ON THE OPTIMALITY OF 3-RESTRICTED ARC CONNECTIVITY FOR DIGRAPHS AND BIPARTITE DIGRAPHS

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Abstract

Let $D$ be a strong digraph. An arc subset $S$ is a $k$-restricted arc cut of $D$ if $D - S$ has a strong component $D'$ with order at least $k$ such that $D[V(D')]$ contains a connected subdigraph with order at least $k$. If such a $k$-restricted arc cut exists in $D$, then $D$ is called $\lambda^k$-connected. For a $\lambda^k$-connected digraph $D$, the $k$-restricted arc connectivity, denoted by $\lambda^k(D)$, is the minimum cardinality over all $k$-restricted arc cuts of $D$. It is known that for many digraphs $\lambda^k(D) \leq \xi^k(D)$, where $\xi^k(D)$ denotes the minimum $k$-degree of $D$. $D$ is called $\lambda^k$-optimal if $\lambda^k(D) = \xi^k(D)$. In this paper, we will give some sufficient conditions for digraphs and bipartite digraphs to be $\lambda^3$-optimal.

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1. Introduction

It is well-known that the network can be modelled as a digraph $D$ with vertices $V(D)$ representing sites and arcs $A(D)$ representing links between sites of the network. Let $v \in V(D)$, the out-neighborhood of $v$ is the set $N^+(v) = \{x \in V(D) : vx \in A(D)\}$ and the out-degree of $v$ is $d^+(v) = |N^+(v)|$. The in-neighborhood of $v$ is the set $N^-(v) = \{x \in V(D) : xv \in A(D)\}$ and the in-degree of $v$ is $d^-(v) = |N^-(v)|$. The neighborhood of $v$ is $N(v) = N^+(v) \cup N^-(v)$.

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Let $\delta^+(D), \delta^-(D)$ and $\delta(D)$ denote, respectively, the minimum out-degree, the minimum in-degree and the minimum degree of $D$.

For a pair nonempty vertex sets $X$ and $Y$ of $D$, $[X,Y] = \{xy \in A(D) : x \in X, y \in Y\}$. Specially, if $Y = X$, where $X = V(D) \setminus X$, then we write $\partial^+(X)$ or $\partial^-(Y)$ instead of $[X,Y]$. For $X \subseteq V(D)$, the subdigraph of $D$ induced by $X$ is denoted by $D[X]$. The underlying graph $U(D)$ of $D$ is the unique graph obtained from $D$ by deleting the orientation of all arcs and keeping one edge of a pair of multiple edges. $D$ is connected if $U(D)$ is connected and $D$ is strongly connected (or, just, strong) if there exists a directed $(x,y)$-path and a directed $(y,x)$-path for any $x,y \in V(D)$. We define a digraph with one vertex to be strong. A connected (strong) component of $D$ is a maximal induced subdigraph of $D$ which is connected (strong). If $D$ has $p$ strong components, then these strong components can be labeled $D_1, \ldots, D_p$ such that there is no arc from $D_j$ to $D_i$ unless $j < i$. We call such an ordering an acyclic ordering of the strong components of $D$.

In a strong digraph $D$, we often use arc connectivity of $D$ to measure the reliability. An arc set $S$ is a $\text{arc cut}$ of $D$ if $D - S$ is not strong. The arc connectivity $\lambda(D)$ is the minimum cardinality over all arc cuts of $D$. The arc cut $S$ of $D$ with cardinality $\lambda(D)$ is called a $\text{\lambda-cut}$. Whitney’s inequality shows $\lambda(D) \leq \delta(D)$. A strong digraph $D$ with $\lambda(D) = \delta(D)$ is called $\lambda$-optimal. However, only using arc connectivity to measure the reliability is not enough. In [12], Volkmann introduced the concept of restricted arc connectivity. An arc subset $S$ of $D$ is a restricted arc cut if $D - S$ has a strong component $D'$ with order at least 2 such that $D \setminus V(D')$ contains an arc. If such an arc cut exists in $D$, then $D$ is called $\lambda'$-connected. For a $\lambda'$-connected digraph $D$, the restricted arc connectivity, denoted by $\lambda'(D)$, is the minimum cardinality over all restricted arc cuts of $D$. The restricted arc cut $S$ of $D$ with cardinality $\lambda'(D)$ is called a $\lambda'$-cut. In [13], Wang and Lin introduced the notion of minimum arc degree. Let $xy \in A(D)$. Then

$$\Omega(\{x, y\}) = \{\partial^+(\{x, y\}), \partial^-(\{x, y\}), \partial^+(\{x\}) \cup \partial^-(\{y\}), \partial^+(\{y\}) \cup \partial^-(\{x\})\}.$$ 

The arc degree of $xy$ is $\xi'(xy) = \min\{\xi(S) : S \in \Omega(\{x, y\})\}$ and the minimum arc degree of $D$ is $\xi'(D) = \min\{\xi'(xy) : xy \in A(D)\}$.

It was proved in [3, 13] that for many $\lambda'$-connected digraphs, $\xi'(D)$ is an upper bound of $\lambda'(D)$. In [13], Wang and Lin introduced the concept of $\lambda'$-optimality. A $\lambda'$-connected digraph $D$ with $\xi'(D) = \lambda'(D)$ is called $\lambda'$-optimal. As a generalization of restricted arc connectivity, in [10], Lin et al. introduced the concept of $k$-restricted arc connectivity.

