*Discussiones Mathematicae*
*Graph Theory xx (xxxx) 1–16*
*doi:10.7151/dmgt.2262*

# FURTHER RESULTS ON PACKING RELATED PARAMETERS IN GRAPHS

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**Abstract**

Given a graph $G = (V, E)$, a set $B \subseteq V(G)$ is a packing in $G$ if the closed neighborhoods of every pair of distinct vertices in $B$ are pairwise disjoint. The packing number $\rho(G)$ of $G$ is the maximum cardinality of a packing in $G$. Similarly, open packing sets and open packing number are defined for a graph $G$ by using open neighborhoods instead of closed ones. We give several results concerning the (open) packing number of graphs in this paper. For instance, several bounds on these packing parameters along with some Nordhaus-Gaddum inequalities are given. We characterize all graphs with equal packing and independence numbers and give the characterization of all graphs for which the packing number is equal to the independence number minus one. In addition, due to the close connection between the open packing and total domination numbers, we prove a new upper bound on the total domination number $\gamma_t(T)$ for a tree $T$ of order $n \geq 2$ improving the upper bound $\gamma_t(T) \leq (n + s)/2$ given by Chellali and Haynes in 2004, in which $s$ is the number of support vertices of $T$.

**Keywords:** packing number, open packing number, independence number, Nordhaus-Gaddum inequality, total domination number.

**2010 Mathematics Subject Classification:** 05C69.

## 1. Introduction and Preliminaries

Throughout this paper, we consider $G$ as a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. We use [23] as a reference for such terminologies and notations which are not explicitly defined here. The *open neighborhood* of a vertex $v$ is denoted by $N_G(v)$, and the *closed neighborhood* of $v$ is $N_G[v] = N_G(v) \cup \{v\}$. The *minimum* and *maximum degree* of $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Let $A$ and $B$ be two subsets of $V(G)$. The *diameter* $diam(G)$ of a graph $G$ is the largest distance between two vertices of $G$. We consider $[A, B]$ as the set of edges having one end point in $A$ and the other in $B$. A vertex $v$ in $V(G)$ is called a *private neighbor* of $u$ with respect to $S \subseteq V(G)$ if $N_G[v] \cap S = \{u\}$. The set of all private neighbors of $u$ with respect to $S$ is denoted by $pn_G(u, S)$. For a positive integer $t$, the $t$*-corona* of $G$ is the graph of order $(t+1)|V(G)|$ obtained from $G$ by attaching a path of length $t$ to each vertex of $G$ so that the resulting paths are vertex disjoint. The *corona* of two graphs $G_1$ and $G_2$ is the graph $G_1 \circ G_2$ formed from one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$, where the $i$th vertex of $G_1$ is adjacent to every vertex in the $i$th copy of $G_2$. The *bistar* $B_{r,r}$ is a graph obtained from $K_2$ by joining $r$ pendant edges to both end vertices of $K_2$. From now on, whenever it is not confusing, we will omit the subindex $G$ in all the notations defined above.

A subset $S$ of $V(G)$ is $k$*-independent* if the maximum degree of the subgraph induced by the vertices in $S$ is less than or equal to $k-1$. The $k$*-independence number* $\alpha_k(G)$ is the maximum cardinality of any $k$-independent set. Note that $\alpha_1(G) = \alpha(G)$ is the well-known *independence number* of $G$.

A set $S \subseteq V(G)$ is a *dominating set* (a *total dominating set*) if each vertex in $V(G) \backslash S$ (in $V(G)$) has at least one neighbor in $S$. The *domination number* $\gamma(G)$ (*total domination number* $\gamma_t(G)$) is the minimum cardinality of a dominating set (a total dominating set) in $G$. For more information on domination and total domination we suggest [10] and [12], respectively.

A subset $B \subseteq V(G)$ is a *packing* in $G$ if for every pair of vertices $u, v \in B$, $N[u] \cap N[v] = \emptyset$. The *packing number* $\rho(G)$ is the maximum cardinality of a packing in $G$. The *open packing*, as it is defined in [11], is a subset $B \subseteq V(G)$ for which the open neighborhoods of the vertices of $B$ are pairwise disjoint in $G$ (clearly, $B$ is an open packing if and only if $|N(v) \cap B| \leq 1$, for all $v \in V(G)$). The *open packing number*, denoted $\rho_o(G)$, is the maximum cardinality among all open packings in $G$.

Gallant *et al.* [8] introduced the concept of *limited packing* in graphs. They exhibited some real-world applications of it to network security, market saturation, NIMBY and codes. In fact, as it is defined in [8], a set of vertices $B \subseteq V(G)$ is called a $k$*-limited packing* in $G$ provided that for all $v \in V(G)$, we have $|N[v] \cap B| \leq k$. The $k$*-limited packing number*, denoted $L_k(G)$, is the largest number of vertices in a $k$-limited packing set. It is easy to see that $L_1(G) = \rho(G)$. For the sake of convenience, given any parameter $\vartheta$ in a graph $G$, a set of vertices of cardinality $\vartheta(G)$ is called a $\vartheta(G)$-set.

Once the concepts of domination and total domination were formally introduced in [17] and [5], respectively, the topics attracted the attention of a large number of researchers during the past decades, and by now, they are very well studied. On the other hand, the concepts of packing and open packing, as their dual versions respectively, are now well known topics in domination theory, although still not very well studied. Based on the inherent properties of these two pairs of dual problems, any advance in one parameter may result in an advance in the dual version. For more information about these parameters and their interactions the reader can consult [10, 11, 14, 21]. It is then our aim to study these two pairs of dual problems throughout finding several relationships between some different kinds of packing parameters and other graph parameters and/or invariants, which show the richness of the so-called packing parameters. We hence remark that some results in this work are precisely showing this richness through the several existent relationships we have found, instead of central results around which the exposition is developed.

