

b-COLORING OF THE MYCIELSKIAN OF SOME CLASSES OF GRAPHS

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Abstract

The b-chromatic number $b(G)$ of a graph G is the maximum k for which G has a proper vertex coloring using k colors such that each color class contains at least one vertex adjacent to a vertex of every other color class. In this paper, we have mainly investigated on the b-chromatic number of the Mycielskian of regular graphs. In particular, we have obtained the exact value of the b-chromatic number of the Mycielskian of some classes of graphs. This includes a few families of regular graphs, graphs with $b(G) = 2$ and split graphs. In addition, we have found bounds for the b-chromatic number of the Mycielskian of some more families of regular graphs in terms of the b-chromatic number of their original graphs. Also we have found b-chromatic number of the generalized Mycielskian of some regular graphs.

Keywords: b-coloring, b-chromatic number, Mycielskian of graphs, regular graphs.

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1. INTRODUCTION

All graphs considered in this paper are simple, finite and undirected. Let G be a graph with vertex set V and edge set E . A b-coloring of a graph G using k colors is a proper coloring of the vertices of G using k colors in which each color class has a color dominating vertex (c.d.v.), that is, a vertex that has a neighbor in each of the other color classes. The b-chromatic number, $b(G)$ of G is the largest k such that G has a b-coloring using k colors. For a given b-coloring of a graph, a set of c.d.v.s., one from each class, is known as a color dominating system (c.d.s.) of that b-coloring. The concept of b-coloring was introduced by

Irving and Manlove [10] in analogy to the achromatic number of a graph G (the maximum number of color classes in a complete coloring of G).

It is clear from the definition of $b(G)$ that the chromatic number, $\chi(G)$ of G is the least k for which G admits a b-coloring using k colors and hence $\chi(G) \leq b(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G . Graphs for which there exists a b-coloring using k colors for every integer k such that $\chi(G) \leq k \leq b(G)$ are known as b-continuous graphs. It can be observed that not all graphs are b-continuous. For instance, it is shown in [15] that Q_3 has a b-coloring using 2 colors and 4 colors but none using 3 colors, and therefore Q_3 is not b-continuous. Hence the natural question that arises is to characterize graphs which are b-continuous. There are a few papers in this direction. Also recently there has been a survey on b-coloring of graphs. For instance, see [2–4, 6–8, 11, 12, 20]. The b-spectrum of a graph G is the set of positive integers k for which G has a b-coloring using k colors and is denoted by $S_b(G)$, that is, $S_b(G) = \{k : G \text{ has a b-coloring using } k \text{ colors}\}$. Clearly, $\{\chi(G), b(G)\} \subseteq S_b(G)$ and G is b-continuous if and only if $S_b(G) = \{\chi(G), \chi(G) + 1, \dots, b(G)\}$.

Let the vertices of a graph G be ordered as v_1, v_2, \dots, v_n such that $d(v_1) \geq d(v_2) \geq \dots \geq d(v_n)$. Then the m -degree, $m(G)$ of G is defined by $m(G) = \max\{i : d(v_i) \geq i - 1, 1 \leq i \leq n\}$. For any graph G , $b(G) \leq m(G) \leq \Delta(G) + 1$. Also for any regular graph, $m(G) = \Delta(G) + 1$.

In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski [19] developed an interesting graph construction as follows. For a graph $G = (V, E)$, the Mycielskian of G , denoted by $\mu(G)$, is the graph with vertex set $V(\mu(G)) = V \cup V' \cup \{u\}$ where $V' = \{x' : x \in V\}$ and the edge set $E(\mu(G)) = E \cup \{xy' : xy \in E\} \cup \{y'u : y' \in V'\}$. The vertex x' is called the twin of the vertex x (and x the twin of x') and the vertex u is known as the root of $\mu(G)$. In $\mu(G)$, if $A \subseteq V$, let A' denotes the set of twin vertices of A in $\mu(G)$ and for every $x \in V$ and any non-negative integer i , define $N_i(x) = \{y \in V : d_G(x, y) = i\}$ where $d_G(x, y)$ is the length of the shortest path joining the vertices x and y in the graph G .

The generalized Mycielskian is defined as follows [17, 18]. Let G be a graph with vertex set $V_0 = \{v_1^0, v_2^0, \dots, v_n^0\}$ and edge set E_0 . Given an integer $m \geq 1$, the m -Mycielskian (also known as the generalized Mycielskian) of G , denoted by $\mu_m(G)$, is the graph whose vertex set is the disjoint union $V^0 \cup V^1 \cup \dots \cup V^m \cup \{u\}$, where $V^i = \{v_j^i : v_j^0 \in V^0\}$ is the i -th copy of V^0 , for $i = 1, 2, \dots, m$, and the edge set $E^0 \cup \left(\bigcup_{i=0}^{m-1} \{v_j^i v_{j'}^{i+1} : v_j^0 v_{j'}^0 \in E^0\} \right) \cup \{v_j^m u : v_j^m \in V^m\}$. For every pair $i, j \in \{0, 1, \dots, m\}$, $i \neq j$, and $s \in \{0, 1, \dots, n-1\}$, the vertices $v_s^i \in V^i$ and $v_s^j \in V^j$ are considered as twins of each other. Also if $S \subseteq V^0$, then $S^i \subseteq V^i$ denotes the twins of the vertices of S in V^i .

In this paper, we have mainly investigated on the b-chromatic number of

the Mycielskian of regular graphs. In particular, we have shown that, if G is a k -regular graph ($k \geq 3$) with girth at least 7 or with girth 5 whose diameter is at least 5 and which contain no C_6 , then $b(\mu(G)) = 2k + 1 = 2b(G) - 1$. Further, if G is a k -regular graph with girth 6, we have shown that $k + \lfloor \frac{k+1}{2} \rfloor \leq b(\mu(G)) \leq 2k + 1$. In addition, we have proved that if G is a k -regular graph with girth at least 8, then $\mu(G)$ is b-continuous. Also, we have found the b-chromatic number of the Mycielskian of split graphs and graphs G with $b(G) = 2$. Finally, we have determined on the b-chromatic number of the generalized Mycielskian of some families of regular graphs.

For notation and terminologies not mentioned in this paper, see [21].

2. b-COLORING OF THE MYCIELSKIAN OF REGULAR GRAPHS

In [1], it has been shown that if G is a graph with b-chromatic number b and for which the number of vertices of degree at least b is at most $2b - 2$, then $b(\mu(G))$ lies in the interval $[b + 1, 2b - 1]$. While considering regular graphs G , in [13, 14] it has been shown that $b(G) = \Delta(G) + 1$, when the girth of G is at least 6 or when the girth is at least 5 with no induced C_6 . For these regular graphs, the number of vertices of degree at least b is 0 and hence $b(\mu(G))$ lies in the interval $[b + 1, 2b - 1]$. What we intend to do in Section 2 is to find the exact value of $b(\mu(G))$ or at least find some better bounds for these families of regular graphs. Also, we would like to investigate on the Mycielskian of k -regular graphs which are b-continuous.

The following are the notations that will be used throughout Section 2.

Let G be a k -regular graph. For $v, w \in V$:

- (i) $N_1(v) = \{v_1, v_2, \dots, v_k\}$ and $N_1(w) = \{w_1, w_2, \dots, w_k\}$.
- (ii) For $1 \leq i \leq k$, let $M(v_i) = \{v_{i,1}, v_{i,2}, \dots, v_{i,k-1}\}$ denote the neighbors of v_i other than v in G . Similarly, for $1 \leq i \leq k$, let $M(w_i) = \{w_{i,1}, w_{i,2}, \dots, w_{i,k-1}\}$ denote the neighbors of w_i other than w in G .
- (iii) For $1 \leq i \leq k$, and $1 \leq j \leq k - 1$, let $M(v_{i,j}) = \{v_{i,j,1}, v_{i,j,2}, \dots, v_{i,j,k-1}\}$ be the neighbors of $v_{i,j}$ other than v_i in G . Similarly, for $1 \leq i \leq k$, and $1 \leq j \leq k - 1$, $M(w_{i,j})$ is defined.

