $d \geq 2$ . Suppose that  $d \geq 3$ . Let  $(u_0, e_1, u_1, \dots, e_d, u_d)$  be a diametral path of  $G$ . Choose  $u \in e_{d-1}$  with  $x_u = \max\{x_v : v \in e_{d-1}\}$ . Let  $G_1$  be the hypertree obtained from  $G$  by moving all edges (except  $e_{d-1}$ ) containing a vertex of  $e_{d-1}$  different from  $u$  from these vertices to  $u$ . By Theorem 6, we have  $\rho_\alpha(G_1) > \rho_\alpha(G)$ , a contradiction. Thus  $d = 2$ , implying that  $G \cong S_{m,k} = H_{m,0,k}$ .

Suppose in the following that  $r \geq 1$ .

If there exists an edge  $e$  with at least three vertices of degree at least 2, then let  $e = \{v_1, \dots, v_k\}$  with  $d_G(v_i) \geq 2$  for  $i = 1, \dots, \ell$ , and  $d_G(v_i) = 1$  for  $i = \ell + 1, \dots, k$ , where  $3 \leq \ell \leq k$ . Let  $G'$  be the hypergraph obtained from  $G$  by moving all edges containing  $v_3, \dots, v_\ell$  except  $e$  from  $v_3, \dots, v_\ell$  to  $v_1$ . Obviously,  $G'$  is a  $k$ -uniform hypercactus with  $m$  edges and  $r$  cycles. By Theorem 9,  $\rho_\alpha(G') > \rho_\alpha(G)$ , a contradiction. Thus, every edge in  $G$  has  $k - 2$  vertices of degree 1.

Suppose that there exist two vertex-disjoint cycles. We choose two such cycles  $C_1$  and  $C_2$  by requiring that  $d_G(C_1, C_2)$  is as small as possible, where  $d_G(C_1, C_2) = \min\{d_G(u, v) : u \in V(C_1), v \in V(C_2)\}$ . Let  $u \in V(C_1)$  and  $v \in V(C_2)$  with  $d_G(C_1, C_2) = d_G(u, v)$ . We may assume that  $x_u \geq x_v$ . Let  $G''$  be the hypergraph obtained from  $G$  by moving edges containing  $v$  in  $C_2$  from  $v$  to  $u$ . Obviously,  $G''$  is a  $k$ -uniform hypercactus with  $m$  edges and  $r$  cycles. By Theorem 6,  $\rho_\alpha(G'') > \rho_\alpha(G)$ , a contradiction. Thus, if  $r \geq 2$ , then all cycles in  $G$  share a common vertex, which we denote by  $w$ . If  $r = 1$ , then  $w$  is a vertex of degree 2 of the unique cycle.

Let  $(v_0, e_1, v_1, \dots, v_{\ell-1}, e_\ell, v_\ell)$  be a cycle of  $G$  of length  $\ell \geq 2$ , where  $v_0 = w$ . Suppose that  $\ell \geq 3$ . Assume that  $x_{v_0} \geq x_{v_2}$ . Let  $G^*$  be the hypergraph obtained from  $G$  by moving the edge  $e_2$  from  $v_2$  to  $v_0$ . Obviously,  $G^*$  is a  $k$ -uniform hypercactus with  $m$  edges and  $r$  cycles. By Theorem 6,  $\rho_\alpha(G^*) > \rho_\alpha(G)$ , a contradiction. Thus, every cycle of  $G$  is of length 2, and there are exactly  $m - 2r$  edges that are not on any cycle.