## 2. Diameter-Related Results

A high number of results on domination theory have relationship with the diameter of graphs (see for instance [10]). In this section we exhibit tight bounds on $L_k(G)$ $(k = 1, 2)$, as a more general parameter than the standard packing number. We also bound the sum and product of the packing and open packing numbers of a graph $G$ and its complement $\overline{G}$ involving the diameter, in which is known in the literature as Nordhaus-Gaddum results. The following well-known lower bound on the domination number of a connected graph $G$ was given in [10]

$$\gamma(G) \geq \left\lceil \frac{1 + diam(G)}{3} \right\rceil. \tag{1}$$

In the next result we bound from below the $k$-limited packing numbers, for $k \in \{1, 2\}$, of a general connected graph $G$, just in terms of $k$ and its diameter.

**Proposition 1.** *For any connected graph $G$ and an integer $k \in \{1, 2\}$,*

$$L_k(G) \geq \left\lceil \frac{k + k \cdot diam(G)}{3} \right\rceil.$$

***Proof.*** Let $P$ be a diametral path in the graph $G$ formed by the set of vertices $V(P) = \{v_1, \ldots, v_{1+diam(G)}\}$. If $k = 1$, then clearly the subset of vertices $V_1(P) =$

$\{v_1, \ldots, v_{3i+1}, \ldots, v_{3\lfloor diam(G)/3\rfloor+1}\}$ is a packing in $G$. That is, if there exists a vertex $v$ adjacent to at least two vertices in $V_1(P)$, then we obtain a path between $v_1$ and $v_{1+diam(G)}$ which passes throughout $v$ and with length less than $diam(G)$, which is a contradiction. Thus, we deduce $\rho(G) \geq |V_1(P)| = \lceil(1 + diam(G))/3\rceil$.

On the other hand, if $k = 2$, then $V_2(P) = V(P) \setminus \{v_3, \ldots, v_{3\lfloor(1+diam(G))/3\rfloor}\}$ is a 2-limited packing in $G$, by a similar fashion. Therefore, $L_2(G) \geq |V_2(P)| = \lceil(2 + 2diam(G))/3\rceil$. ∎

We must recall at this point the following fact. In [13], Kang provided a counter-example to a crucial assertion in the proof of (1) given in [10], and then presented a correct proof to it. Since $\rho(G) \leq \gamma(G)$ (see [8]), Proposition 1 provides another proof of (1) and improves it, simultaneously. It is immediate from the definitions that $\rho(G) \leq \rho_o(G)$, for each graph $G$. So, Proposition 1 also improves the following theorem given in [19].

**Theorem 2** [19]**.** *For any connected graph $G$, $\rho_o(G) \geq \left\lceil \frac{1+diam(G)}{3} \right\rceil$.*

Nordhaus and Gaddum [16] in 1956, gave lower and upper bounds on the sum and product of the chromatic numbers of a graph and its complement in terms of the order. Since then, bounds on $\Psi(G)+\Psi(\overline{G})$ or $\Psi(G)\Psi(\overline{G})$ are called Nordhaus-Gaddum inequalities, where $\Psi$ is any graph parameter. For more information about this subject the reader can consult [1].

Nordhaus-Gaddum inequalities for limited packing parameters were initiated by exhibiting the sharp upper bound $L_2(G) + L_2(\overline{G}) \leq n + 2$ in [21]. Next we establish upper bounds on the sum and product of the packing and open packing numbers of a graph and its complement. To do so, we first need the following useful observation.

**Observation 3.** *For any graph $G$, $\rho(G) = 1$ if and only if $diam(G) \leq 2$.*

Clearly $\rho(G) + \rho(\overline{G}) = 2$ ($\rho(G)\rho(\overline{G}) = 1$) if and only if both $diam(G)$ and $diam(\overline{G})$ are at most 2, by Observation 3. Thus, we restrict our attention to the case $\max\{diam(G), diam(\overline{G})\} \geq 3$. First, for the sake of convenience we define $M$ and $\Delta'$ as follows.

$$M = \max\{diam(G), diam(\overline{G})\}$$

and

$$\Delta' = \begin{cases} \Delta(G) & \text{if } diam(G) > diam(\overline{G}), \\ \Delta(\overline{G}) & \text{if } diam(\overline{G}) > diam(G). \end{cases}$$

Since $diam(G) \geq 3$ results in $diam(\overline{G}) \leq 3$, it is clear that $diam(\overline{G}) = diam(G)$ only when $diam(\overline{G}) = 3 = diam(G)$. Accordingly, we consider this case separately. That is, we did not consider the case $diam(G) = diam(\overline{G})$ in the definition of $\Delta'$.

We need to recall now that a partial version of the following theorem can be found in [15], although the difference between both results are not that high. Taking into account that we further on present a total version of it (see Theorem 6), where we refer to the proof of our next theorem, and for the sake of completeness of our exposition, we point it out in detail.

**Theorem 4.** *Let $G$ and $\overline{G}$ be both connected with $M \geq 3$.*

(i) $\rho(G) + \rho(\overline{G}) = \rho(G)\rho(\overline{G}) = 4$ *if and only if* $diam(G) = diam(\overline{G}) = 3$.

(ii) *If* $diam(G) \neq diam(\overline{G})$, *then*

$$\rho(G)+\rho(\overline{G}) \leq n - \left\lceil \frac{2M + 3\Delta' - 11}{3} \right\rceil \text{ and } \rho(G)\rho(\overline{G}) \leq n - \left\lceil \frac{2M + 3\Delta' - 8}{3} \right\rceil.$$

*Furthermore, these bounds are sharp.*

***Proof.*** (i) Let $diam(G) = diam(\overline{G}) = 3$ and let $u$ and $v$ be the end vertices of a diametral path of length 3 in $G$. It is easy to see that $\{u, v\}$ is a dominating set in $\overline{G}$. Therefore, $\gamma(\overline{G}) \leq 2$. On the other hand, $\rho(\overline{G}) \leq \gamma(\overline{G})$. Now, Observation 3 implies that $\rho(\overline{G}) = 2$. A similar argument shows that $\rho(G) = 2$.