**Definition** [10]. Let $D$ be a strong digraph. An arc subset $S$ is a $k$-restricted arc cut of $D$ if $D - S$ has a strong component $D'$ with order at least $k$ such that
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Let \( D \setminus V(D') \) contains a connected subdigraph with order at least \( k \). If such a \( k \)-restricted arc cut exists in \( D \), then \( D \) is called \( \lambda^k \)-connected. For a \( \lambda^k \)-connected digraph \( D \), the \( k \)-restricted arc connectivity, denoted by \( \lambda_k(D) \), is the minimum cardinality over all \( k \)-restricted arc cuts of \( D \). The \( k \)-restricted arc cut \( S \) of \( D \) with cardinality \( \lambda_k(D) \) is called a \( \lambda^k \)-cut.

**Definition** [10]. Let \( D \) be a strong digraph. For any \( X \subseteq V(D) \), let \( \Omega(X) = \{ \partial^+(X_1) \cup \partial^-(X \setminus X_1) : X_1 \subseteq X \} \) and \( \xi(X) = \min \{|S| : S \in \Omega(X)\} \). Define the minimum \( k \)-degree of \( D \) to be

\[
\xi^k(D) = \min \{ \xi(X) : X \subseteq V(D), |X| = k, D[X] \text{ is connected} \}.
\]

Clearly, \( \lambda^1(D) = \lambda(D) \), \( \lambda^2(D) = \lambda'(D) \), \( \xi^1(D) = \delta(D) \) and \( \xi^2(D) = \xi'(D) \).

Let \( D \) be a \( \lambda^k \)-connected digraph, where \( k \geq 2 \). Then \( D \) is \( \lambda^{k-1} \)-connected and \( \lambda^{k-1}(D) \leq \lambda^k(D) \). It was shown in [10] that \( \xi^k(D) \) is an upper bound of \( \lambda^k(D) \) for many digraphs. And a \( \lambda^3 \)-connected digraph \( D \) with \( \lambda^k(D) = \xi^k(D) \) is called \( \lambda^k \)-optimal.

The research on the \( \lambda^k \)-optimality of digraph \( D \) is considered to be a hot issue. In [11], Hellwig and Volkmann concluded many sufficient conditions for digraphs to be \( \lambda \)-optimal. Besides, sufficient conditions for digraphs to be \( \lambda^3 \)-optimal were also given by several authors, for example by Balbuena et al. [1–4], Chen et al. [5,6], Grüter and Guo [7,8], Liu and Zhang [9], Volkmann [12] and Wang and Lin [13]. However, closely related conditions for \( \lambda^3 \)-optimal digraphs have received little attention until recently. In [10], Lin et al. gave some sufficient conditions for digraphs to be \( \lambda^3 \)-optimal. In this paper, we will give some sufficient conditions for digraphs to be \( \lambda^3 \)-optimal. As corollaries, degree conditions or degree sum conditions for a digraph or a bipartite digraph to be \( \lambda^3 \)-optimal are given. The main contributions in this paper are as following.

**Theorem 1.** Let \( D \) be a digraph with \( |V(D)| \geq 6 \). If \( |N^+(u) \cap N^-(v)| \geq 5 \) for any \( u, v \in V(D) \) with \( uv \notin A(D) \), then \( D \) is \( \lambda^3 \)-optimal.

**Theorem 2.** Let \( D = (X, Y, A(D)) \) be a bipartite digraph with \( |V(D)| \geq 6 \). If \( |N^+(u) \cap N^-(v)| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \) for any \( u, v \in V(D) \) in the same partite, then \( D \) is \( \lambda^3 \)-optimal.

2. Proof of Theorem 1

We first introduce three useful lemmas.

**Lemma 3** (Theorem 1.4 in [10]). Let \( D \) be a strong digraph with \( \delta^+(D) \geq 2k - 1 \) or \( \delta^-(D) \geq 2k - 1 \). Then \( D \) is \( \lambda^k \)-connected and \( \lambda^k(D) \leq \xi^k(D) \).
Lemma 4. Let $D$ be a strong digraph with $\delta^+(D) \geq 2k - 1$ or $\delta^-(D) \geq 2k - 1$, and let $S = \partial^+(X)$ be a $\lambda^k$-cut of $D$, where $X$ is a subset of $V(D)$. If $D[X]$ contains a connected subdigraph $B$ with order $k$ such that $|N^+(x) \cap X| \geq k$ for any $x \in X \setminus V(B)$ or $D[\overline{X}]$ contains a connected subdigraph $C$ with order $k$ such that $|N^-(y) \cap X| \geq k$ for any $y \in \overline{X} \setminus V(C)$, then $D$ is $\lambda^k$-optimal.

Proof. By Lemma 3, $D$ is $\lambda^k$-connected and $\lambda^k(D) \leq \xi^k(D)$. By reason of symmetry, we only prove the case that $D[X]$ contains a connected subdigraph $B$ with order $k$ such that $|N^+(x) \cap X| \geq k$ for any $x \in X \setminus V(B)$. The hypotheses imply that

$$
\xi^k(D) \leq |\partial^+(V(B))| = |(V(B), X \setminus V(B))| + |(V(B), \overline{X})|
$$

$$
\leq k|X \setminus V(B)| + |(V(B), \overline{X})| \leq \sum_{x \in X \setminus V(B)} |N^+(x) \cap X| + |(V(B), \overline{X})|
$$

$$
= |(X \setminus V(B), \overline{X})| + |(V(B), \overline{X})| = |X, \overline{X}| = |S| = \lambda^k(D).
$$

Thus $\lambda^k(D) = \xi^k(D)$ and $D$ is $\lambda^k$-optimal. \hfill \blacksquare

Lemma 5 (Lemma 4.1 in [10]). Let $D$ be a strong digraph with $|V(D)| \geq 6$ and $\delta(D) \geq 4$, and let $S$ be a $\lambda^3$-cut of $D$. If $D$ is not $\lambda^3$-optimal, then there exists a subset of vertices $X \subset V(D)$ such that $S = \partial^+(X)$ and both induced subdigraphs $D[X]$ and $D[\overline{X}]$ contain a connected subdigraph with order 3.