Let us start with the following observations on k -regular graphs with girth at least 7.

Observation 2.1. *Let G be a k -regular graph with girth at least 7. For $v \in V$, we have the following.*

- (i) $N_1(v)$ and $N_2(v)$ are independent sets.
- (ii) For $y, z \in N_2(v)$, $[N_1(y) \cap N_1(z)] \cap N_3(v) = \emptyset$ and there exists at most one edge between $N_1(y)$ and $N_1(z)$ (otherwise, we will get a C_6 or a C_4).

(iii) For $w \in N_1(v)$ and $x \in N_3(v)$, there exists at most one edge between x and $N_2(w)$.

Theorem 2.2. For $k \geq 3$, if G is a k -regular graph with girth at least 7, then $b(\mu(G)) = 2k + 1 = 2b(G) - 1$.

Proof. Let G be a k -regular graph with girth at least 7 and $k \geq 3$. It can be easily seen that $m(\mu(G)) = 2k + 1$. Hence it is enough to produce a b-coloring using $2k + 1$ colors. Let $\{0, 1, 2, \dots, 2k\}$ be the set of $2k + 1$ colors. Let $v \in V$.

Let us first partially color the graph to get c.d.vs. for each of the color classes. This is done by defining a coloring c for $\mu(G)$ as follows.

- (i) $c(u) = k$, $c(v) = 0$, $c(v') = 2k$, $c(v_{1,1}) = 2k$.
- (ii) For $1 \leq i \leq k$

$$c(v_i) = i,$$

$$c(v'_i) = k + i.$$
- (iii) For $2 \leq i \leq k - 1$, $1 \leq j \leq k - 1$

$$c(v_{i,j}) = \begin{cases} j & \text{for } i > j, \\ j + 1 & \text{for } i \leq j, \end{cases}$$

$$c(v'_{i,j}) = k + j,$$

$$c(v_{k,j}) = k + j,$$

$$c(v'_{k,j}) = j.$$

This partial coloring makes v, v_2, v_3, \dots, v_k as c.d.vs. for the color classes $0, 2, 3, \dots, k$, respectively. We have to extend this partial coloring in such a way that we get c.d.vs. for the remaining color classes, namely $1, k + 1, k + 2, \dots, 2k$. Let us do this by making $v_{1,1}, v_{2,1}, v_{k,1}, v_{k,2}, \dots, v_{k,k-1}$ as c.d.vs. for the color classes $2k, 1, k + 1, k + 2, \dots, 2k - 1$, respectively. Now, let us divide the proof into 2 cases.

Case 1. $k \geq 4$. Let us assign the colors $\{2, 3, \dots, k\}$ to the vertices of $M(v_{1,1})$ and the colors $\{k + 2, k + 3, \dots, 2k - 1, 0\}$ to the vertices of $(M(v_{1,1}))'$ in any order. Next, let us assign the colors $\{3, 4, \dots, k, 0\}$ to the vertices of $M(v_{2,1})$ and the colors $\{k + 1, k + 3, k + 4, \dots, 2k\}$ to the vertices of $(M(v_{2,1}))'$ (in any order), in such a way that the color 0 and $2k$ are assigned to a vertex in $M(v_{2,1})$ and its twin, respectively. Note that by using (ii) of Observation 2.1, there can be at most one edge between $M(v_{1,1})$ and $M(v_{2,1})$ and thereby a possibility of an edge between two vertices with the same color. Even in such a situation, we can permute the colors given for $M(v_{1,1})$ and $(M(v_{1,1}))'$ in such a way that the given partial coloring becomes proper. The coloring c has been given in Figure 1.

Next, for $1 \leq i \leq k - 1$, let us assign the colors $\{k + 1, k + 2, \dots, 2k - 1, 0\} \setminus \{k + i\}$ to the vertices of $M(v_{k,i})$ (in any order) in G . Note that by using (i) of Observation 2.1 for the vertex v_k , for $1 \leq i, j \leq k - 1$, there will be no edge between the vertices of $M(v_{k,i})$ and $M(v_{k,j})$ and by using (ii) of Observation 2.1,

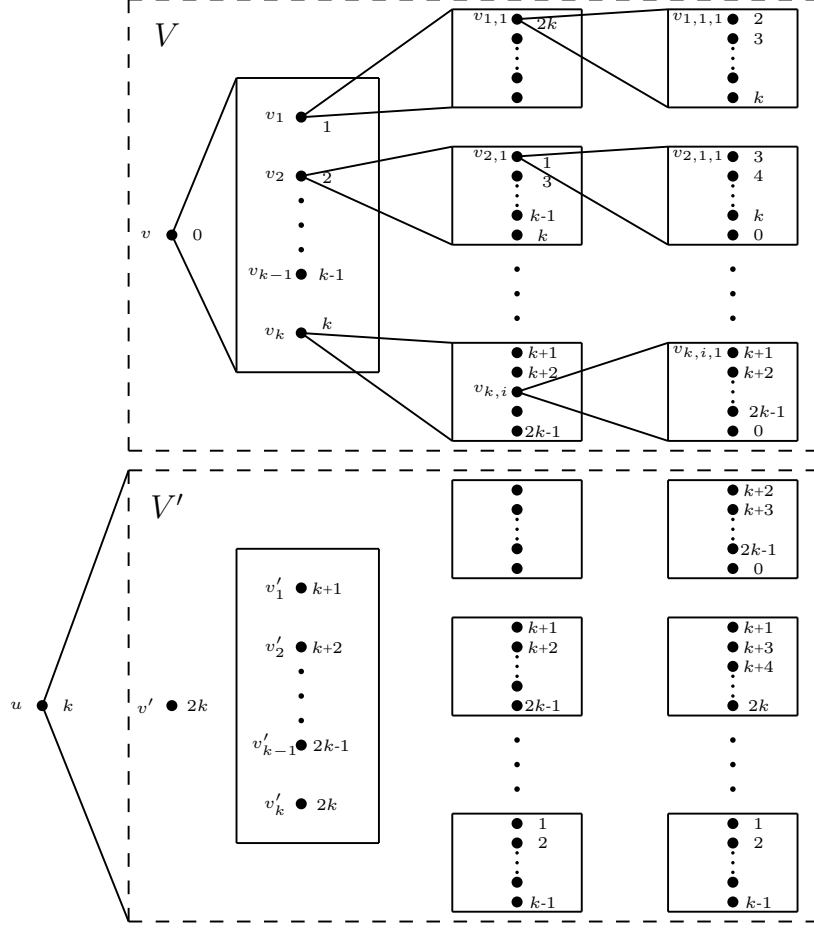


Figure 1. b-coloring of k -regular graph with girth at least 7 where $k \geq 4$.

for $1 \leq i \leq k - 1$, there exist at most one edge between $M(v_{1,1})$ and $M(v_{k,i})$ and one edge between $M(v_{2,1})$ and $M(v_{k,i})$. Even in the worst case, a vertex in $M(v_{k,i})$ can have at most 4 colored neighbors in $N_3(v)$. Namely, one in $M(v_{1,1})$ and its twin in $(M(v_{1,1}))'$ and one in $M(v_{2,1})$ and its twin in $(M(v_{2,1}))'$. But the colors given to $M(v_{1,1})$ will never create a problem while coloring $M(v_{k,i})$, $1 \leq i \leq k - 1$ and none of the colors in $M(v_{2,1})$ will create a problem except 0. Since the twin of the vertex with color 0 is given color $2k$, the color of at most one vertex from $M(v_{2,1}) \cup (M(v_{2,1}))'$ creates a problem for a vertex in $M(v_{k,i})$. Thus altogether, the color of at most 2 vertices from $M(v_{1,1}) \cup (M(v_{1,1}))' \cup M(v_{2,1}) \cup (M(v_{2,1}))'$ can create a problem for a vertex in $M(v_{k,i})$, $1 \leq i \leq k - 1$. Since $k \geq 4$, for $1 \leq i \leq k - 1$, we can permute the colors of $M(v_{k,i})$ to get a proper partial coloring.