Conversely, $\rho(G) + \rho(\overline{G}) = \rho(G)\rho(\overline{G}) = 4$ implies $\rho(G) = \rho(\overline{G}) = 2$. Therefore, by Observation 3, we have $diam(G) \geq 3$ and $diam(\overline{G}) \geq 3$. On the other hand, $diam(G) \geq 3$ implies $diam(\overline{G}) \leq 3$ (see [23]). Therefore, $diam(G) = diam(\overline{G}) = 3$.

(ii) Assume $diam(G) \neq diam(\overline{G})$. Without loss of generality, we may assume that $diam(G) > diam(\overline{G})$. Since $diam(G) \geq 3$ implies $diam(\overline{G}) \leq 3$, we have $diam(G) \geq 3$ and $diam(\overline{G}) \leq 2$ since they are different. Thus, $\rho(\overline{G}) = 1$. Now, let $B$ be a maximum packing in $G$ and let $u$ be a vertex of maximum degree. Then, at most one of the vertices in $N[u]$ belongs to $B$. Let $x$ and $y$ be the end vertices of a diametral path $P$ of length $\ell(P) = diam(G) \geq 3$ in $G$. Since $diam(G[N[u]]) \leq 2$, at least one of the end vertices, say $x$, is in $G \setminus N[u]$ and at most three vertices of $P$ are in $N[u]$. Let $P_x$ and $P_y$ be the largest subpaths of $P \setminus N[u]$ beginning at $x$ and $y$, respectively. Clearly, $P_y = \emptyset$ if $y \in N[u]$. Also note that $V(P \setminus N[u])$ satisfies the following.

- It is formed by the disjoint union of $V(P_x)$ and $V(P_y)$, or
- it is formed by the disjoint union of $V(P_x)$, $V(P_y)$ and a singleton $\{z\}$, for some $z \in V(P)$.

Therefore, at most three vertices of $P$ do not belong to $P \setminus N[u]$ and so, $|V(P_x)| + |V(P_y)| \geq diam(G) - 2$.

Since $\rho(P_m) = \lceil \frac{m}{3} \rceil$ (see [8]), at most $\lceil |V(P_x)|/3 \rceil + \lceil |V(P_y)|/3 \rceil$ vertices of $P_x \cup P_y$ belong to $B$ and so, at least $\lfloor 2|V(P_x)|/3 \rfloor + \lfloor 2|V(P_y)|/3 \rfloor$ vertices of $P_x \cup P_y$ belong to $V(G) \setminus B$. Thus,

$$\begin{aligned} |V(G) \setminus B| &\geq \Delta(G) + \lfloor 2|V(P_x)|/3 \rfloor + \lfloor 2|V(P_y)|/3 \rfloor \\ &= \Delta(G) + \lceil (2|V(P_x)| - 2)/3 \rceil + \lceil (2|V(P_y)| - 2)/3 \rceil \\ &\geq \Delta(G) + \lceil 2(|V(P_x)| + |V(P_y)| - 2)/3 \rceil \\ &\geq \Delta(G) + \lceil (2diam(G) - 8)/3 \rceil. \end{aligned}$$

Therefore, $\rho(G) = |B| \leq n - \left\lceil \frac{2diam(G)+3\Delta(G)-8}{3} \right\rceil$. This implies the upper bounds.

That these bounds are sharp may be seen as follows. Let $H$ be a graph obtained from the star $K_{1,t}$, $t \geq 3$, with central vertex $u$, by adding new edges among its pendant vertices provided that there exist two non-adjacent vertices $u_1$ and $u_2$ in $N(u)$ and a vertex $w \in N(u)$ which is neither adjacent to $u_1$ nor to $u_2$. We add two paths $P'_x$ and $P'_y$ with end vertices $x, x'$ and $y, y'$, respectively, whose lengths satisfy that $\ell(P'_x) \geq \ell(P'_y)$ and $\ell(P'_x) \equiv 0 \pmod 3$. Then, we add edges between $x'$ and $u_1$, and between $y'$ and $u_2$. Hence, $\Delta(H) = t$, $diam(H) = d(x, y)$ and the three vertices $u$, $u_1$ and $u_2$ of the diametral path between $x$ and $y$ belong to $N[u]$. It is easy to see that the subset $B \subseteq V(H)$ containing one vertex of $N[u]$, say $w$, and $\lceil |V(P'_x)|/3 \rceil + \lceil |V(P'_y)|/3 \rceil$ vertices of $V(P'_x) \cup V(P'_y)$ is a maximum packing in $H$. Hence, we have

$$|V(H) \setminus B| = \Delta(H) + \lceil (2|V(P'_x)| - 2)/3 \rceil + \lceil (2|V(P'_y)| - 2)/3 \rceil. \tag{2}$$

Moreover, since $\ell(P'_x) \equiv 0 \pmod 3$ and $|V(P'_x)| + |V(P'_y)| = diam(H) - 2$, we deduce

$$\begin{aligned} |V(H) \setminus B| &= \Delta(H) + \lceil 2(|V(P'_x)| + |V(P'_y)| - 2)/3 \rceil \\ &= \Delta(H) + \lceil (2diam(H) - 8)/3 \rceil, \end{aligned}$$

by (2). Hence, $|B| = n - \left\lceil \frac{2diam(H)+3\Delta(G)-8}{3} \right\rceil$. Taking into account this, the sharpness of the upper bounds follows from the fact that $\rho(\overline{H}) = 1$. ■

A result somehow similar to Theorem 4 (but different in details) can be proved in connection with the open packing number. To this end, we first need the following lemma.

**Lemma 5** [18]. *Let $G$ be a graph of order at least 3. Then $\rho_o(G) = 1$ if and only if $diam(G) \leq 2$ and every edge of $G$ lies on a triangle.*

Before presenting the next theorem, we have the following straightforward fact. If $diam(G) \leq 2$, then $\rho_o(G) \leq 2$. Now, as an immediate consequence we have $\rho_o(G) + \rho_o(\overline{G}), \rho_o(G)\rho_o(\overline{G}) \leq 4$ when $M \leq 2$. In this sense, from now on we are only interested in the case $M \geq 3$.