Proof of Theorem 1. Clearly, $D$ is a strong digraph with $\delta(D) \geq 5$. By Lemma 3, $D$ is $\lambda^3$-connected and $\lambda^3(D) \leq \xi^3(D)$. Suppose, on the contrary, that $D$ is not $\lambda^3$-optimal, that is, $\lambda^3(D) < \xi^3(D)$. Let $S$ be a $\lambda^3$-cut of $D$. By Lemma 5, there exists a subset of vertices $X \subset V(D)$ such that $S = \partial^+(X)$ and both induced subdigraphs $D[X]$ and $D[\overline{X}]$ contain a connected subdigraph with order 3.

Let $Y = \overline{X}$, and let $X_i = \{x \in X : |N^+(x) \cap Y| = i\}$, $Y_i = \{y \in Y : |N^-(y) \cap X| = i\}$, $i = 0, 1, 2$, and let $X_3 = \{x \in X : |N^+(x) \cap Y| \geq 3\}$, $Y_3 = \{y \in Y : |N^-(y) \cap X| \geq 3\}$.

Claim 1. $\min\{|X|, |Y|\} \geq 4$.

Proof. Suppose that $|X| = 3$. Then $\lambda^3(D) = |S| = |\partial^+(X)| \geq \xi(X) \geq \xi^3(D)$, contrary to the assumption. Suppose that $|Y| = 3$. Then $\lambda^3(D) = |S| = |\partial^-(Y)| \geq \xi(Y) \geq \xi^3(D)$, contrary to the assumption. Claim 1 follows. \hfill \Box

Claim 2. $X_0 = Y_0 = \emptyset$.

Proof. For the reason of symmetry, we only prove that $X_0 = \emptyset$ by contradiction. Suppose $X_0 \neq \emptyset$ and let $x \in X_0$. Then for any $\overline{x} \in Y$, $x \overline{x} \notin A(D)$ and we have that $5 \leq |N^+(x) \cap N^-(\overline{x})| = |N^+(x) \cap N^-(\overline{x}) \cap X| + |N^+(x) \cap N^-(\overline{x}) \cap Y| \leq$

\[ |N^-(x) \cap X| + |N^+(x) \cap Y| = |N^-(x) \cap X|. \] It implies that \(|N^-(x) \cap X| \geq 5\). Therefore \(Y \subseteq Y_3\). So \(D\) is \(\lambda^3\)-optimal by Lemma 4, a contradiction to our assumption.

Combining Claim 2 with Lemma 4, we have that \(Y_1 \cup Y_2 \neq \emptyset\) and \(X_1 \cup X_2 \neq \emptyset\). Otherwise we will obtain that \(D\) is \(\lambda^3\)-optimal, which is a contradiction. Next, we consider two cases.

Case 1. \(X_1 \neq \emptyset\). Let \(x' \in X_1\) and suppose \(N^+(x') \cap Y = \{y\}\). Then for any \(y \in Y \setminus \{y\}, x'y \notin A(D)\), so we have that \(5 \leq |N^+(x') \cap N^-(y)| = |N^+(x') \cap N^-(y) \cap X| + |N^+(x') \cap N^-(y) \cap Y| \leq |N^-(y) \cap X| + |N^+(x') \cap Y| = |N^-(y) \cap X| + 1\). So \(|N^-(y) \cap X| \geq 4\). If \(Y_1 \cup Y_2 \neq \emptyset\), then \(y' \in Y_1 \cup Y_2\). Besides, \(5 \leq \delta(D) \leq \delta(y') = |N^-(y')| = |N^-(y') \cap X| + |N^-(y') \cap Y| \geq 3\). Let \(y_1, y_2 \in N^-(y') \cap Y\), then \(D[y', y_1, y_2]\) is connected and \(|N^-(y) \cap X| \geq 4\) for any \(y \in Y \setminus \{y', y_1, y_2\}\). By Lemma 4, we have that \(D\) is \(\lambda^3\)-optimal, a contradiction.

Case 2. \(X_2 \neq \emptyset\). Let \(x' \in X_2\) and suppose \(N^+(x') \cap Y = \{y', y''\}\). Then for any \(y \in Y \setminus \{y', y''\}, x'y \notin A(D)\), thus \(5 \leq |N^+(x') \cap N^-(y)| = |N^+(x') \cap N^-(y) \cap X| + |N^+(x') \cap N^-(y) \cap Y| \leq |N^-(y) \cap X| + |N^+(x') \cap Y| = |N^-(y) \cap X| + 2\). So \(|N^-(y) \cap X| \geq 3\) and \(Y \setminus \{y', y''\} \subseteq Y_3\). On the other hand, since \(Y_1 \cup Y_2 \neq \emptyset\), \(y' \in Y_1 \cup Y_2\) or \(y'' \in Y_1 \cup Y_2\). If \(|Y_1 \cup Y_2| = 1\), then we can prove that \(D\) is \(\lambda^3\)-optimal by a proof similar to Case 1, which is a contradiction. If \(Y_1 \cup Y_2 = \{y', y''\}\), then we consider two subcases.

Subcase 2.1. \(y'y'' \notin A(D)\) or \(y''y' \notin A(D)\). Since \(y'' \in Y_1 \cup Y_2\) and \(\delta(D) \geq 5\), then there exists \(y_1 \in N^-(y'') \cap Y\) such that \(y_1 \neq y'\). Therefore \(D[y', y'', y_1]\) is connected and \(|N^-(y) \cap X| \geq 3\) for any \(y \in Y \setminus \{y', y'', y_1\}\). By Lemma 4, we have that \(D\) is \(\lambda^3\)-optimal, a contradiction.