Finally, let us assign the colors $\{1, 2, \dots, k-1\}$ to the vertices of $(M(v_{k,i}))'$ (in any order). Similar to the previous argument, for $1 \leq i \leq k-1$, every vertex in $(M(v_{k,i}))'$ has at most two neighbors in $M(v_{1,1}) \cup M(v_{2,1})$. For the same reason, since $k \geq 4$, for $1 \leq i \leq k-1$, we can permute the colors of $(M(v_{k,i}))'$ to get a proper partial coloring. This partial coloring will ensure that $v_{1,1}, v_{2,1}, v_{k,1}, v_{k,2}, \dots, v_{k,k-1}$ are c.d.vs. for the color classes $2k, 1, k+1, k+2, \dots, 2k-1$, respectively.

Case 2. $k = 3$. Let $c(v_{1,1,1}) = 5$ and $c(v_{1,1,2}) = 3$. Let us assign the colors $\{6, 3\}$ to the vertices of $M(v_{2,1})$ and the colors $\{5, 2\}$ to the vertices of $M(v_{3,1})$. By (ii) of Observation 2.1, $M(v_{1,1})$ can only be adjacent to at most one vertex in $M(v_{3,1})$ and one vertex in $M(v_{2,1})$ and hence we can permute the colors to get a proper partial coloring. Next, let us assign the colors $\{4, 1\}$, $\{0, 2\}$, $\{0, 4\}$, $\{0, 1\}$ and $\{0, 2\}$ to the vertices of $M(v_{3,2})$, $(M(v_{1,1}))'$, $(M(v_{2,1}))'$, $(M(v_{3,1}))'$ and $(M(v_{3,2}))'$, respectively. Again for the same reason, we can permute the colors to get a proper coloring. In this case also it can be seen that $v_{1,1}, v_{2,1}, v_{3,1}, v_{3,2}$ are c.d.vs. for the color classes $6, 1, 4, 5$, respectively.

In both cases, we have ensured that $v_{1,1}, v_{2,1}, v_{k,1}, v_{k,2}, \dots, v_{k,k-1}$ are c.d.vs. for the color classes $2k, 1, k+1, k+2, \dots, 2k-1$, respectively. For the remaining uncolored vertices, since the degree of each of the uncolored vertex is at most $2k$, we can apply greedy coloring to get a proper coloring for the whole $\mu(G)$ using $2k+1$ colors. ■

Let us recall the concept of System of Distinct Representatives (SDR) for a family of subsets of a given finite set. Let $\mathcal{F} = \{A_\alpha : \alpha \in J\}$ be a family of sets. An SDR for the family \mathcal{F} is a set of elements $\{x_\alpha : \alpha \in J\}$ such that $x_\alpha \in A_\alpha$ for every $\alpha \in J$ and $x_\alpha \neq x_\beta$ whenever $\alpha \neq \beta$. Theorem 2.3 gives a necessary and sufficient condition for the existence of an SDR for a given family of finite sets.

Theorem 2.3 [9]. *Let $\mathcal{F} = \{A_i : 1 \leq i \leq r\}$ be a family of finite sets. Then \mathcal{F} has an SDR if and only if the union of any k members of \mathcal{F} , $1 \leq k \leq r$, contains at least k elements.*

Let us next consider k -regular graphs with girth at least 6.

Observation 2.4. *Let G be a k -regular graph with girth at least 6. For $v \in V$, we have the following.*

- (i) $N_1(v)$ and $N_2(v)$ are independent sets.
- (ii) Any two vertices can have at most one common neighbor.

Theorem 2.5. *If G is a k -regular graph with girth at least 6, then $k + \lfloor \frac{k+1}{2} \rfloor \leq b(\mu(G)) \leq 2k+1$.*

Proof. Let G be a k -regular graph with girth at least 6. For $k = 1, 2$, the result is trivial. So let us assume that $k \geq 3$. For graphs with girth at least 7, by using Theorem 2.2, we see that $b(\mu(G)) = 2k + 1$. So let us consider G to be a regular graph with girth exactly 6. Here it can be easily seen that $m(\mu(G)) = 2k + 1$. Let $\{0, 1, 2, \dots, 2k\}$ be the set of $2k + 1$ colors. Let $v \in V$.

Let us start by defining a proper coloring using $2k + 1$ colors, in such a way that for each $i \in \{0, 1, 2, \dots, k + \lfloor \frac{k-1}{2} \rfloor\}$ there exists a vertex with color i which has a neighbor in each of the other color classes.

Let us begin by defining c as done in Theorem 2.2.

$$(i) \quad c(u) = k, \quad c(v) = 0, \quad c(v') = 2k.$$

For $1 \leq i \leq k$

$$\begin{aligned} c(v_i) &= i, \\ c(v'_i) &= k + i. \end{aligned}$$

$$(ii) \quad \text{For } 1 \leq i \leq k - 1, 1 \leq j \leq k - 1$$

$$\begin{aligned} c(v'_{i,j}) &= k + j, \\ c(v_{k,j}) &= k + j, \\ c(v'_{k,j}) &= j. \end{aligned}$$

$$(iii) \quad \text{For } 1 \leq i \leq \lfloor \frac{k-1}{2} \rfloor, 1 \leq j \leq k - 1$$

$$c(v_{k,i,j}) = \begin{cases} j - 1 & \text{for } i > j - 1, \\ j & \text{for } i \leq j - 1. \end{cases}$$

This partial coloring is proper.

For, $1 \leq i \leq k - 1$, let $C_i = \{1, 2, \dots, k\} \setminus \{i\}$ and for $1 \leq j \leq k - 1$, let A_{ij} denote the set of colors in C_i which are not assigned to the neighbors of $v_{i,j}$. That is, $A_{ij} = C_i \setminus \{\text{set of colors given to the neighbors of } v_{i,j}\}$.

For, $1 \leq i, j \leq k - 1$, it is easy to observe that, if a color of A_{ij} is assigned to the vertex $v_{i,j}$, then the coloring is proper. Thus we shall show that for $1 \leq i, j \leq k - 1$, a color of A_{ij} is available to the vertex $v_{i,j}$ and that the vertices in $M(v_i)$ receive distinct colors.

For $1 \leq i \leq k - 1$, let $\mathcal{F}_i = \{A_{ij} : 1 \leq j \leq k - 1\}$. Considering \mathcal{F}_i as a family of finite sets, if we show that \mathcal{F}_i has an SDR, then for $1 \leq i, j \leq k - 1$, we have proved that a color of A_{ij} is available to the vertex $v_{i,j}$ and that the vertices in $M(v_i)$ receive distinct colors.