**Theorem 6.** *Let $G$ and $\overline{G}$ be both connected with $M \geq 3$.*

(i) *If $diam(G) = diam(\overline{G}) = 3$, then $\rho_o(G) + \rho_o(\overline{G}) = \rho_o(G)\rho_o(\overline{G}) = 4$.*

(ii) *If $diam(G) \neq diam(\overline{G})$, then*

$$\rho_o(G) + \rho_o(\overline{G}) \leq n - \left\lceil \frac{M + 2\Delta' - 10}{2} \right\rceil$$

*and*

$$\rho_o(G)\rho_o(\overline{G}) \leq 2n - 2\left\lceil \frac{M + 2\Delta' - 6}{2} \right\rceil.$$

*Furthermore, these bounds are sharp.*

***Proof.*** (i) Let $diam(G) = diam(\overline{G}) = 3$ and let $u, v$ be the end vertices of a diametral path in $G$ (in $\overline{G}$) of length 3. It is easy to see that $\{u, v\}$ is a total dominating set in $\overline{G}$ (in $G$). Therefore, $\gamma_t(\overline{G}) \leq 2$ ($\gamma_t(G) \leq 2$). On the other hand, $\rho_o(\overline{G}) \leq \gamma_t(\overline{G}) \leq 2$ ($\rho_o(G) \leq \gamma_t(G) \leq 2$). Thus, $\rho_o(G) + \rho_o(\overline{G}) \leq 4$ and $\rho_o(G)\rho_o(\overline{G}) \leq 4$. Since $\rho(G) \leq \rho_o(G)$ for all graph $G$, Theorem 4 implies $\rho_o(G) + \rho_o(\overline{G}) = \rho_o(G)\rho_o(\overline{G}) = 4$.

(ii) Let $diam(G) \neq diam(\overline{G})$. Similarly to the proof of Theorem 4, we may assume that $diam(G) \geq 3$ and $diam(\overline{G}) \leq 2$. Thus, $\rho_o(\overline{G}) \leq 2$. Now, let $B$ be a maximum open packing in $G$ and consider $u, x, y, P, P_x$ and $P_y$ defined in the same way as in the proof of Theorem 4. Note that at most two vertices in $N[u]$ belong to $B$. Since

$$\rho_o(P_n) = \gamma_t(P_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor \leq \lfloor n/2 \rfloor + 1 \tag{3}$$

(see [7]), at most $\lfloor |V(P_x)|/2 \rfloor + \lfloor |V(P_y)|/2 \rfloor + 2$ vertices of $P_x \cup P_y$ belong to $B$ and so, at least $\lceil |V(P_x)|/2 \rceil + \lceil |V(P_y)|/2 \rceil - 2$ vertices of $P_x \cup P_y$ belong to $V(G) \setminus B$. In order to complete our proof, let us make a claim.

**Claim A.** *The following three statements do not hold simultaneously.*

(a) $|V(P_x) \cap B| = \lfloor |V(P_x)|/2 \rfloor + 1$ *and* $|V(P_y) \cap B| = \lfloor |V(P_y)|/2 \rfloor + 1$,

(b) $\lceil |V(P_x)|/2 \rceil + \lceil |V(P_y)|/2 \rceil = \lceil (|V(P_x)| + |V(P_y)|)/2 \rceil$,

(c) $|N[u] \cap B| = 2$.

***Proof.*** Suppose, to the contrary, that (a), (b) and (c) hold simultaneously. It follows from (a) and the inequality (3) that the cardinalities $|V(P_x)|$ and $|V(P_y)|$ are congruent to 1, 2 or 3 (mod 4). Also, (b) implies that at least one of $|V(P_x)|$ and $|V(P_y)|$, say $|V(P_x)|$, is an even number. Thus, $|V(P_x)| = 4k + 2$ for some integer $k \geq 0$. Since $|V(P_x) \cap B| = \lfloor |V(P_x)|/2 \rfloor + 1 = 2k + 2$, we note that the first two and the last two vertices of $P_x$ must be in $B$. On the other hand, $|N[u] \cap B| = 2$ implies that $u \in B$. Assume now that $z$ is a vertex in $N(u)$ which is adjacent to the end vertex $x'$ of $P_x$. Hence $\{u, x'\} \subseteq N(z) \cap B$, which is a contradiction. This completes the proof of Claim A. $\square$

The argument before Claim A yields the following.

$$\begin{aligned} |V(G) \setminus B| &\geq |N[u]| - 2 + \lceil |V(P_x)|/2 \rceil + \lceil |V(P_y)|/2 \rceil - 2 \\ &\geq \Delta(G) - 3 + \lceil (|V(P_x)| + |V(P_y)|)/2 \rceil \\ &\geq \Delta(G) - 3 + \lceil (diam(G) - 2)/2 \rceil \\ &= \lceil (2\Delta + diam(G) - 8)/2 \rceil. \end{aligned} \tag{4}$$

We infer now from Claim A, that at least one of the first two inequalities in (4) is strict. Thus, $\rho_o(G) = |B| \leq n - \lceil (2\Delta + diam(G) - 8)/2 \rceil - 1$. This implies the upper bound.

To see that the bounds are sharp, consider the star $K_{1,t}$ for $t \geq 2$ with central vertex $u$. Identify the end vertex $v_4$ of the path $v_1v_2v_3v_4$ to a vertex in $N(u)$. Let $G$ be the obtained graph. Clearly, $\rho_o(G) = 4$. Moreover, $\{u, v_2\}$ is a dominating set in $G$. Therefore, the edge $uv_2$ does not lie on a triangle in $\overline{G}$. Now, Lemma 5 shows that $\rho_o(\overline{G}) = 2$, and the upper bounds are attained for such graph $G$. This completes the proof. $\blacksquare$

In contrast with Theorem 4(i), we observe that the converse of Theorem 6(i) does not hold. To see this, it suffices to consider $G = C_5 = \overline{G}$.