Subcase 2.2. \(y'y'' \notin A(D)\) and \(y''y' \notin A(D)\). Since \(y'' \notin A(D)\) and \(y''y' \notin A(D)\), then \(5 \leq |N^+(y'') \cap N^-(y')| = |N^+(y') \cap N^-(y'') \cap X| + |N^+(y') \cap N^-(y'') \cap Y| \leq |N^-(y') \cap X| + |N^+(y') \cap N^-(y'') \cap X| \leq 2 + |N^+(y') \cap N^-(y'') \cap Y|.\) Therefore \(|N^+(y') \cap N^-(y'') \cap X| \geq 3\). Let \(y_1 \in N^+(y') \cap N^-(y'') \cap Y\). Then \(D[y', y'', y_1]\) is connected and \(|N^-(y) \cap X| \geq 3\) for any \(y \in Y \setminus \{y', y'', y_1\}\). By Lemma 4, we have that \(D\) is \(\lambda^3\)-optimal, a contradiction.

The proof is complete.

From Theorem 1, we have following corollaries.

**Corollary 6.** Let \(D\) be a digraph with \(|V(D)| \geq 6\). If \(d^+(u) + d^-(v) \geq |V(D)| + 3\) for any \(u, v \in V(D)\) with \(uv \notin A(D)\), then \(D\) is \(\lambda^3\)-optimal.

**Corollary 7 (Theorem 1.7 in [10]).** Let \(D\) be a digraph with \(|V(D)| \geq 6\). If \(\delta(D) \geq \frac{|V(D)|+3}{2}\), then \(D\) is \(\lambda^3\)-optimal.
Remark 8. To show the condition that “$|N^+(u) \cap N^-(v)| \geq 5$ for any $u, v \in V(D)$ with $uv \notin A(D)$” in Theorem 1 is sharp, we give a class of digraphs. Let $m, k$ be positive integers with $m \geq 3$, and let $D$ be a digraph with $|V(D)| = 4m + 4$. Define the vertex set of $D$ as $V(D) = B \cup C$, where $B = \{x_0, \ldots, x_m, w_0, \ldots, w_m\}$ and $C = \{y_0, \ldots, y_m, z_0, \ldots, z_m\}$. And define the arc set of $D$ as $A(D) = A(D[B]) \cup A(D[C]) \cup M_1 \cup M_2 \cup M_3 \cup M_4$, where $A(D[B]) \cup A(D[C]) = \{uw : \text{for any } u, v \in B \text{ or } C\}$, $M_1 = \{x_iy_k(\text{mod } m+1) : 0 \leq i \leq m \text{ and } 0 \leq k-i \leq 1\}$, $M_2 = \{w_ix_k(\text{mod } m+1) : 0 \leq i \leq m \text{ and } 0 \leq k-i \leq 2\}$, $M_3 = \{y_ix_k(\text{mod } m+1) : 0 \leq i \leq m \text{ and } 0 \leq k-i \leq 2\}$ and $M_4 = \{z_ix_k(\text{mod } m+1) : 0 \leq i \leq m \text{ and } 0 \leq k-i \leq 2\}$.

Clearly, $D$ is strong and there exists $0 \leq i, j \leq m$ such that $|N^+(x_i) \cap N^-(y_j)| = 4$ and $x_iy_j \notin A(D)$. And $\partial^+(B)$ is a 3-restricted edge cut with $|\partial^+(B)| = (2+3) \cdot (m+1) = 5m+5$. On the other hand, $\xi^3(D) = \xi(\{x_i, x_p, x_q\}) = |\partial^+(\{x_i, x_p, x_q\})| = 3 \cdot (2m+3) - 6 = 6m + 3$, where $0 \leq l, p, q \leq m$. So $\lambda^3(D) \leq |\partial^+(B)| = 6m + 3 < 6m + 3 + \xi^3(D)$ for $m \geq 3$. Thus $D$ is not $\lambda^3$-optimal.

Besides, in $D$, there exists $0 \leq i, j \leq m$ such that $x_iy_j \notin A(D)$ and $d^+(x_i) + d^-(y_j) = 2 \cdot (2m + 3) = |V(D)| + 2 < |V(D)| + 3$, and $\delta(D) = 2m + 3 = \frac{|V(D)|}{2} + 1 < \frac{|V(D)| + 3}{2}$. So this example also shows that the conditions that “$d^+(u) + d^-(v) \geq |V(D)| + 3$ for any $u, v \in V(D)$ with $uv \notin A(D)$” in Corollary 6 and “$\delta(D) \geq \frac{|V(D)|}{2} + 3$” in Corollary 7 are sharp.

3. Proof of Theorem 2

We first introduce several useful lemmas.

Lemma 9 (Lemma 2.1 in [10]). Let $D$ be a strong digraph and $X_1, Y_1$ disjoint subsets of $V(D)$. If $D[X_1]$ contains a connected subdigraph with order at least $k$ and $D[Y_1]$ contains a strong subdigraph with order at least $k$, then $D$ is $\lambda^k$-connected and each arc set in $\{\partial^+(Y_1), \partial^+(Y_1)\} \cup \Omega(X_1)$ is a $k$-restricted arc cut of $D$.

Lemma 10. Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. Then $D$ is $\lambda^3$-connected and $\lambda^3(D) \leq \xi^3(D)$.

Proof. By reason of symmetry, we only consider the case that $\delta^-(D) \geq 3$. Let $X'$ be a subset of $V(D)$ with $|X'| = 3$ such that $D[X']$ is connected and $\xi^3(D) = \xi(X')$. Without loss of generality, assume that $|X' \cap X| = 1$ and $|X' \cap Y| = 2$. Let $X' \cap X = \{x\}$ and $X' \cap Y = \{y, z\}$. Let $D_1, \ldots, D_p$ be an acyclic ordering of the strong components of $D[X']$.