By using Theorem 2.3, it is enough to prove that, for $1 \leq i \leq k - 1$, the union of any t ($1 \leq t \leq k - 1$) members of \mathcal{F}_i contains at least t elements. Let $\mathcal{E} = \{A_{i\alpha_1}, A_{i\alpha_2}, \dots, A_{i\alpha_t}\}$ be a class of any t members of \mathcal{F}_i , $1 \leq i \leq k - 1$.

Case 1. $t \leq \lfloor \frac{k-1}{2} \rfloor$. By (ii) of Observation 2.4, every vertex in $M(v_i)$ can be adjacent to at most $\lfloor \frac{k-1}{2} \rfloor$ colored neighbors in $N_3(v)$. So for any j , $1 \leq j \leq k - 1$, $|A_{ij}| \geq \lceil \frac{k-1}{2} \rceil$. Thus $|\bigcup_{p=1}^t A_{i\alpha_p}| \geq \lfloor \frac{k-1}{2} \rfloor \geq t$.

Case 2. $t \geq \lfloor \frac{k-1}{2} \rfloor + 1$. Suppose $|\bigcup_{p=1}^t A_{i\alpha_p}| \leq t - 1 \leq k - 2$. Then there

exists at least one color say $s \in C_i$, such that $s \notin A_{i\alpha_p}$, for $1 \leq p \leq t$. Then for $1 \leq p \leq t$, v_{i,α_p} is adjacent to a vertex with the color s . But by (ii) of Observation 2.4, every vertex with color s in $N_3(v)$ has at most one neighbor in $M(v_i)$. Also there are only $\lfloor \frac{k-1}{2} \rfloor$ vertices in $N_3(v)$ with color s . This is a contradiction to the fact that for every $1 \leq p \leq t$, v_{i,α_p} is adjacent to a vertex with color s . Thus $|\bigcup_{p=1}^t A_{i\alpha_p}| \geq t$.

Thus \mathcal{F}_i has a SDR and this is true for any i such that $1 \leq i \leq k-1$. Hence the coloring c can be extended to a proper coloring including the vertices in $\bigcup_{i=1}^{k-1} M(v_i)$.

Next, for $1 \leq j \leq \lfloor \frac{k-1}{2} \rfloor$, let us color the vertices of $(M(v_{k,j}))'$ with the colors $\{j\} \cup \{k+1, k+2, \dots, k+j-1, k+j+1, \dots, 2k-1\}$ as follows. One can see that j is the only color that creates a problem while coloring the vertices of $(M(v_{k,j}))'$. Since no two vertex can have more than one common neighbor, for $1 \leq j \leq \lfloor \frac{k-1}{2} \rfloor$, there can be at most $k-2$ vertices of $(M(v_{k,j}))'$ that can be adjacent to the vertices of $N_2(v) \setminus M(v_k)$ which are colored j (the number of vertices with color j in $N_2(v)$ is $k-2$). Even in the worst case, there exists a vertex in $(M(v_{k,j}))'$ which is not adjacent to a vertex with color j and hence by assigning the color j to that vertex and the rest of the colors in any order, we can extend the coloring c to the vertices of $\bigcup_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} (M(v_{k,j}))'$. This partial coloring guarantees that $\{v, v_1, v_2, \dots, v_k, v_{k,1}, v_{k,2}, \dots, v_{k, \lfloor \frac{k-1}{2} \rfloor}\}$ are c.d.vs. of the colors $\{0, 1, 2, \dots, k + \lfloor \frac{k-1}{2} \rfloor\}$. Let us color the remaining uncolored vertices by using greedy coloring technique. For $k + \lfloor \frac{k+1}{2} \rfloor \leq q \leq 2k$, if each of the vertices with color q is non-adjacent to the vertices of some color class, then by assigning one of the available color to each of the vertices with color q , we can eliminate the color q . If not, there exists a c.d.v. for the color q . Repeat this process for every q , $k + \lfloor \frac{k+1}{2} \rfloor \leq q \leq 2k$. This will yield a b-coloring using at least $k + \lfloor \frac{k+1}{2} \rfloor$ colors. Thus $b(\mu(G)) \geq k + \lfloor \frac{k+1}{2} \rfloor$. ■

Let us next consider k -regular graphs with girth 5.

Theorem 2.6. *For $k \geq 3$, if G is a k -regular graph with girth 5, diameter at least 5 and which contains no cycle of length 6, then $b(\mu(G)) = 2k + 1$.*

Proof. Let G be a k -regular graph with girth 5, diameter at least 5 and which contains no cycle of length 6. Here also it can be seen that $m(\mu(G)) = 2k + 1$ and hence it is enough to produce a b-coloring using $2k + 1$ colors. Let $\{0, 1, 2, \dots, 2k\}$ be the set of colors.

First, let us consider the case when $\text{diam}(G) = 5$. Also, let $v, w \in V$ such that $d(v, w) = 5$.

As done in Theorem 2.2, let us first partially color the graph to get c.d.vs. for each of the color classes. Let us do this by making $v, v_1, v_2, \dots, v_{k-1}, u, w_2, w_3, \dots$,

w_k, w as c.d.vs. for the color classes $0, 1, 2, \dots, 2k$. Let us start by defining a coloring c for $\mu(G)$ as follows.

- (i) $c(u) = k, c(v) = 0, c(v') = 2k, c(w) = 2k$ and $c(w') = 0$.
- (ii) For $1 \leq i \leq k$
 - $c(v_i) = i,$
 - $c(v'_i) = k + i.$

Now, let us color the vertices of $N_2(v) \setminus N(v_k)$. For $1 \leq i \leq k-1$, let us assign the colors $C_i = \{1, 2, \dots, k\} \setminus \{i\}$ to the vertices of $M(v_i)$ by using induction on i . For $i = 1$, let us assign the colors $\{2, 3, \dots, k\}$ to the vertices of $M(v_1)$ in any order. Let s be a positive integer such that $2 \leq s \leq k-1$. Let us assume that for each ℓ such that $1 \leq \ell \leq s-1$, the vertices of $M(v_\ell)$ are assigned distinct colors from C_ℓ such that the coloring is proper. Now, let us assign the colors of C_s to the vertices of $M(v_s)$ by using SDR technique as done in Theorem 2.5.

For $1 \leq j \leq k-1$, let $A_{sj} = C_s \setminus \{\text{set of colors given to the neighbors of } v_{s,j}\}$.

Let $\mathcal{F}_s = \{A_{sj} : 1 \leq j \leq k-1\}$. By using Theorem 2.3, it is enough to prove that the union of any t ($1 \leq t \leq k-1$) members of \mathcal{F}_i contains at least t elements. Let $\mathcal{E} = \{A_{s\alpha_1}, A_{s\alpha_2}, \dots, A_{s\alpha_t}\}$ be a class of t members of \mathcal{F}_s .

Since the girth of G is 5 and G contains no cycle of length 6, each of the vertices in $M(v_s)$ has at most one neighbor in $\bigcup_{\ell=1}^{s-1} M(v_\ell)$. So $|A_{sj}| \geq k-2$. Therefore for $t \leq k-2$, the union of t members of \mathcal{F}_s contains at least $k-2$ elements. Also the number of colored neighbors of $M(v_s)$ in $\bigcup_{\ell=1}^{s-1} M(v_\ell)$ is at most $k-2$. So there exists a vertex v_{sj_0} in $M(v_s)$ which has no colored neighbor in $\bigcup_{\ell=1}^{s-1} M(v_\ell)$ and hence $|A_{sj_0}| = k-1$. Thus even when $t = k-1$, the union contains at least $k-1$ elements. Thus, for $1 \leq i \leq k-1$, the vertices of $M(v_i)$ can be properly colored with distinct colors of C_i .