Suppose that  $G \not\cong H_{m,r,k}$ . Then there exists a vertex  $z$  such that  $d_G(w, z) = 2$ . Let  $z'$  be the unique vertex such that  $d_G(w, z') = d_G(z', z) = 1$ . There are two cases. First suppose that  $z'$  lies on some cycle. Let  $e_1$  and  $e_2$  be the cycle containing  $w$  and  $z'$ . Let  $H$  be the hypergraph obtained from  $G$  by moving all edges containing  $z'$  except  $e_1$  and  $e_2$  from  $z'$  to  $w$  if  $x_w \geq x_{z'}$ , and the hypergraph obtained from  $G$  by moving all edges containing  $w$  except  $e_1$  and  $e_2$  from  $w$  to  $z$  otherwise. Now suppose that  $z'$  does not lie on any cycle. Let  $e$  be the edge containing  $w$  and  $z'$ . Let  $H$  be the hypergraph obtained from  $G$  by moving all edges containing  $z'$  except  $e$  from  $z'$  to  $w$  if  $x_w \geq x_{z'}$ , and the hypergraph obtained from  $G$  by moving all edges containing  $w$  except  $e$  from  $w$  to  $z$  otherwise. In either case,  $H$  is a  $k$ -uniform hypercactus with  $m$  edges and  $r$  cycles. By Theorem 6,  $\rho_\alpha(H) > \rho_\alpha(G)$ , a contradiction. It follows that  $G \cong H_{m,r,k}$ . ■

**Corollary 11.** *Suppose that  $k \geq 2$ .*

- (i) *If  $G$  is a  $k$ -uniform hypertree with  $m \geq 1$  edges, then  $\rho_\alpha(G) \leq \rho_\alpha(S_{m,k})$  for  $0 \leq \alpha < 1$  with equality if and only if  $G \cong S_{m,k}$ .*
- (ii) *If  $G$  is a  $k$ -uniform unicyclic hypergraphs with  $m \geq 2$  edges, then  $\rho_\alpha(G) \leq \rho_\alpha(H_{m,1,k})$  for  $0 \leq \alpha < 1$  with equality if and only if  $G \cong H_{m,1,k}$ .*

The cases when  $\alpha = 0$  in Corollary 11 (i) and (ii) have been known in [8, 2].

For  $2 \leq d \leq m$ , let  $S_{m,d,k}$  be the  $k$ -uniform hypertree obtained from the  $k$ -uniform loose path  $P_{d,k} = (v_0, e_1, v_1, \dots, v_{d-1}, e_d, v_d)$  by attaching  $m - d$  pendant edges at  $v_{\lfloor \frac{d}{2} \rfloor}$ . Obviously,  $S_{m,2,k} \cong S_{m,k}$ .

**Theorem 12.** *For  $k \geq 2$ , let  $G$  be a  $k$ -uniform hypertree with  $m$  edges and diameter  $d \geq 2$ . For  $0 \leq \alpha < 1$ , we have  $\rho_\alpha(G) \leq \rho_\alpha(S_{m,d,k})$  with equality if and only if  $G \cong S_{m,d,k}$ .*

**Proof.** It is trivial for  $d = 2$ . Suppose that  $d \geq 3$ .

Let  $G$  be a  $k$ -uniform hypertree with maximum  $\alpha$ -spectral radius among hypertrees with  $m$  edges and diameter  $d$ .

Let  $P = (v_0, e_1, v_1, \dots, e_d, v_d)$  be a diametral path of  $G$ . Let  $x$  be the  $\alpha$ -Perron vector of  $G$ .