First, we claim that $V(D_1) \cap Y \neq \emptyset$. Otherwise, we have that $V(D_1) \subseteq X$ and $|V(D_1)| = 1$. Let $V(D_1) = \{u\}$. Then $N^-(u) \subseteq \{y, z\}$. So $3 \leq \delta^-(D) \leq d^-(u) = |N^-(u)| \leq |\{y, z\}| = 2$, a contradiction. Next, we aim to prove $|V(D_1)| \geq 3$. 
Since $N^-(v) \subseteq \{x\} \cup (V(D_1) \cap X)$ for any $v \in V(D_1) \cap Y$, we have $3 \leq \delta^-(D) \leq \delta^-(y) = |N^-(v)| \leq |\{x\} \cup (V(D_1) \cap X)| = |\{x\}| + |V(D_1) \cap X| = 1 + |V(D_1) \cap X|$. Thus $|V(D_1) \cap X| \geq 2$ and $|V(D_1)| = |V(D_1) \cap X| + |V(D_1) \cap Y| \geq 2 + 1 = 3$. It follows that $|V(D_1)| \geq 3$. Since $D[X']$ is connected and $D[X'] \subseteq D \setminus V(D_1)$, by Lemma 9, each arc set in $\Omega(X')$ is a 3-restricted arc cut of $D$. Therefore, $D$ is $\lambda^3$-connected and $\lambda^3(D) \leq \xi(X') = \xi^3(D)$.

**Lemma 11.** Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$, and let $S = \partial^+(X')$ be a $\lambda^3$-cut of $D$, where $X'$ is a subset of $V(D)$. If $D[X']$ contains a connected subdigraph $B$ with order 3 such that $|N^+(x) \cap X'| \geq 2$ for any $x \in X' \setminus V(B)$ or $D[\overline{X}]$ contains a connected subdigraph $C$ with order 3 such that $|N^-(y) \cap X'| \geq 2$ for any $y \in X' \setminus V(C)$, then $D$ is $\lambda^3$-optimal.

**Proof.** By Lemma 10, $D$ is $\lambda^3$-connected and $\lambda^3(D) \leq \xi^3(D)$. By reason of symmetry, we only prove the case that $D[X']$ contains a connected subdigraph $B$ with order 3 such that $|N^+(x) \cap X'| \geq 2$ for any $x \in X' \setminus V(B)$. The hypotheses imply that

$$\xi^3(D) \leq |\partial^+(V(B))| = |[V(B), X' \setminus V(B)]| + |[V(B), \overline{X}]|,$$

$$\leq 2|X' \setminus V(B)| + |[V(B), \overline{X}]| \leq \sum_{x \in X' \setminus V(B)} |N^+(x) \cap X'| + |[V(B), \overline{X}]| = |[X' \setminus V(B), \overline{X}]| + |[V(B), \overline{X}]| = |X', \overline{X}]| = |S| = \lambda^3(D).$$

Thus $\lambda^3(D) = \xi^3(D)$ and $D$ is $\lambda^3$-optimal.

By a proof similar to that of Lemma 4.1 shown in [10], we can get the following Lemma 12.

**Lemma 12.** Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta(D) \geq 3$, and let $S$ be a $\lambda^3$-cut of $D$. If $D$ is not $\lambda^3$-optimal, then there exists a subset of vertices $X' \subset V(D)$ such that $S = \partial^+(X')$ and both induced subdigraphs $D[X']$ and $D[\overline{X}]$ contain a connected subdigraph with order 3.

**Proof of Theorem 2.** Since $|V(D)| \geq 6$, for any $u, v \in V(D)$ in the same partite, $|N^+(u) \cap N^-(v)| \geq \left[\frac{|V(D)|}{4}\right] + 1 \geq 3$. Therefore $D$ is strong and $\delta(D) \geq 3$. By Lemma 10, $D$ is $\lambda^3$-connected and $\lambda^3(D) \leq \xi^3(D)$. Suppose, on the contrary, that $D$ is not $\lambda^3$-optimal, that is, $\lambda^3(D) < \xi^3(D)$. Let $S$ be a $\lambda^3$-cut of $D$. Then by Lemma 12, there exists a subset of vertices $X' \subset V(D)$ such that $S = \partial^+(X')$ and both induced subdigraphs $D[X']$ and $D[\overline{X}]$ contain a connected subdigraph with order 3.

Let $X' = X''$, and let $X'_x = X' \cap X$, $X'_{Y} = X' \cap Y$, $X''_X = X'' \cap X$ and $X''_Y = X'' \cap Y$. And let $X'_x = \{x \in X'_x : |N^+(x) \cap X'| = i\}$, $X'_{Y} = \{x \in X'_{Y} : |N^+(x) \cap Y| = i\}$, $X''_X = \{x \in X''_X : |N^-(x) \cap X'_{Y}| = i\}$, $X''_{Y} = \{x \in X''_{Y} : |N^-(x) \cap Y| = i\}$.
$|N^-(y) \cap X'_X| = i$, $i = 0, 1$, and $X'_{X_2} = \{ x \in X'_X : |N+(x) \cap X''_X| \geq 2 \}$, $X''_{Y_2} = \{ y \in X'_Y : |N+(y) \cap X''_X| \geq 2 \}$, $X''_{Y_2} = \{ x \in X'_X : |N^-(x) \cap X'_X| \geq 2 \}$, $X''_{Y_2} = \{ y \in X'_Y : |N^-(y) \cap X'_X| \geq 2 \}$.