Next, let us color the vertices in $N_1(w) \cup N_2(w)$. Here also one can observe that, for $1 \leq i \leq k-1$, each of the vertices in $M(w_i)$ has at most one neighbor in $N_2(v)$ and similarly each of the vertices in $M(v_i)$ has at most one neighbor in $N_2(w)$. Hence the set of vertices in $N_2(w)$ has at most $k-1$ neighbors in $N_2(v)$ that are colored k . Without loss of generality, let w_1, w_2, \dots, w_k be the neighbors of w in G such that for $1 \leq i \leq k-1$, the number of neighbors of $M(w_i)$ in $N_2(v)$ which are colored k is at least the number of neighbors of $M(w_{i+1})$ in $N_2(v)$ which are colored k . Hence $M(v_k)$ has no neighbor in $N_2(v)$ which is colored k . Now, let us extend the coloring c as follows.

- For $1 \leq i \leq k$
 - $c(w_i) = k + i - 1,$
 - $c(w'_i) = i - 1.$

For $2 \leq i \leq k$, let us assign the colors $D_i = \{k, k+1, \dots, 2k-1\} \setminus \{k+i-1\}$ to the vertices of $M(w_i)$ by induction on i . For $i = 2$, let us assign the colors $\{k, k+2, k+3, \dots, 2k-1\}$ to the vertices of $M(w_2)$ as follows. Since the number

of neighbors of $M(w_1)$ in $N_2(v)$ which are colored k is the maximum, $M(w_2)$ will have at least a vertex which has no neighbor in $N_2(v)$ which is colored k and hence color k can be assigned to that vertex and the remaining colors can be assigned in any order to the remaining vertices of $M(w_2)$. Let r be a positive integer such that $3 \leq r \leq k-1$. Let us assume that for each ℓ such that $2 \leq \ell \leq r-1$, the vertices of $M(w_\ell)$ are assigned distinct colors from D_ℓ such that the coloring is proper. Now, let us assign the colors of D_r to the vertices of $M(w_r)$ by using SDR technique as done in Theorem 2.5.

For $1 \leq j \leq k-1$, let $B_{rj} = D_r \setminus \{\text{set of colors given to the neighbors of } w_{r,j}\}$.

Let $\mathcal{H}_r = \{B_{rj} : 1 \leq j \leq k-1\}$. By using Theorem 2.3, it is enough to prove that the union of any t ($1 \leq t \leq k-1$) members of \mathcal{H}_r contains at least t elements. Let $\mathcal{S} = \{B_{r\beta_1}, B_{r\beta_2}, \dots, B_{r\beta_t}\}$ be a class of any t members of \mathcal{H}_r .

Since the girth of G is 5 and G contains no cycle of length 6, each of the vertices in $M(w_r)$ has at most one colored neighbor in $\bigcup_{\ell=2}^{r-1} M(w_\ell)$ and has at most one colored neighbor in $N_2(v)$. So $|B_{rj}| \geq k-3$. Therefore the union of any $t \leq k-3$ members of \mathcal{H}_r contains at least $k-3$ elements. Also the number of colored neighbors of $M(w_r)$ in $\bigcup_{\ell=2}^{r-1} M(w_\ell)$ is at most $k-3$. So there exist two vertices w_{rj_1}, w_{rj_2} in $M(w_r)$ which have no colored neighbor in $\bigcup_{\ell=2}^{r-1} M(w_\ell)$ and hence $|B_{rj_1}| \geq k-2$ and $|B_{rj_2}| \geq k-2$. Hence the union of any $k-2$ members of \mathcal{H}_r will also contain at least $k-2$ elements.

Suppose the union of all the $k-1$ members of \mathcal{H}_r contains only $k-2$ elements. Then all the vertices of $M(w_r)$ have distinct neighbors with some particular color. Since $r-1 \leq k-2$, the only possibility for this color is k . Also $M(w_r)$ has at most $r-2$ vertices that have neighbors in $N_2(w)$ which are colored k . Depending on r , let us consider two cases.

Case 1. $r \leq \lceil \frac{k-1}{2} \rceil$. Then $M(w_r)$ has at least $(k-1) - (r-2) \geq (k-1) - (\lceil \frac{k-1}{2} \rceil - 2) = \lfloor \frac{k-1}{2} \rfloor + 2$ neighbors in $N_2(v)$ with color k . We know that, for $1 \leq j \leq r$, the number of neighbors of $M(w_j)$ in $N_2(v)$ with color k is at least the number of neighbors of $M(w_r)$ in $N_2(v)$ with color k . Since $r \geq 3$, the number of neighbors of $M_2(w)$ in $N_2(v)$ with color k is at least $3(\lfloor \frac{k-1}{2} \rfloor + 2) > k-1$, a contradiction.

Case 2. $r \geq \lceil \frac{k-1}{2} \rceil + 1$. Then $M(w_r)$ has at least $(k-1) - (r-2) \geq r - (r-2) = 2$ neighbors in $N_2(v)$ which are colored k . For the same reason as mentioned in Case 1, for $1 \leq j \leq r$, the number of neighbors of $M(w_j)$ in $N_2(v)$ with color k is at least 2 and hence $k-1 \geq 2r \geq 2(\lceil \frac{k-1}{2} \rceil + 1)$, a contradiction.

Finally, for $M(w_k)$, since none of the vertices in $M(w_k)$ has a neighbor in $N_2(v)$ which is colored k , argument similar to those given for coloring $M(v_{k-1})$ will also work in coloring the vertices of $M(w_k)$ with distinct color of D_k . Therefore for $2 \leq i \leq k$, we can assign the colors $\{k, k+1, \dots, 2k-1\} \setminus \{k+i-1\}$ properly to the vertices of $M(w_i)$.

Also the SDR technique will ensures that, for $1 \leq i \leq k-1$ and $2 \leq j \leq k$, we can assign the colors $\{k+1, k+2, \dots, 2k-1\}$ and $\{1, 2, \dots, k-1\}$ to the vertices of $(M(v_i))'$ and $(M(w_j))'$, respectively and still the coloring is proper. This partial coloring ensures that $v, v_1, v_2, \dots, v_{k-1}, u, w_2, w_3, \dots, w_k, w$ are the c.d.vs. for the color classes $0, 1, 2, \dots, 2k$, respectively. By using greedy coloring technique, the remaining uncolored vertices can be given a proper coloring using $2k+1$ colors. When the $\text{diam}(G) \geq 6$, it can be easily seen that none of the vertices in $N_2(v)$ can have a neighbor in $N_2(w)$ and hence a similar coloring will still yield a b-coloring using $2k+1$ colors. ■

Let us next find the b-spectrum of Mycielskian of k -regular graph with girth at least 7.

Theorem 2.7. *If G is a k -regular graph with girth at least 7, then $\{k+3, k+4, \dots, 2k\} \subseteq S_b(\mu(G))$.*

Proof. Let G be a k -regular graph with girth at least 7. Let $s \in \{k+3, k+4, \dots, 2k\}$ and $\{0, 1, 2, \dots, s-1\}$ be the set of colors. Let us now define a b-coloring c for $\mu(G)$ using s colors as follows. Let $v \in V(G)$.