## 3. Other Packing-Related Parameters

We next continue by relating the packing and open packing numbers of graphs with other domination parameter, that is, with the Roman domination number. The concepts concerning Roman domination in graphs were formally defined by Cockayne *et al.* in [6] motivated, in some sense, by an article in *Scientific American* of Ian Stewart entitled "Defend the Roman Empire" [22]. A *Roman dominating function* on a graph $G$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ such that every vertex $u$ with $f(u) = 0$ is adjacent to a vertex $v$ with $f(v) = 2$. The *weight* of $f$ is defined as $\omega(f) = \sum_{v \in V(G)} f(v)$. The *Roman domination number*, denoted $\gamma_R(G)$, is the minimum weight of any Roman dominating function on $G$. A Roman dominating function $f$, generates three sets $V_i^f = \{v \in V(G)\colon f(v) = i\}$ for $i = 0, 1, 2$. Since these sets determine $f$ and viceversa, we can equivalently write $f = \left(V_0^f, V_1^f, V_2^f\right)$.

**Observation 7.** *Let $G$ be a graph of order $n$ and minimum degree $\delta$. Then the following statements hold.*

(i) *If $\delta \geq 2$, then $\rho(G) \leq \left\lfloor \frac{n - \gamma_R(G)}{\delta - 1} \right\rfloor$.*

(ii) *If* $\delta \geq 3$, *then* $\rho_o(G) \leq \left\lfloor \frac{n-\gamma_R(G)}{\delta-2} \right\rfloor$.

*Furthermore, these bounds are tight.*

***Proof.*** Let $B$ and $B'$ be a packing set and an open packing set of cardinality $\rho(G)$ and $\rho_o(G)$, respectively. According to the definitions, it is readily seen that the function $f = (N[B] \setminus B, V(G) \setminus N[B], B)$ and $f' = (N[B'] \setminus B', V(G) \setminus N[B'], B')$ are Roman dominating functions. Thus,

$$\gamma_R(G) \leq w(f) = 2|B| + |V(G)| - |N[B]| \leq 2|B| + n - (\delta+1)|B| = n - (\delta-1)|B|,$$

and

$$\gamma_R(G) \leq w(f') = 2|B'| + |V(G)| - |N[B']| \leq 2|B'| + n - \delta|B'| = n - (\delta-2)|B'|.$$

Since $|B| = \rho(G)$ and $|B'| = \rho_o(G)$, we deduce the bounds. The tightness of the bounds can be easily checked by considering any graph of maximum degree $n-1$ and minimum degree larger than $n/2+1$. In such a case, it clearly happens $\gamma_R(G) = 2$ and $\rho(G) = \rho_o(G) = 1$. Also, any cycle of order $3t$ satisfies $\gamma_R(C_{3t}) = 2t$ and $\rho(C_{3t}) = t$, which again shows the tightness of the first bound. ■

Other parameter closely related to the packing number is the independence number of graphs, and clearly, $\rho(G) \leq \alpha(G)$ for every graph $G$. In what follows, we characterize all graphs $G$ for which the upper bound holds with equality. For this purpose, we first construct the family $\Omega$ of graphs $G$ as follows. Let $G'$ be a graph and $K_{n_1}, \dots, K_{n_r}$ be a decomposition of it into complete subgraphs. Let $G$ be obtained from $G'$ by adding $r$ new vertices $v_1, \dots, v_r$ and joining $v_i$ to all vertices of $K_{n_i}$, for all $1 \leq i \leq r$, and some isolated vertices.

On the other hand, it follows directly from the definition that $\rho_o(G) \leq \alpha_2(G)$, for every graph $G$. We introduce the family $\Omega_o$ of all graphs $G$ constructed as follows in order to characterize all graphs for which the equality holds. Let $H' = aP_1 + bP_2 + cP_3 + dC_4$ (disjoint union) for some non-negative integers $a, b, c$ and $d$. We denote the $i$th copy of the path $P_3$ by $v_{1i}v_{2i}v_{3i}$ and the $j$th copy of the cycle $C_4$ by $u_{1j}u_{2j}u_{3j}u_{4j}u_{1j}$. Now, let $G$ be the graph obtained from $H$ by adding some edges with one end in $\{v_{3i}\}_{i=1}^{c}$ and the other end in $\{u_{3j}, u_{4j}\}_{j=1}^{d}$.

**Theorem 8.** *Let $G$ be a graph of order $n \geq 2$. Then, the following statements hold.*

(i) $\rho(G) = \alpha(G)$ *if and only if* $G \in \Omega$.

(ii) $\rho_o(G) = \alpha_2(G)$ *if and only if* $G \in \Omega_o$.

***Proof.*** (i) We prove (i) in the case of graphs with no isolated vertices since the isolated vertices belong to every maximum packing and maximum independent set.

We assume first that $G \in \Omega$. Clearly, $B = \{v_1, \dots, v_r\}$ is a packing in $G$. Thus, it suffices to prove that $B$ is an $\alpha(G)$-set. Suppose, to the contrary, that there exists an independent set $A$ in $G$ with $|A| > |B|$. So, $A = B_1 \cup A_1$ for some $\emptyset \neq B_1 \subseteq B$ and $\emptyset \neq A_1 \subseteq V(G) \setminus B$. Since $A$ is an independent set and $|A| > |B|$, the vertices in $A_1 \subseteq A$ must be chosen from $\bigcup_{v \in B \setminus B_1} pn(v, B \setminus B_1)$ and at least two vertices of $pn(u, B \setminus B_1)$, for some $u \in B \setminus B_1$, belong to $A$. This contradicts the independence of $A$. Therefore, $B$ is an $\alpha(G)$-set, implying the equality.

Conversely, let $\rho(G) = \alpha(G)$. Suppose that $B$ is a $\rho(G)$-set. So, $B$ is an $\alpha(G)$-set as well. Therefore, each vertex in $V(G) \setminus B$ has exactly one neighbor in $B$. Suppose that there is $u \in B$ for which there exist two non-adjacent vertices $x, y \in pn(u, B)$. Now, $B' = (B \setminus \{u\}) \cup \{x, y\}$ is an independent set with $|B'| > |B|$ which is a contradiction. Therefore, $pn(u, B)$ induces a clique, for all $u \in B$ and so, $G \in \Omega$.