**Claim 1.** $\min\{|X'_X|, |X'_Y|, |X''_X|, |X''_Y|\} \geq 2$.

**Proof.** If, on the contrary $|X'_X| = 1$, let $X'_X = \{ v \}$. Then $|N(v) \cap X'_Y| \geq 2$ for $D[X']$ contains a connected subgraph with order 3. Let $y_1, y_2 \in N(v) \cap X'_Y$. Then $D[v, y_1, y_2]$ is connected, and for any $x' \in X'\{v, y_1, y_2\}$, $N^+(x') \subseteq \{ v \} \cup (N^+(x') \cap X'')$, we have $3 \leq \delta(D) \leq d^+(x') = |N^+(x')| \leq |\{ v \}| + |N^+(x') \cap X''| = 1 + |N^+(x') \cap X''|$. Therefore $|N^+(x') \cap X''| \geq 2$. By Lemma 11, $D$ is $\lambda^3$-optimal, a contradiction to our assumption. Thus $|X'_X| \geq 2$. Similarly, we can prove that $\min\{|X'_Y|, |X''_X|, |X''_Y|\} \geq 2$. \hfill \Box

**Claim 2.** Either $X''_{X_0} = \emptyset$ or $X''_{X_0} = \emptyset$ and either $X''_{Y_0} = \emptyset$ or $X''_{Y_0} = \emptyset$.

**Proof.** If $X''_{X_0} \neq \emptyset$ and $X''_{X_0} \neq \emptyset$, then there exists $x \in X''_{X_0} \subseteq X$ and $\overline{x} \in X''_{X_0} \subseteq X$ such that $|N^+(x) \cap N^-(\overline{x})| \geq \left\lceil \frac{|V(D)|}{4}\right\rceil + 1$. On the other hand, since $x \in X''_{X_0}$ and $\overline{x} \in X''_{X_0}$, $N^+(x) \subseteq X'_X$ and $N^-(\overline{x}) \subseteq X'_Y$, which implies that $N^+(x) \cap N^-(\overline{x}) = \emptyset$, a contradiction. Thus either $X''_{X_0} = \emptyset$ or $X''_{X_0} = \emptyset$. Similarly, we can obtain that either $X''_{Y_0} = \emptyset$ or $X''_{Y_0} = \emptyset$. \hfill \Box

We consider the following two cases.

**Case 1.** $X'_{X_0} = X'_{Y_0} = \emptyset$ or $X''_{X_0} = X''_{Y_0} = \emptyset$. By reason of symmetry, we only prove the case that $X'_{X_0} = X'_{Y_0} = \emptyset$.

**Claim 1.1.** Either $X'_{X_1} = \emptyset$ and $X'_{Y_1} \neq \emptyset$ or $X'_{X_1} \neq \emptyset$ and $X'_{Y_1} = \emptyset$.

**Proof.** Since $D$ is not $\lambda^3$-optimal, by Lemma 11, we have that $X'_{X_1} \cup X'_{Y_1} \neq \emptyset$. Suppose $X'_{X_1} \neq \emptyset$ and $X'_{Y_1} \neq \emptyset$. Take $x_1 \in X'_{X_1}$. Then for any $\overline{x} \in X''_X$, we have that $\left\lceil \frac{|V(D)|}{4}\right\rceil + 1 \leq |N^+(x_1) \cap N^-(\overline{x})| = |N^+(x_1) \cap N^-(\overline{x}) \cap X'| + |N^+(x_1) \cap N^-(\overline{x}) \cap X''| \leq |N^-(\overline{x}) \cap X'| + |N^+(x_1) \cap X''| = |N^-(\overline{x}) \cap X'| + 1$. It implies that $|N^-(\overline{x}) \cap X'| \geq \left\lceil \frac{|V(D)|}{4}\right\rceil \geq 2$. So $X''_X \subseteq X''_{Y_2}$. By a similar proof, we can also prove that $X''_Y \subseteq X''_{Y_2}$. Therefore $D$ is $\lambda^3$-optimal by Lemma 11, a contradiction. The proof of Claim 1.1 is complete. \hfill \Box

Without loss of generality, let $X'_{X_1} \neq \emptyset$ and $X'_{Y_1} = \emptyset$.

**Case 1.1.** $|X'_{X_1}| = 1$. Let $x_1 \in X'_{X_1}$. Then $3 \leq \delta(D) \leq d^+(x_1) = |N^+(x_1) \cap X'_X| + |N^+(x_1) \cap X''_X| = |N^+(x_1) \cap X'_X| + 1$, therefore $|N^+(x_1) \cap X'_X| \geq 2$. Let $y_1, y_2 \in N^+(x_1) \cap X'_Y$. Then $D[x_1, y_1, y_2]$ is connected, and for any $v \in X'\{x_1, y_1, y_2\}$, $|N^+(v) \cap X''| \geq 2$. By Lemma 11, $D$ is $\lambda^3$-optimal, a contradiction.

**Case 1.2.** $|X'_{X_1}| \geq 2$. Let $x_1, x_2 \in X'_{X_1}$. Then $\left\lceil \frac{|V(D)|}{4}\right\rceil + 1 \leq |N^+(x_1) \cap N^-(x_2)| = |N^+(x_1) \cap N^-(x_2) \cap X'_X| + |N^+(x_1) \cap N^-(x_2) \cap X''_X| \leq |N^+(x_1) \cap N^-(x_2)| + |N^+(x_1) \cap N^-(x_2) \cap X''_X| \leq |N^+(x_1) \cap N^-(x_2)| + 1$, therefore $|N^+(x_1) \cap N^-(x_2)| \geq 2$. We can prove that $X''_{Y_2} = \emptyset$. By a similar proof, we can also prove that $X''_{X_0} = \emptyset$. Therefore $D$ is $\lambda^3$-optimal by Lemma 11, a contradiction. The proof of Claim 1.2 is complete. \hfill \Box
\[ N^-(x_2) \cap X'_X \mid + \mid N^+(x_1) \cap X'_Y \mid = \mid N^+(x_1) \cap N^-(x_2) \cap X'_Y \mid + 1. \] So \( \mid N^+(x_1) \cap N^-(x_2) \cap X'_Y \mid \geq \left[ \frac{|V(D)|}{4} \right] \geq 2. \] Let \( y_1 \in N^+(x_1) \cap N^-(x_2) \cap X'_Y. \) Then