- (i) $c(u) = k, c(v) = 0$.
- (ii) For $1 \leq i \leq k$

$$c(v_i) = i.$$
- (iii) For $1 \leq i \leq s-k-1$

$$c(v'_i) = k+i.$$
- (iv) For $1 \leq i \leq k-1, 1 \leq j \leq k-1$

$$c(v_{i,j}) = \begin{cases} j & \text{for } i > j, \\ j+1 & \text{for } i \leq j, \end{cases}$$

$$c(v'_{k,j}) = j.$$
- (v) For $1 \leq i \leq k-1, 1 \leq j \leq s-k-1$

$$c(v'_{i,j}) = k+j,$$

$$c(v_{k,j}) = k+j.$$
- (vi) For $1 \leq i \leq s-k-1, 2 \leq j \leq s-k-1$

$$c(v_{k,i,1}) = 0,$$

$$c(v_{k,i,j}) = \begin{cases} k+j-1 & \text{for } i \geq j, \\ k+j & \text{for } i < j. \end{cases}$$
- (vii) For $1 \leq i \leq s-k-1, 1 \leq j \leq k-1$

$$c(v'_{k,i,j}) = j.$$

Since girth of G is at least 7, the sets $N_1(v)$, $N_2(v)$ and $N_2(v_k) \cap N_3(v)$ are independent. Also for $1 \leq i \leq 3$, every vertex in $N_i(v)$ will have exactly one neighbor in $N_{i-1}(v)$. This guarantees that the given partial coloring c is proper and that the vertices $v, v_1, v_2, \dots, v_k, v_{k,1}, v_{k,2}, \dots, v_{k,s-k-1}$ are c.d.vs. for the color classes $0, 1, 2, \dots, s-1$, respectively.

Next, let us color the remaining uncolored vertices of V . Let w be an uncolored vertex in V . Note that an uncolored vertex in V can have at most k colored neighbors in V . Let us consider the number of colored neighbors of w in V' . Clearly $w \notin N_1(v)$. Let us assume that $w \in N_2(v)$. Since $N_2(v)$ is independent and no neighbors of w in $N_3(v)$ are colored, there can be at most one colored neighbor of w in $(N_1(v))'$ and hence in this case, there is at most 1 colored neighbor of w in V' . Next, let us assume that $w \in N_3(v)$. Recall that, every vertex in $N_3(v)$ has at most one neighbor in $N_2(v)$ and hence one colored neighbor in $(N_2(v))'$. In $N_3(v)$, by using (iii) of Observation 2.1, w can have at most one neighbor in $N_2(v_k) \cap N_3(v)$ and hence at most one colored neighbor in $(N_3(v))'$. Hence in this case, there are at most 2 colored neighbors of w in V' . When $w \in N_4(v)$, it is easy to observe that the number of colored neighbors in $(N_3(v))'$ is at most 1 and when $w \in N_i(v)$, $i \geq 5$, w has no colored neighbors in V' . Therefore, for any uncolored vertex in V , the number of colored neighbors in V is at most k and in V' is at most 2 and hence is at most $k+2$ in $\mu(G)$. Since $s \geq k+3$, we always have an available color for all the uncolored vertices of V . Finally, since the degree of any vertex in V' is $k+1$, we can extend this partial coloring c to a b-coloring of the whole graph $\mu(G)$ using s colors. Therefore, $\{k+3, k+4, \dots, 2k\} \subseteq S_b(\mu(G))$. ■

Let us recall a sufficient condition for the b-continuity of regular graphs given in [4].

Theorem 2.8 [4]. *If G is a k -regular graph with girth at least 6 having no cycles of length 7, then G is b-continuous.*

As a consequence of Theorem 2.7 and Theorem 2.8, we see that the Mycielskian of all k -regular graphs with girth at least 8 are b-continuous.

Theorem 2.9. *If G is a k -regular graph with girth at least 8, then $\mu(G)$ is b-continuous.*

Proof. Let G be a k -regular graph with girth at least 8. By using Theorem 2.8, $S_b(G) = \{\chi(G), \chi(G)+1, \dots, b(G) = k+1\}$ and hence for every $\ell \in S_b(G)$, there exists a b-coloring for G using ℓ colors. This can be extended to a b-coloring for $\mu(G)$ using $\ell+1$ colors by coloring each of the twin vertex with the color of its corresponding vertex and by coloring the root vertex with $\ell+1$.

Hence $\{\chi(\mu(G)) = \chi(G)+1, \chi(G)+2, \dots, b(G)+1 = k+2\} \subseteq S_b(\mu(G))$. Also by using Theorem 2.2 and Theorem 2.7, we see that $\{k+3, k+4, \dots, 2k, 2k+1 = b(\mu(G))\} \subseteq S_b(\mu(G))$ and hence $\mu(G)$ is b-continuous. ■

3. EXACT VALUE OF $b(\mu(G))$ FOR SOME FAMILIES OF GRAPHS

In [1], it has been shown that the b-chromatic number of the Mycielskian of split graph and $K_{n,n}$ minus a perfect matching of the graph lies in the interval $[b+1, 2b-1]$. In Section 3, we find the exact values of $b(\mu(G))$ of these families of graphs. In addition, we find the exact value of $b(\mu(G))$ when $b(G) = 2$. For a vertex $v \in V$, let $\bar{N}(v) = \{w \in V : vw \notin E, w \neq v\}$.

Theorem 3.1 [16]. *Let G be bipartite and G_1, G_2, \dots, G_r be its connected components such that $|G_i| \geq 3$ for $1 \leq i \leq r$. Then $b(G) \geq 3$ if and only if*

- (i) $r = 1$ and $X \subseteq \bigcup_{v \in Y} \bar{N}(v)$ or $Y \subseteq \bigcup_{v \in X} \bar{N}(v)$ where X and Y are the bipartite classes of G_1 , or
- (ii) $r = 2$ and at least one of G_1, G_2 is not complete bipartite or
- (iii) $r \geq 3$.

Equivalently, we can say that for a bipartite graph G with connected components G_1, G_2, \dots, G_r , $b(G) = 2$ if and only if

- (i) $r = 1$ and there exist vertices $x_0 \in X$ and $y_0 \in Y$ such that $N(x_0) = Y$ and $N(y_0) = X$ where X and Y are the bipartite classes of G_1 (we denote these graphs as type (i)), or
- (ii) for $r \geq 2$
 - (a) G_1 is of type (i) and every other component is either a K_2 or a K_1 , or
 - (b) for $1 \leq i \leq r$, G_i is a complete bipartite graph (with at least one G_i such that $|G_i| \geq 2$) and at least $r-2$ components being K_2 or K_1 .

Theorem 3.2. *If G is a graph with $b(G) = 2$, then $b(\mu(G)) = 3$.*

Proof. Let G be a graph with $b(G) = 2$. Then $3 \leq \chi(G) + 1 = \chi(\mu(G)) \leq b(\mu(G))$. Let us first assume that G is connected and let $V = X \cup Y$ where X and Y are the bipartite classes of G . Then by using (i) of the equivalent form of Theorem 3.1, there exist vertices $x_0 \in X$ and $y_0 \in Y$ such that $N(x_0) = Y$ and $N(y_0) = X$. Suppose $b(\mu(G)) = \ell \geq 4$. Then there exists a b-coloring say c of $\mu(G)$ using ℓ colors. Let $\{1, 2, \dots, \ell\}$ be the set of colors. Without loss of generality, let 1 and 2 be the colors given to x_0 and y_0 . Since the neighborhood set of any vertex in X (likewise Y) is a subset of $N(x_0)$ ($N(y_0)$), no vertices in X or Y can be c.d.vs. for any color class $m \geq 3$. So, the c.d.vs. for any color class $m \geq 3$ must be in $X' \cup Y' \cup \{u\}$.

Case 1. u is a c.d.v. Without loss of generality, let $c(u) = 3$. Since every vertex in $X' \cup Y'$ is not adjacent to either color 1 or color 2, none of the vertices in $\mu(G)$ can be a c.d.v. of any color class $m \geq 4$, a contradiction.