(ii) We prove (ii) in the case of graphs with no isolated vertices and disjoint copies of $P_2$ since all isolated vertices and disjoint copies of $P_2$ belong to every maximum open packing and maximum 2-independent set. Hence, when dealing with $\Omega_o$, we have $H' = cP_3 + dC_4$.

Let $G \in \Omega_o$. Clearly, $B = V(H') \setminus \left(\left(\bigcup_{i=1}^{c} \{v_{3i}\}\right) \cup \left(\bigcup_{j=1}^{d} \{u_{3j}, u_{4j}\}\right)\right)$ is an open packing in $G$. It is not difficult to see that $B$ is an $\alpha_2(G)$-set. Therefore, $\alpha_2(G) \leq \rho_o(G)$ implying the equality.

Conversely, assume that the equality holds and $B$ is a $\rho_o(G)$-set. So, $B$ is an $\alpha_2(G)$-set as well. Therefore, every vertex in $V(G) \setminus B$ has exactly one neighbor in $B$. Now, consider an edge $uv$ with end vertices in $B$. Suppose one of the end vertices, say $u$, has at least two private neighbors, say $x$ and $y$, lying outside $B$. Then, $B' = (B \setminus \{u\}) \cup \{x, y\}$ is a 2-independent set in $G$ with $|B'| > |B|$, which is a contradiction. Therefore, each vertex in $G[B]$ has at most one private neighbor lying outside $B$. Suppose now that both the end vertices $u$ and $v$ of the edge $uv$ have such private neighbors, say $u'$ and $v'$, respectively. Then $u'$ must be adjacent to $v'$, for otherwise $(B \setminus \{u\}) \cup \{u', v'\}$ would be a 2-independent set of cardinality $|B| + 1$, a contradiction. These arguments show that $G \in \Omega_o$. ■

The following classic result was proved by Meir and Moon in 1975.

**Theorem 9** [14]. *If $T$ is a tree, then $\rho(T) = \gamma(T)$.*

We now deduce the following result due to Borowiecki (see [2]), as an immediate consequence of Part (i) of Theorem 8 and Theorem 9.

**Corollary 10** [2]. If $T$ is a tree, then $\gamma(T) = \alpha(T)$ if and only if $T = K_1$ or $T = T' \circ K_1$ for some tree $T'$.

By Theorem 8, $\rho(G) + 1 \leq \alpha(G)$ whether $G \notin \Omega$. In the next theorem, we characterize those graphs $G$ for which $\rho(G) + 1 = \alpha(G)$. For this purpose, we introduce a couple of special families of graphs.

$\Theta_1$: The family of all graphs $G$ constructed from a graph $H \in \Omega$, with the set of isolated vertices $I$, by adding a new clique $K(H)$ and some edges with one end vertex in $V(H) \setminus (\{v_1, \ldots, v_r\} \cup I)$ and the other in $V(K(H))$ such that $V(H)$ dominates $V(K(H))$.

$\Theta_2$: The family of all graphs $G$ constructed as follows. Let $H_1, \ldots, H_r$ be a sequence of graphs such that $\alpha(H_i) \leq 2$ for all $1 \leq i \leq r$, and $\alpha(H_i) = 2$ for at least one index $1 \leq i \leq r$. Add new vertices $v_1, \ldots, v_r$ and join $v_i$ to all vertices of $H_i$, for all $1 \leq i \leq r$. We now add some edges with one end vertex in $V(H_i)$ and the other in $V(H_j)$, $1 \leq i \neq j \leq r$, such that for every two non-adjacent vertices $u_i, w_i \in V(H_i)$, $\{u_i, w_i\}$ dominates at least one vertex of the other similar type of (non-adjacent) vertices in $V(H_j)$ if they exist, for $1 \leq i \neq j \leq r$. Now let $G$ be obtained as above by adding some isolated vertices.

$\Theta_3$: The family of all graphs $G$ constructed from a graph $H \in \Theta_2$, with the set of isolated vertices $I$, by adding a new clique $K(H)$ and some edges with one end vertex in $V(H) \setminus (\{v_1, \ldots, v_r\} \cup I)$ and the other one in $V(K(H))$ such that for every two non-adjacent vertices $u_i, w_i \in V(H_i)$, $\{u_i, w_i\}$ dominates all vertices of $K(H)$.

We are now in a position to present the characterization theorem above mentioned.

**Theorem 11.** *Let $G$ be a graph with no isolated vertices. Then, $\rho(G)+1 = \alpha(G)$ if and only if $G \in \Theta_1 \cup \Theta_2 \cup \Theta_3$.*

***Proof.*** Without loss of generality, we may consider just the graphs with no isolated vertices. Let $\rho(G) + 1 = \alpha(G)$ and let $B = \{v_1, \ldots, v_{|B|}\}$ be a $\rho(G)$-set. We distinguish two cases depending on $N[B]$.

*Case* 1. $N[B] = V(G)$. Since $B$ is a packing of $G$, $N(v_1), \ldots, N(v_{|B|})$ are pairwise vertex-disjoint. If $G_i = G[N(v_i)]$ is a complete graph for each $1 \leq i \leq |B|$, then $G \in \Omega$ and so, $\rho(G) = \alpha(G)$ by Theorem 8, which is not possible. Therefore, $\alpha(G_i) \geq 2$ for at least one index $1 \leq i \leq |B|$. Suppose that there exists an index $i$ for which $\alpha(G_i) \geq 3$. Let $x_1, x_2$ and $x_3$ be three independent vertices of $G_i$. It is easy to see that $B' = (B \setminus \{v_i\}) \cup \{x_1, x_2, x_3\}$ is an independent set with $\alpha(G) \geq |B'| = \rho(G) + 2$, a contradiction. So, $\alpha(G_i) \leq 2$, for all $1 \leq i \leq |B|$.