\[
\xi^3(D) \leq \xi(\{x_1, x_2, y_1\}) \leq |\partial^+(\{x_1, x_2, y_1\})|
= \mid \{x_1, X'_Y\{y_1\}\} \mid + \mid \{x_1, X'_X\}\mid + \mid \{x_2, X'_Y\}\mid + \mid \{y_1, X'_X\}\mid
+ 2 \cdot (\mid X'_Y \mid - 1) + 2 + \mid X'_X \mid - 2 + \mid \{y_1, X'_X\}\mid \leq |S| = \lambda^3(D).
\]

Thus \( D \) is \( \lambda^3 \)-optimal, a contradiction.

Case 2. \( X'_{X_0} = X''_{X_0} = \emptyset \) or \( X''_{X_0} = X'_{X_0} = \emptyset. \) By reason of symmetry, we only prove the case that \( X'_{X_0} = X''_{X_0} = \emptyset. \) Without loss of generality, we may assume that \( X'_{X_0} \neq \emptyset \) and \( X''_{X_0} \neq \emptyset. \) Otherwise, by Case 1, \( D \) is \( \lambda^3 \)-optimal, a contradiction. On the other hand, since for any \( u \in X'_{X_0}, N^+(u) \subseteq X'_Y, \) we have

\[
\left[ \frac{|V(D)|}{4} \right] + 1 \leq \delta(D) \leq d^+(u) = \mid N^+(u) \mid \leq \mid X'_X \mid.
\]

Therefore \( \mid X'_X \mid \geq \left[ \frac{|V(D)|}{4} \right] + 1. \) Similarly, we can also prove that \( \mid X'_Y \mid \geq \left[ \frac{|V(D)|}{4} \right] + 1. \) Thus

\[
\mid X'_Y \mid + \mid X'_X \mid = \mid V(D) \mid - \mid X'_X \mid - \mid X'_Y \mid \leq \mid V(D) \mid - 2 \cdot \left( \left[ \frac{|V(D)|}{4} \right] + 1 \right) \leq \frac{|V(D)|}{2} - 2.
\]

Claim 2.1. \( \mid X'_X \mid \geq \mid X'_Y \mid + 1 \) or \( \mid X'_Y \mid \geq \mid X'_X \mid + 1. \)

Proof. Otherwise, we have that \( \mid X'_Y \mid + \mid X'_X \mid \geq \mid X'_X \mid + \mid X'_Y \mid \geq 2 \cdot \left( \left[ \frac{|V(D)|}{4} \right] + 1 \right) \geq \frac{|V(D)|}{2} + 2, \) a contradiction to (1). \( \Box \)

Without loss of generality, we assume that \( \mid X'_X \mid \geq \mid X'_Y \mid + 1 \) in the following discussion.

Claim 2.2. \( \mid N^+(x) \cap X'_X \mid \geq 3 \) and \( \mid N^-(y) \cap X'_X \mid \geq 3 \) for any \( x \in X'_X \) and \( y \in X'_Y. \)

Proof. By reason of symmetry, we only prove that for any \( x \in X'_X, \mid N^+(x) \cap X'_X \mid \geq 3. \) Since \( X''_{X_0} \neq \emptyset, \) for any \( x \in X'_X \) and \( x \in X''_{X_0}, \) \( \left[ \frac{|V(D)|}{4} \right] + 1 \leq \mid N^+(x) \cap N^-(x) \mid = \mid N^+(x) \cap N^-(x) \cap X'_X \mid + \mid N^+(x) \cap N^-(x) \cap X'_Y \mid \leq \mid N^-(x) \cap X'_X \mid + \mid N^+(x) \cap X'_Y \mid = \mid N^+(x) \cap X'_X \mid, \) so \( \mid N^+(x) \cap X'_X \mid \geq \left[ \frac{|V(D)|}{4} \right] + 1 \geq 3. \) \( \Box \)

Claim 2.3. \( X'_Y = X''_{X_2} = \emptyset. \)

Proof. Here, we only prove that \( X'_Y = \emptyset. \) The proof of the statement that \( X''_{X_2} = \emptyset \) is similar. Suppose, by a contradiction, there exists \( y \in X''_{X_2}. \) Let
$x_1, x_2 \in N^+(y) \cap X''_X$. Then

$$\xi^3(D) \leq \xi(\{x_1, x_2, y\}) \leq |\partial^+ (\{y\}) \cup \partial^- (\{x_1, x_2\})|$$

$$= |\partial^+ (\{y\})| + |\partial^- (\{x_1, x_2\})| - 2 = |\{y\}, X'_X| + |\{y\}, X''_X| + |\{x_1\}, X'_X| + |\{x_1\}, X''_X| + |\{x_2\}, X'_X| + |\{x_2\}, X''_X| - 2$$

$$\leq |X'_X| + |\{y\}, X'_X| + |\{y\}, X''_X| + 2|X'_X| + |\{x_1\}, X'_X| + |\{x_2\}, X'_X| - 2$$

$$\leq 3 \max \{|X'_X|, |X''_X|\} + |\{y\}, X'_X| + |\{y\}, X''_X| + |\{x_1\}, X'_X| + |\{x_2\}, X'_X| - 2$$

So $D$ is $\lambda^3$-optimal, a contradiction.