Case 2. u is not a c.d.v. The c.d.vs. of the color class 3 are in $X' \cup Y'$. Without loss of generality, let X' contain one of the c.d.vs. of the color class 3. Then $c(u) = 1$. Clearly $c(x'_0)$ cannot be 1 or 2. Also the neighborhood set of any vertex in X' is a subset of $N(x'_0)$. Thus, $c(x'_0) = 3$ and this in turn implies that no vertex in X' can be c.d.v. of any color class $m \geq 4$. Thus the c.d.vs. of color class 4 are only in Y' . But this is not possible, as none of the vertices in Y' is adjacent with a vertex with color 2. Thus there exists no c.d.v. for the color class 4, a contradiction.

Thus we see that when G is connected, $b(\mu(G)) \leq 3$ and hence $b(\mu(G)) = 3$. Next, let us consider the case when G is not connected. Then by using (ii) of the equivalent form of Theorem 3.1, for some $r \geq 2$, $G = \bigcup_{1 \leq i \leq r} G_i$, such that (a) G_1 is of type (i) and every other component is either a K_2 or a K_1 , or (b) for $1 \leq i \leq r$, G_i is a complete bipartite graph (with at least one G_i such that $|G_i| \geq 2$) and at least $r - 2$ components being K_2 or K_1 .

Let us consider the first possibility. Here for $2 \leq i \leq r$, the degree of the vertices in $G_i \cup G'_i$ is at most 2. Thus by using arguments similar to those given in the connected case, G will have no b-coloring using 4 colors. Thus $b(\mu(G)) \leq 3$.

Let us next consider the second possibility. Suppose $b(\mu(G)) = p \geq 4$. Then there exists a b-coloring, say ϕ of $\mu(G)$ using p colors. Let $\{1, 2, \dots, p\}$ be the set of colors. Let $\phi(u) = 1$. Since $G_j = K_2$ or K_1 , for $j \geq 3$, none of the c.d.vs. will be in $G_j \cup G'_j$. So, either $G_1 \cup G'_1$ or $G_2 \cup G'_2$ will contain at least two c.d.vs. of distinct color classes (other than 1).

Without loss of generality, let $G_1 \cup G'_1$ contain two c.d.vs. say for color classes 2 and 3. Let $G_1 = X_1 \cup Y_1$, where X_1 and Y_1 are the bipartition classes of G_1 . Let us first consider the case when the c.d.v. of either 2 or 3 is in $X_1 \cup Y_1$, say $x_1 \in X_1$ with $\phi(x_1) = 2$. Then every color other than 2 is present in $Y_1 \cup Y'_1$. In particular, 1 is present in Y_1 . This guarantees that no vertex in $Y_1 \cup (X_1 \setminus \{x_1\})$ is a c.d.v. of any color class $q \geq 3$. Since the neighbors (excluding u) of any vertex in X'_1 is a subset of $N(x_1)$, no vertex in X'_1 is a c.d.v. of a color class $m \geq 3$. Finally, no vertex in Y'_1 can have a neighbor with color $m \geq 3$ and hence cannot be a c.d.v. of a color class $m \geq 3$. Thus the c.d.vs. of 2 and 3 are in $X'_1 \cup Y'_1$. But even in this case, with similar techniques we can show that it is also not possible. Thus $b(\mu(G)) = 3$. ■

Theorem 3.3. *If G is a split graph, then $b(\mu(G)) = b(G) + 1 = \omega(G) + 1$.*

Proof. Let G be a split graph. Then $\omega(G) + 1 \leq \chi(\mu(G)) \leq b(\mu(G))$. Then the vertex set V can be partitioned into two sets, one inducing a clique and the other

inducing an independent set. Let $V = A \cup B$ where A induces a maximum clique and B is an independent set. Clearly $|A| = \omega(G)$.

Suppose $b(\mu(G)) = \ell \geq \omega(G) + 2$. Then there exists a b-coloring say c of $\mu(G)$ using ℓ colors. Let $\{1, 2, \dots, \ell\}$ be the set of colors. Without loss of generality, let $1, 2, \dots, \omega(G)$ be the colors assigned to the vertices of A . Since the degree of the vertices in B is at most $\omega(G) - 1$, none of the vertices in B' can be a c.d.v. in $\mu(G)$.

Case (i) B contains a c.d.v. Let $v \in B$ be a c.d.v. of the color, say $\omega(G) + 1$. Since A is a maximum clique, there exists at least one vertex $w \in A$ which is not adjacent to v . It can also be observed that $N(v) \subseteq N(w)$ and hence v cannot be adjacent to a vertex whose color is $c(w)$, a contradiction.

Case (ii) B contains no c.d.vs. This concludes that all the c.d.vs. must be in $A \cup A' \cup \{u\}$. Since $|A| = \omega(G)$ and $\ell \geq \omega(G) + 2$, it can be seen that A' contains at least one c.d.v., say, w'_1 of a color $\omega(G) + 1$. Then $c(u) = c(w_1)$ and hence the c.d.v. of $\omega(G) + 2$ must also be in A' , say w'_2 . This again forces $c(u) = c(w_2)$, a contradiction. ■

It can be observed that, not all k -regular graphs of girth 4 have $b(G) = k + 1$, see for instance [5]. While considering k -regular graphs with girth 4 and $b(G) = k + 1$, we shall show that these assumptions does not imply that $b(\mu(G))$ is very close to $2k + 1$.

Theorem 3.4. *If $G = K_{n,n} - PM$ where PM is a perfect matching of $K_{n,n}$, then $b(\mu(G)) = n + \lceil \frac{n-1}{2} \rceil$, for $n \geq 3$.*

Proof. Let $G = K_{n,n} - PM$ where PM is a perfect matching of $K_{n,n}$. Let $V = X \cup Y$ where $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ be the bipartition of G and $\{x_1y_1, x_2y_2, \dots, x_ny_n\}$ be the PM .

Let us first show that $b(\mu(G)) \leq n + \lceil \frac{n-1}{2} \rceil$. On the contrary, let us suppose that $b(\mu(G)) = \ell \geq n + \lceil \frac{n-1}{2} \rceil + 1$ and let c be a b-coloring using ℓ colors. Let C denote a c.d.s. of c . Without loss of generality, let $c(u) = 1$. Let us start with the following observations on c .

- (i) The c.d.vs. of c can only be present in $X \cup Y \cup \{u\}$.
- (ii) Since $\ell \geq n + \lceil \frac{n-1}{2} \rceil + 1 \geq n + 2$, there exist at least one c.d.v. in X and at least one c.d.v. in Y .
- (iii) There exists an $i \in \{1, 2, \dots, n\}$, say $i = 1$, such that $c(x_1) = c(y_1) = 1$ (Otherwise, none of the vertices in X and Y can be adjacent to the color 1 and hence there will be no c.d.v. in X or Y or both, a contradiction).

Let $S = \{i \in \{1, 2, \dots, n\} : x_i \text{ and } y_i \text{ belong to } C\}$. Suppose $|S| = p \leq \lceil \frac{n-1}{2} \rceil$. Then the number of c.d.vs. of distinct colors present in $X \cup Y$ is at most $n + \lceil \frac{n-1}{2} \rceil$.

By using observation (iii), we see that $c(x_1) = c(y_1) = c(u) = 1$ and hence the number of c.d.vs. of distinct color classes present in $X \cup Y \cup \{u\}$ is at most $n + \lceil \frac{n-1}{2} \rceil$, a contradiction. Thus $|S| \geq \lceil \frac{n-1}{2} \rceil + 1$.