Now let $u_i$ and $w_i$ be two non-adjacent vertices of $G_i$, for some $1 \leq i \leq |B|$. Then, $B'' = (B \setminus \{v_i\}) \cup \{u_i, w_i\}$ is an independent set of the cardinality $|B''| = \alpha(G) = \rho(G) + 1$. If there exist two non-adjacent vertices $u_j$ and $w_j$ in $V(G_j)$ for which $[\{u_i, w_i\}, \{u_j, w_j\}] = \emptyset$ (note that $[\{u_i, w_i\}, \{u_j, w_j\}]$ stands for the set of edges having one endpoint in $\{u_i, w_i\}$ and other in $\{u_j, w_j\}$), then $B''' =$

$(B'' \setminus \{v_j\}) \cup \{u_j, w_j\}$ is an independent set with $|B'''| > \alpha(G)$, a contradiction. Therefore, $\{u_i, w_i\}$ dominates at least one vertex in the other similar type of (non-adjacent) vertices in $V(H_j)$, for each $1 \le j \ne i \le |B|$. Thus, $G \in \Theta_2$.

*Case* 2. $N[B] \ne V(G)$. Let $S = V(G) \setminus N[B]$. We now consider two subcases.

*Subcase* 2.1. Suppose first that $G_1, \ldots, G_{|B|}$ are complete graphs. If $G[S]$ is not complete, choose two non-adjacent vertices $x, y \in S$. Then, $B \cup \{x, y\}$ would be an independent set of cardinality $|B| + 2$ which is a contradiction. So, $G[S]$ is a clique. On the other hand, if $N(B)$ does not dominate a vertex $x \in S$, then $G$ is an element of $\Omega$. This contradicts $\rho(G) + 1 = \alpha(G)$. Therefore, $G \in \Theta_1$.

*Subcase* 2.2. Suppose that $\alpha(G_i) \ge 2$, for some $1 \le i \le |B|$. Similar to Subcase 2.1, we deduce that $G[S]$ is a clique. Also, similar to Case 1, for each two non-adjacent vertices $u_i, w_i \in V(G_i)$, $\{u_i, w_i\}$ dominates at least one vertex in the other similar type of (non-adjacent) vertices in $V(H_j)$, for $1 \le j \ne i \le r$. Therefore, $G - S \in \Theta_2$. Suppose that there are two non-adjacent vertices $u_i, w_i$ of $G_i$ such that $\{u_i, w_i\}$ does not dominate a vertex $x \in S$. Then, $B' = (B \setminus \{v_i\}) \cup \{x, u_i, w_i\}$ is an independent set in $G$ of cardinality $|B'| = \rho(G) + 2$, a contradiction. Therefore, $G \in \Theta_3$.

Conversely, let $G \in \Theta_1 \cup \Theta_2 \cup \Theta_3$. Then we have $\rho(G) + 1 \le \alpha(G)$, by Theorem 8. We now consider two cases.

*Case* 3. $G \in \Theta_1$. Applying an argument similar to that of the proof of Theorem 8, we find an $\alpha(G)$-set $\{v_1, \ldots, v_r\} \cup \{x\}$ of cardinality $r + 1$, in which $x$ is a vertex in $S$. Since $\{v_1, \ldots, v_r\}$ is a packing, $\rho(G) + 1 \ge \alpha(G)$ implying the equality.

*Case* 4. $G \in \Theta_2 \cup \Theta_3$. Choose an index $1 \le i \le r$ for which $\alpha(G_i) \ge 2$. We claim that for each two non-adjacent vertices $u_i, w_i$ of $G_i$, the independent set $B_i = (\{v_1, \ldots, v_r\} \setminus \{v_i\}) \cup \{u_i, w_i\}$ is an $\alpha(G)$-set. Otherwise, there exists an independent set $B'_i$ such that $|B'_i| > |B_i|$. It is not difficult to show that such a set must contain either at least two pairs $u_j$,$w_j$ and $u_k$,$w_k$ of non-adjacent vertices of $H_j$ and $H_k$, respectively, for some $1 \le j \ne k \le r$, or a vertex $x \in S$ and a pair of such non-adjacent vertices $u_j$ and $w_j$. In such cases, there would be at least one edge in $[\{u_j, w_j\}, \{u_k, w_k\}]$ or $[\{u_j, w_j\}, \{x\}]$, respectively. This contradicts the independence of $B'_i$. So, $B_i$ is an $\alpha(G)$-set. Thus, $\rho(G) + 1 \ge r + 1 = \alpha(G)$. Note that in the case $G \in \Theta_3$, $\{v_1, \ldots, v_r\} \cup \{x\}$ is an $\alpha(G)$-set of cardinality at most $\rho(G) + 1$ as well, in which $x$ is an arbitrary vertex in $S$. Therefore, again $\rho(G) + 1 \ge r + 1 = \alpha(G)$. This completes the proof. ∎

## 4. Packing-Related Parameters in Trees

The following inequality was independently proved by Gentner and Rautenbach [9], and Desormeaux and Henning [7].

$$\gamma_t(G) \geq \min\{k \mid d_1 + \cdots + d_k \geq n\}, \tag{5}$$

for any graph $G$ with no isolated vertices and order $n$ with the non-increasing degree sequence $d_1 \geq \cdots \geq d_n$. The authors in [7] denoted the lower bound in (5) by $ord_s(G)$ and called it the *order-sum number* of $G$. They proved that

$$ord_s(T) \geq (n - \ell + 2)/2,$$

in which $T$ is a tree of order $n \geq 2$ with $\ell$ leaves, which strengthens $\gamma_t(T) \geq (n - \ell + 2)/2$ given in [3].

The discussion above motivates us to introduce a graph parameter in order to give a new upper bound on $\gamma_t(T)$. Let $T$ be a tree of order $n \geq 2$ with $s$ support vertices and let $L(T)$ be the set of leaves of $T$. Let $d_1 \leq \cdots \leq d_p$ be the non-decreasing degree sequence of vertices in $V(T) \setminus L(T)$. We define the parameter $\widehat{\rho}_o$ for a tree $T$ as

$$\widehat{\rho}_o(T) = \max\{k \mid d_1 + \cdots + d_{k-s} \leq n - s\}.$$

We make use of the following result due to Rall [20] which will be useful for our purposes.