The proof is complete.

Claim 2.4. For any $x \in X'_X$, $|N(x) \cap X''_X| \geq 2$.

Proof. Let $X'_Y = \{y_1, y_2, \ldots, y_p\}$ and let $S^* = \{s^* : s^* \in N^+(y_i) \cap N^-(y_j) \cap X'_X\}$, where $i, j \in \{1, \ldots, p\}$ and $i \neq j$. Then $D[S^* \cup X'_X]$ is strong. Besides, by Claim 2.3, we have that for any $i, j \in \{1, \ldots, p\}$ and $i \neq j$, $y_i, y_j \in X'_Y$.

Therefore

$$\left| \frac{|V(D)|}{4} \right| + 1 \leq |N^+(y_i) \cap N^-(y_j)| = |N^+(y_i) \cap N^-(y_j) \cap X'_X| + |N^+(y_i) \cap N^-(y_j) \cap X''_X| \leq |N^+(y_i) \cap N^-(y_j) \cap X'_X| + 1.$$ 

So $|N^+(y_i) \cap N^-(y_j) \cap X'_X| \geq \left| \frac{|V(D)|}{4} \right| + 2$. Similarly, we can prove that $|N^+(y_j) \cap N^-(y_i) \cap X'_X| \geq 2$. On the other hand, since $|X'_X| \geq 2$, we have $|S^* \cup X'_X| \geq 3$. For any $x \in S^*$, clearly, $|N(x) \cap X''_X| \geq 2$. Next, we claim that for any $x \in X'_X \setminus S^*$, $|N(x) \cap X''_X| \leq \left| \{X'_Y, \{x\}\} \right|$. Suppose there exists $x^* \in X'_X \setminus S^*$ such that $|N^+(x^*) \cap X''_X| > \left| \{X'_Y, \{x^*\}\} \right|$. Since $D[S^* \cup X'_X]$ is strong and $|S^* \cup X'_X| \geq 3$, we have $X'_X \setminus \{x^*\}$ is a 3-restricted edge cut. Therefore $|\partial^+(X'_X \setminus \{x^*\})| = |S| - |N^+(x^*) \cap X''_X| + |\{X'_Y, \{x^*\}\}| \leq 3$, a contradiction to the minimality of $S$. Thus $\left| \{X'_Y, \{x\}\} \right| \geq |N^+(x) \cap X''_X|$. By Claim 2.2, we have that $\left| \{X'_Y, \{x\}\} \right| \geq 3$. The proof of Claim 2.4 is complete.

Let $x_1 \in X'_X$ such that $|N^+(x_1) \cap X''_X| \leq |N^+(u) \cap X''_X|$ for any $u \in X'_X$, and let $y_1, y_2 \in N(x_1) \cap X''_X$. Then

$$\xi^3(D) \leq |\partial^+(\{x_1, y_1, y_2\})| = \left| \{x_1, y_1, y_2\}, X'_X \setminus \{x_1, y_1, y_2\} \right| + \left| \{x_1, y_1, y_2\}, X''_X \right|$$

$$\leq 2|X'_X| - \left| X'_X \right| + \left| \{x_1\}, X'_X \right| + \left| \{y_1\}, X'_X \right| + \left| \{y_2\}, X'_X \right|$$

$$\leq 3|X'_X| - 5 + \left| \{x_1\}, X'_X \right| + \left| \{y_1\}, X'_X \right| + \left| \{y_2\}, X'_X \right|$$

$$+ \left| \{y_2\}, X'_X \right| \left( |X'_X| - \left| X'_X \right| \right) + \left| \{x_1\}, X'_X \right| + \left| \{x_2\}, X'_X \right| \leq |S| = \lambda^3(D).$$

So $D$ is $\lambda^3$-optimal, a contradiction.

The proof is complete.
From Theorem 2, we have following corollaries.

**Corollary 13.** Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta(D) \geq 3$. If for any $u, v \in V(D)$ in the same partite, $d^+(u) + d^-(v) \geq |V(D)| - 1$, then $D$ is $\lambda^3$-optimal.

**Corollary 14.** Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $|V(D)| \geq 6$. If $\delta(D) \geq \left\lfloor \frac{|V(D)|}{2} \right\rfloor$, then $D$ is $\lambda^3$-optimal.

**Remark 15.** To show that the condition “$|N^+(u) \cap N^-(v)| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1$ for any $u, v \in V(D)$ in the same partite” in Theorem 2 is sharp, we consider the digraph $T$ shown in Figure 1. Clearly, $|V(T)| \geq 6$ and $D$ is strong. There exists $x_1, y_1$ in the same partite such that $|N^+(x_1) \cap N^-(y_1)| = 2 < 3 = \left\lceil \frac{|V(T)|}{4} \right\rceil + 1$. Clearly, $\partial^+ \left( \{x_1, x_2, x_3, x_4\} \right)$ is a 3-restricted edge cut and $\xi^3(T) = |\partial^+ \left( \{x_1, x_2, x_3\} \right)| = 5$. Therefore, $\lambda^3(T) \leq |\partial^+ \left( \{x_1, x_2, x_3, x_4\} \right)| = 4 < 5 = \xi^3(T)$ and $T$ is not $\lambda^3$-optimal.

Besides, since $d^+(x_3) + d^-(y_4) = 6 < 7 = |V(T)| - 1$ and $\delta(T) = 3 < 4 = \left\lfloor \frac{|V(D)|}{2} \right\rfloor$, this example also shows that the conditions “$d^+(u) + d^-(v) \geq |V(D)| - 1$ for any $u, v \in V(D)$ in the same partite” in Corollary 13 and “$\delta(D) \geq \left\lceil \frac{|V(D)|}{2} \right\rceil$” in Corollary 14 are sharp.

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**References**


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