ON THE OPTIMALITY OF 3-RESTRICTED ARC CONNECTIVITY FOR DIGRAPHS AND BIPARTITE DIGRAPHS

YAoyao Zhang and Jixiang Meng\(^1\)

College of Mathematics and System Sciences
Xinjiang University
Urumqi 830046, P.R. China

e-mail: yoyoyame@126.com
mjxxju@sina.com

Abstract

Let \(D\) be a strong digraph. An arc subset \(S\) is a \(k\)-restricted arc cut of \(D\) if \(D - S\) has a strong component \(D'\) with order at least \(k\) such that \(D'\setminus V(D')\) contains a connected subdigraph with order at least \(k\). If such a \(k\)-restricted arc cut exists in \(D\), then \(D\) is called \(\lambda^k\)-connected. For a \(\lambda^k\)-connected digraph \(D\), the \(k\)-restricted arc connectivity, denoted by \(\lambda^k(D)\), is the minimum cardinality over all \(k\)-restricted arc cuts of \(D\). It is known that for many digraphs \(\lambda^k(D) \leq \xi^k(D)\), where \(\xi^k(D)\) denotes the minimum \(k\)-degree of \(D\). \(D\) is called \(\lambda^k\)-optimal if \(\lambda^k(D) = \xi^k(D)\). In this paper, we will give some sufficient conditions for digraphs and bipartite digraphs to be \(\lambda^3\)-optimal.

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1. Introduction

It is well-known that the network can be modelled as a digraph \(D\) with vertices \(V(D)\) representing sites and arcs \(A(D)\) representing links between sites of the network. Let \(v \in V(D)\), the out-neighborhood of \(v\) is the set \(N^+(v) = \{x \in V(D) : vx \in A(D)\}\) and the out-degree of \(v\) is \(d^+(v) = |N^+(v)|\). The in-neighborhood of \(v\) is the set \(N^-(v) = \{x \in V(D) : xv \in A(D)\}\) and the in-degree of \(v\) is \(d^-(v) = |N^-(v)|\). The neighborhood of \(v\) is \(N(v) = N^+(v) \cup N^-(v)\).

\(^1\)Corresponding author.
Let $\delta^+(D), \delta^-(D)$ and $\delta(D)$ denote, respectively, the minimum out-degree, the minimum in-degree and the minimum degree of $D$.

For a pair nonempty vertex sets $X$ and $Y$ of $D$, $[X,Y] = \{xy \in A(D) : x \in X, y \in Y\}$. Specially, if $Y = \overline{X}$, where $\overline{X} = V(D) \setminus X$, then we write $\partial^+(X)$ or $\partial^-(Y)$ instead of $[X,Y]$. For $X \subseteq V(D)$, the subdigraph of $D$ induced by $X$ is denoted by $D[X]$. The underlying graph $U(D)$ of $D$ is the unique graph obtained from $D$ by deleting the orientation of all arcs and keeping one edge of a pair of multiple edges. $D$ is connected if $U(D)$ is connected and $D$ is strongly connected (or, just, strong) if there exists a directed $(x,y)$-path and a directed $(y,x)$-path for any $x, y \in V(D)$. We define a digraph with one vertex to be strong. A connected (strong) component of $D$ is a maximal induced subdigraph of $D$ which is connected (strong). If $D$ has $p$ strong components, then these strong components can be labeled $D_1, \ldots, D_p$ such that there is no arc from $D_j$ to $D_i$ unless $j < i$. We call such an ordering an acyclic ordering of the strong components of $D$.

In a strong digraph $D$, we often use arc connectivity of $D$ to measure the reliability. An arc set $S$ is a arc cut of $D$ if $D - S$ is not strong. The arc connectivity $\lambda(D)$ is the minimum cardinality over all arc cuts of $D$. The arc cut $S$ of $D$ with cardinality $\lambda(D)$ is called a $\lambda$-cut. Whitney’s inequality shows $\lambda(D) \leq \delta(D)$. A strong digraph $D$ with $\lambda(D) = \delta(D)$ is called $\lambda$-optimal. However, only using arc connectivity to measure the reliability is not enough. In [12], Volkmann introduced the concept of restricted arc connectivity. An arc subset $S$ of $D$ is a restricted arc cut if $D - S$ has a strong component $D'$ with order at least 2 such that $D \setminus V(D')$ contains an arc. If such an arc cut exists in $D$, then $D$ is called $\lambda'$-connected. For a $\lambda'$-connected digraph $D$, the restricted arc connectivity, denoted by $\lambda'(D)$, is the minimum cardinality over all restricted arc cuts of $D$. The restricted arc cut $S$ of $D$ with cardinality $\lambda'(D)$ is called a $\lambda'$-cut. In [13], Wang and Lin introduced the notion of minimum arc degree. Let $xy \in A(D)$. Then

$$\Omega\{xy\} = \{\partial^+\{x,y\}, \partial^-(\{x,y\}), \partial^+\{\{x\}\} \cup \partial^-\{\{y\}\}, \partial^+\{\{y\}\} \cup \partial^-\{\{x\}\}\}.$$ 

The arc degree of $xy$ is $\xi'(xy) = \min\{|S| : S \in \Omega\{xy\}\}$ and the minimum arc degree of $D$ is $\lambda'(D) = \min\{\xi'(xy) : xy \in A(D)\}$.