**Claim 1.** *Every edge of  $G$  has at least  $k - 2$  vertices of degree 1.*

**Proof.** Suppose that there is at least one edge with at least three vertices of degree at least 2. Let  $f = \{u_1, \dots, u_k\}$  be such an edge. First suppose that  $f$  is not an edge on  $P$ . We may assume that  $d_G(u_1, P) = d_G(u_i, P) - 1$  for  $i = 2, \dots, k$ , where  $d_G(w, P) = \min\{d_G(w, v) : v \in V(P)\}$ . Then  $d_G(u_1) \geq 2$ . We may assume that  $d_G(u_i) \geq 2$  for  $i = 2, \dots, r$  and  $d_G(u_i) = 1$  for  $i = r + 1, \dots, k$ , where  $3 \leq r \leq k$ . Let  $G'$  be the hypertree obtained from  $G$  by moving all edges containing  $u_3, \dots, u_r$  except  $f$  from  $u_3, \dots, u_r$  to  $u_1$ . Obviously,  $G'$  is a hypertree with  $m$  edges and diameter  $d$ . By Theorem 9,  $\rho_\alpha(G') > \rho_\alpha(G)$ , a contradiction. Thus  $f$  is an edge on  $P$ , i.e.,  $f = e_i$  for some  $i$  with  $2 \leq i \leq d - 1$ . Let  $e_i \setminus \{v_{i-1}, v_i\} = \{v_{i,1}, \dots, v_{i,k-2}\}$ . We may assume that  $v_{i,1}, \dots, v_{i,s}$  are precisely those vertices with degree at least 2 among  $v_{i,1}, \dots, v_{i,k-2}$ , where  $1 \leq s \leq k - 2$ . Let  $G''$  be the hypertree obtained from  $G$  by moving all edges containing  $v_{i,1}, \dots, v_{i,s}$  except  $e_i$  from  $v_{i,1}, \dots, v_{i,s}$  to  $v_i$ . Obviously,  $G''$  is a hypertree with  $m$  edges and diameter  $d$ . By Theorem 9,  $\rho_\alpha(G'') > \rho_\alpha(G)$ , also a contradiction. It follows that all edges of  $G$  have at most two vertices of degree at least 2. Claim 1 follows.  $\square$

**Claim 2.** *Any edge not on  $P$  is a pendant edge.*

**Proof.** Suppose that  $e$  is an edge not on  $P$  and it is not a pendant edge. Then there are two vertices, say  $u$  and  $v$ , in  $e$  such that  $d_u \geq 2$  and  $d_v \geq 2$ . Suppose

without loss of generality that  $d_G(u, P) < d_G(v, P)$ . Let  $w$  be the vertex on  $P$  with  $d_G(u, P) = d_G(u, w)$ . Let  $G^*$  be the hypertree obtained from  $G$  by moving all edges containing  $v$  except  $e$  from  $v$  to  $w$  if  $x_w \geq x_v$ , and the hypertree obtained from  $G$  by moving all edges containing  $w$  (except the edge in the path connecting  $w$  and  $v$ ) from  $w$  to  $v$  otherwise. By Theorem 6,  $\rho_\alpha(G^*) > \rho_\alpha(G)$ , a contradiction. This proves Claim 2.  $\square$

**Claim 3.** *There is at most one vertex of degree greater than two in  $G$ .*

**Proof.** Suppose that there are two vertices, say  $s$  and  $t$ , on  $P$  with degree greater than two. We may assume that  $x_s \geq x_t$ . Let  $H$  be the hypertree obtained from  $G$  by moving all pendant edges containing  $t$  from  $t$  to  $s$ . By Theorem 6, we have  $\rho_\alpha(H) > \rho_\alpha(G)$ , a contradiction. Claim 3 follows.  $\square$

Combing Claims 1–3,  $G$  is a hypertree obtained from the path  $P$  by attaching  $m - d$  pendant edges at some  $v_i$  with  $1 \leq i \leq d - 1$ , and by Theorem 8, we have  $G \cong S_{m,d,k}$ .  $\blacksquare$

The above result for  $\alpha = 0$  has been proved in [25] by a relation between the 0-spectral radius of a power hypergraph and the 0-spectral radius of its graph. Recall that for  $\alpha = 0$  and  $k = 2$ , Simić and one author of this paper [23] determined the tree on  $n$  vertices and diameter  $d$  with  $k$ th largest 0-spectral radius for  $k = 1, \dots, \lfloor \frac{d}{2} \rfloor + 1$  if  $4 \leq d \leq n - 4$  and for  $k = 1, \dots, \lfloor \frac{d}{2} \rfloor$  if  $d = n - 3$ .