**Lemma 12** [20]**.** *For any tree $T$ of order at least two, $\gamma_t(T) = \rho_o(T)$.*

**Theorem 13.** *Let $T$ be a tree of order $n \geq 2$ with $s$ support vertices. Then, the following statements hold.*

(i) $\gamma_t(T) \leq \widehat{\rho}_o(T) \leq (n + s)/2$.

(ii) $\widehat{\rho}_o(T) = (n + s)/2$ *if and only if $n + s$ is even and $d_{(n-s)/2} = 2$.*

(iii) *If $T$ has the non-increasing degree sequence $d_1 \geq \cdots \geq d_n$, then $ord_s(T) \leq \gamma_t(T) \leq 2ord_s(T) - 2$ (see [7]). Moreover, all integer values between the lower and upper bounds are realizable.*

***Proof.*** (i) Let $B$ be a maximum open packing in $G$. Let $u$ be a support vertex and $L_u$ be the set of all leaves adjacent to $u$. If $L_u \cap B = \emptyset$, then there exists a vertex $w \in B \cap N(u)$, for otherwise $B \cup \{v\}$ is an open packing in $G$ in which $v \in L_u$, and this is a contradiction. Now, $(B \setminus \{w\}) \cup \{v\}$ is an open packing in $G$. So, in what follows we may assume that $B$ contains exactly one leaf adjacent to each support vertex of $T$. Let $\{v_1, \ldots, v_s\}$ be the set of such leaves and $B = \{v_1, \ldots, v_s, v_{s+1}, \ldots, v_{|B|}\}$ be the maximum open packing. By Lemma 12, we have $\gamma_t(T) = |B|$. On the other hand, the definition of open packing implies

$$\begin{aligned} s + d_1 + \cdots + d_{|B|-s} &\leq s + deg(v_{s+1}) + \cdots + \deg(v_{|B|}) \\ &= \deg(v_1) + \cdots + deg(v_{|B|}) \\ &= \sum_{i=1}^{|B|} |N(v_i) \cap B| + \sum_{i=1}^{|B|} |N(v_i) \cap (V(G) \setminus B)| \\ &\leq |B| + n - |B| = n. \end{aligned} \tag{6}$$

So, $\gamma_t(T) = |B| \leq \widehat{\rho}_o(T)$.

Now let $t = \widehat{\rho}_o(T)$. Then,

$$2(t-s) \leq d_1 + \cdots + d_{t-s} \leq n-s. \tag{7}$$

This results in the desired upper bound.

(ii) Let $\widehat{\rho}_o(T) = (n+s)/2$ and $t = \widehat{\rho}_o(T)$. Clearly, $n+s$ is even. By using $(n+s)/2$ instead of $t$ in (7), we have

$$d_1 + \cdots + d_{(n-s)/2} = n-s. \tag{8}$$

Since $2 \leq d_1 \leq \cdots \leq d_{(n-s)/2}$, the equality (8) implies $d_1 = \cdots = d_{(n-s)/2} = 2$.

Conversely, since $d_{(n-s)/2} = 2$, we get $d_1 + \cdots + d_{(n-s)/2} = n-s$. Therefore, $(n+s)/2 \leq \widehat{\rho}_o(T)$.

(iii) The lower and upper bounds were proved in [7]. To show that all integer values between the lower and upper bounds are realizable, it suffices to prove the following claim.

**Claim B.** *For any integers $a \geq 2$ and $0 \leq b \leq a-2$, there exists a tree $T$ such that $ord_s(T) = a$ and $\gamma_t(T) = a+b$.*

***Proof.*** Let $T$ be a tree obtained from $T' = P_a \circ \overline{K_{2a}}$ by subdividing exactly one pendant edge for each one of the first $b$ support vertices on $P_a$. Then, $n = 2a^2+a+b$. Let $\{v_1, \ldots, v_a\}$ be the set of vertices of the path $P_a$ and $V(T) \backslash V(T') = \{u_1, \ldots, u_b\}$. We have,

$$\sum_{i=1}^{a} \deg(v_i) = 2a^2 + 2a - 2 \geq 2a^2 + a + b = n.$$

Therefore, $ord_s(T) \leq a$.

Now let us suppose $ord_s(T) = k \leq a-1$ and let $\{w_1, \ldots, w_k\}$ be the set of vertices for which $\sum_{i=1}^{k} \deg(w_i) \geq n$. Hence,

$$\sum_{i=1}^{k} \deg(w_i) \leq \sum_{i=1}^{a-1} \deg(v_i) = 2a^2 - 3 < n,$$

which is a contradiction. Thus, $ord_s(T) = a$.

On the other hand, it is easy to check that $\{v_1, \ldots, v_a\} \cup \{u_1, \ldots, u_b\}$ is a minimum total dominating set in $T$. So, $\gamma_t(T) = a+b$. This completes the proof. ∎

Note that, as an immediate consequence of Theorem 13(i), we have improved the following result, that was proved by Chellali and Haynes by induction on the order $n$.

**Theorem 14** [4]**.** *If $T$ is a tree of order $n \geq 3$ with $s$ support vertices, then $\gamma_t(T) \leq (n+s)/2$ and this bound is sharp.*

Note that the difference between $(n+s)/2$ and $\widehat{\rho}_o(T)$ can be arbitrary large. Indeed, for any positive integer $b$ there exists a tree $T$ for which $(n+s)/2 - \widehat{\rho}_o(T) = b$. To see this, it suffices to consider the bistar $B_{b+1,b+1}$.

**Acknowledgement**

This work was partially done while the second author (B. Samadi) was visiting the University of Cadiz, Spain, supported by the University of Mazandaran and the Ministry of Science, Research and Technology of Iran, under the program Ph.D. Research Opportunity.

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Received 19 November 2018
Revised 28 August 2019
Accepted 17 October 2019