For $i, j \in S$, x_i must have a neighbor with the color of x_j and vice versa. The only possibility for this to happen is that $c(y'_i) = c(x_i)$ and $c(y'_j) = c(x_j)$. Thus, for every $i \in S$, $c(y'_i) = c(x_i)$ and for similar reasons $c(x'_i) = c(y_i)$. We know that, for every $i \in S$, x_i is a c.d.v. and hence must have a neighbor whose color is $c(y_i)$. Since for $i \in S$, $c(x'_i) = c(y_i)$, the color of the vertices in $Y \setminus \{y_i\}$ cannot be $c(y_i)$ and hence one of the vertices in X' must have received the color $c(y_i)$. Thus $n = |Y'| \geq 2|S| \geq 2(\lceil \frac{n-1}{2} \rceil + 1)$, a contradiction. Hence $b(\mu(G)) \leq n + \lceil \frac{n-1}{2} \rceil$. Let us now show that we can define a b-coloring ϕ using $n + \lceil \frac{n-1}{2} \rceil$ colors as follows. Let $\{1, 2, \dots, n + \lceil \frac{n-1}{2} \rceil\}$ be the set of colors.

- (i) $\phi(u) = 1$.
- (ii) $\phi(x_i) = i$ for $1 \leq i \leq n$.
- (iii) $\phi(y_i) = \begin{cases} n + i - 1 & \text{for } 2 \leq i \leq \lceil \frac{n-1}{2} \rceil + 1, \\ i & \text{for } \lceil \frac{n-1}{2} \rceil + 2 \leq i \leq n \text{ and } i = 1. \end{cases}$
- (iv) $\phi(x'_i) = \begin{cases} 2 & \text{for } i = 1, \\ n + i - 1 & \text{for } 2 \leq i \leq \lceil \frac{n-1}{2} \rceil + 1, \\ i - \lfloor \frac{n-1}{2} \rfloor & \text{for } \lceil \frac{n-1}{2} \rceil + 2 \leq i \leq n. \end{cases}$
- (v) $\phi(y'_i) = \begin{cases} n + 1 & \text{for } i = 1, \\ i & \text{for } 2 \leq i \leq \lceil \frac{n-1}{2} \rceil + 1, \\ i + \lceil \frac{n-1}{2} \rceil & \text{for } \lceil \frac{n-1}{2} \rceil + 2 \leq i \leq n. \end{cases}$

In a routine way, one can check that the given coloring ϕ is proper and $x_1, x_2, x_3, \dots, x_n, y_2, y_3, \dots, y_{\lceil \frac{n-1}{2} \rceil + 1}$ are the c.d.vs. of the color classes $1, 2, 3, \dots, n + \lceil \frac{n-1}{2} \rceil$, respectively. ■

4. b-COLORING OF GENERALIZED MYCIELSKIAN OF SOME GRAPHS

In Section 4, we show that the results in Section 2 can be generalized to the generalized Mycielskian of regular graphs. For $m \geq 2$, while considering the generalized Mycielskian of k -regular graphs, it can be seen that the number of vertices with degree $2k$ is $(m-1)n$ and hence it can be shown that $b(\mu_m(G)) = 2k + 1$ even when G is a k -regular graph with girth at least 6.

Theorem 4.1. *For $m \geq 2$, if G is a k -regular graph with girth at least 6, then $b(\mu_m(G)) = 2k + 1$.*

Proof. Let $G = (V^0, E^0)$ be a k -regular graph with girth at least 6. Here also

$m(\mu_m(G)) = 2k + 1$ and hence it is enough to show that there exists a b-coloring using $2k + 1$ colors. Let $\{0, 1, \dots, 2k\}$ be the set of colors. Let us first partially color the graph to get c.d.vs. for each of the color classes. This is done by defining a coloring c for $\mu_m(G)$ as follows. Let $v^0 \in V^0$.

$$(i) \quad c(u) = k + 1, \quad c(v^0) = 0, \quad c(v^1) = 2k, \quad c(v^2) = k,$$

$$(ii) \quad \text{for } 1 \leq i \leq k$$

$$c(v_i^0) = i,$$

$$c(v_i^1) = k + i,$$

$$(iii) \quad \text{for } 1 \leq i \leq k - 1, 1 \leq j \leq k - 1$$

$$c(v_{i,j}^0) = \begin{cases} k + j & \text{for } i \neq j, \\ j + 1 & \text{for } i = j \text{ and } i \neq k - 1, \end{cases}$$

$$c(v_{k-1,k-1}^0) = 1,$$

$$c(v_{k,j}^0) = k + j,$$

$$c(v_{i,j}^1) = \begin{cases} k + j & \text{for } i = j, \\ k & \text{for } i = j - 1 \quad \text{or } (i, j) = (k - 1, 1), \\ j & \text{for } i \neq j, i \neq j - 1 \quad \text{and } (i, j) \neq (k - 1, 1), \end{cases}$$

$$c(v_{k,j}^1) = j,$$

$$c(v_{i,j}^2) = \begin{cases} 2k & \text{for } i = j - 1 \quad \text{or } (i, j) = (k - 1, 1), \\ j & \text{for } i \neq j - 1 \quad \text{and } (i, j) \neq (k - 1, 1), \end{cases}$$

$$c(v_{k,j}^2) = j.$$

One can easily see that the given partial coloring is proper and the vertices $v^0, v_1^0, v_2^0, \dots, v_k^0, v_1^1, v_2^1, \dots, v_k^1$ are the c.d.vs. for the color classes $0, 1, 2, \dots, 2k$, respectively. Since the degree of each of the uncolored vertex is at most $2k$, we can apply greedy coloring to get a proper coloring for the remaining vertices of $\mu_m(G)$ using $2k + 1$ colors. \blacksquare

Theorem 4.2. *If G is a k -regular graph with girth at least 7, then $\{k + 3, k + 4, \dots, 2k\} \subseteq S_b(\mu_m(G))$.*

Proof. Let $G = (V^0, E^0)$ be a k -regular graph with girth at least 7. Let $s \in \{k + 3, k + 4, \dots, 2k\}$ and $\{0, 1, 2, \dots, s - 1\}$ be the set of colors. By Theorem 2.7, the result is true for $m = 1$. So, let us assume that $m \geq 2$. While coloring the vertices of $\mu_m(G)$, we can use the same technique used in Theorem 2.7 to color the vertices of $V^0 \cup V^1 \cup \{u\}$. Now color the vertices of V^2, V^3, \dots, V^m successively. For $2 \leq p \leq m - 1$, the number of colored neighbors of any vertex in V^p is at most k and the number of colored neighbors of any vertex in V^m is at most $k + 1$. Since $s \geq k + 3$, all the vertices in V^2, V^3, \dots, V^m can be properly colored.

Therefore $\mu_m(G)$ has a b -coloring using s colors. Hence $\{k+3, k+4, \dots, 2k\} \subseteq S_b(\mu_m(G))$. ■

As a consequence of Theorem 2.8, Theorem 4.2 and by using similar technique as used in Theorem 2.9, we see that the generalized Mycielskian of all k -regular graph with girth at least 8 are b -continuous.

Corollary 4.3. *If G is a k -regular graph with girth at least 8, then $\mu_m(G)$ is b -continuous.*

In a similar way, combining the techniques used in Theorem 2.6 and Theorem 4.2, we can establish Corollary 4.4.

Corollary 4.4. *If G is a k -regular graph with girth 5, diameter at least 5 and containing no cycles of length 6, then $b(\mu_m(G)) = 2k + 1$.*

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