Suppose that  $m \geq d \geq 3$ . Let  $H$  be the hypergraph obtained from  $S_{m,d,k}$  by moving edge  $e_d$  from  $v_{d-1}$  to  $v_{\lfloor \frac{d}{2} \rfloor}$  if  $x_{v_{\lfloor \frac{d}{2} \rfloor}} \geq x_{v_{d-1}}$ , and the hypergraph obtained from  $S_{m,d,k}$  by moving edges containing  $v_{\lfloor \frac{d}{2} \rfloor}$  except  $e_{\lfloor \frac{d}{2} \rfloor + 1}$  from  $v_{\lfloor \frac{d}{2} \rfloor}$  to  $v_{d-1}$  otherwise. Obviously,  $H \cong S_{m,d-1,k}$ . By Theorem 6,  $\rho_\alpha(S_{m,d,k}) < \rho_\alpha(S_{m,d-1,k})$ . Now by Theorem 12, Corollary 11(i) follows. Moreover, if  $G$  is a  $k$ -uniform hypertree with  $m \geq 3$  edges and  $G \not\cong S_{m,k}$ ,  $\rho_\alpha(G) \leq \rho_\alpha(S_{m,3,k})$  with equality if and only if  $G \cong S_{m,3,k}$ , which has been known for  $\alpha = 0$  in [8].

For  $2 \leq t \leq m$ , let  $T_{m,t,k}$  be the  $k$ -uniform hypertree consisting of  $t$  pendant paths of almost equal lengths (i.e.,  $t - (m - t \lfloor \frac{m}{t} \rfloor)$  pendant paths of length  $\lfloor \frac{m}{t} \rfloor$  and  $m - t \lfloor \frac{m}{t} \rfloor$  pendant paths of length  $\lfloor \frac{m}{t} \rfloor + 1$ ) at a common vertex. Particularly,  $T_{m,2,k}$  is just the  $k$ -uniform loose path  $P_{m,k}$ .

**Theorem 13.** *Let  $G$  be a  $k$ -uniform hypertree with  $m$  edges and  $t \geq 2$  pendant edges. For  $0 \leq \alpha < 1$ , we have  $\rho_\alpha(G) \leq \rho_\alpha(T_{m,t,k})$  with equality if and only if  $G \cong T_{m,t,k}$ .*

**Proof.** Let  $G$  be a  $k$ -uniform hypertree with maximum  $\alpha$ -spectral radius among hypertrees with  $m$  edges and  $t$  pendant edges. Let  $x$  be the  $\alpha$ -Perron vector of  $G$ .

Suppose that there exists an edge  $e = \{u_1, \dots, u_k\}$  with at least three vertices of degree at least 2. Assume that  $d_G(u_i) \geq d_G(u_{i+1})$  for  $i = 1, \dots, k - 1$ . Let  $G'$  be the hypertree obtained from  $G$  by moving all edges containing  $u_3, \dots, u_k$  except

$e$  from these vertices to  $u_1$ . Obviously,  $G'$  is a hypertree with  $m$  edges and  $t$  pendant edges. By Theorem 9,  $\rho_\alpha(G') > \rho_\alpha(G)$ , a contradiction. It follows that each edge of  $G$  has at most two vertices of degree at least 2.

Suppose that there are two vertices, say  $u, v$  with degree greater than 2. We may assume that  $x_u \geq x_v$ . Let  $H$  be the hypertree obtained from  $G$  by moving an edge not on the path connecting  $u$  and  $v$  containing  $v$  from  $v$  to  $u$ . By Theorem 6, we have  $\rho_\alpha(H) > \rho_\alpha(G)$ , a contradiction. Thus, there is at most one vertex of degree greater than 2 in  $G$ .

If there is no vertex of degree greater than 2, then  $t = 2$ , and  $G$  is the  $k$ -uniform loose path  $P_{m,k}$ . If there is exactly one vertex of degree greater than 2, then  $t \geq 3$ ,  $G$  is a hypertree consisting of  $t$  pendant paths at a common vertex, and by Theorem 8, we have  $G \cong T_{m,t,k}$ . ■

For  $\alpha = 0$ , this is known in [26, 30].

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