It was proved in [3, 13] that for many $\lambda'$-connected digraphs, $\xi'(D)$ is an upper bound of $\lambda'(D)$. In [13], Wang and Lin introduced the concept of $\lambda'$-optimality. A $\lambda'$-connected digraph $D$ with $\xi'(D) = \lambda'(D)$ is called $\lambda'$-optimal.

As a generalization of restricted arc connectivity, in [10], Lin et al. introduced the concept of $k$-restricted arc connectivity.

**Definition** [10]. Let $D$ be a strong digraph. An arc subset $S$ is a $k$-restricted arc cut of $D$ if $D - S$ has a strong component $D'$ with order at least $k$ such that
$D\setminus V(D')$ contains a connected subdigraph with order at least $k$. If such a $k$-restricted arc cut exists in $D$, then $D$ is called $\lambda^k$-connected. For a $\lambda^k$-connected digraph $D$, the $k$-restricted arc connectivity, denoted by $\lambda^k(D)$, is the minimum cardinality over all $k$-restricted arc cuts of $D$. The $k$-restricted arc cut $S$ of $D$ with cardinality $\lambda^k(D)$ is called a $\lambda^k$-cut.

**Definition** [10]. Let $D$ be a strong digraph. For any $X \subseteq V(D)$, let $\Omega(X) = \{\partial^+(X_1) \cup \partial^-(X \setminus X_1) : X_1 \subseteq X\}$ and $\xi(X) = \min\{|S| : S \in \Omega(X)\}$. Define the minimum $k$-degree of $D$ to be

$$\xi^k(D) = \min\{\xi(X) : X \subseteq V(D), |X| = k, D[X] \text{ is connected}\}.$$  

Clearly, $\lambda^1(D) = \lambda(D)$, $\lambda^2(D) = \lambda'(D)$, $\xi^1(D) = \delta(D)$ and $\xi^2(D) = \xi'(D)$. Let $D$ be a $\lambda^k$-connected digraph, where $k \geq 2$. Then $D$ is $\lambda^{k-1}$-connected and $\lambda^{k-1}(D) \leq \lambda^k(D)$. It was shown in [10] that $\xi^k(D)$ is an upper bound of $\lambda^k(D)$ for many digraphs. And a $\lambda^k$-connected digraph $D$ with $\lambda^k(D) = \xi^k(D)$ is called $\lambda^k$-optimal.

The research on the $\lambda^k$-optimality of digraph $D$ is considered to be a hot issue. In [11], Hellwig and Volkmann concluded many sufficient conditions for digraphs to be $\lambda$-optimal. Besides, sufficient conditions for digraphs to be $\lambda^3$-optimal were also given by several authors, for example by Balbuena et al. [1–4], Chen et al. [5,6], Grüter and Guo [7,8], Liu and Zhang [9], Volkmann [12] and Wang and Lin [13]. However, closely related conditions for $\lambda^3$-optimal digraphs have received little attention until recently. In [10], Lin et al. gave some sufficient conditions for digraphs to be $\lambda^3$-optimal. In this paper, we will give some sufficient conditions for digraphs to be $\lambda^3$-optimal. As corollaries, degree conditions or degree sum conditions for a digraph or a bipartite digraph to be $\lambda^3$-optimal are given. The main contributions in this paper are as following.

**Theorem 1.** Let $D$ be a digraph with $|V(D)| \geq 6$. If $|N^+(u) \cap N^-(v)| \geq 5$ for any $u, v \in V(D)$ with $uv \notin A(D)$, then $D$ is $\lambda^3$-optimal.

**Theorem 2.** Let $D = (X, Y, A(D))$ be a bipartite digraph with $|V(D)| \geq 6$. If $|N^+(u) \cap N^-(v)| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1$ for any $u, v \in V(D)$ in the same partite, then $D$ is $\lambda^3$-optimal.

2. **Proof of Theorem 1**

We first introduce three useful lemmas.

**Lemma 3** (Theorem 1.4 in [10]). Let $D$ be a strong digraph with $\delta^+(D) \geq 2k - 1$ or $\delta^-(D) \geq 2k - 1$. Then $D$ is $\lambda^k$-connected and $\lambda^k(D) \leq \xi^k(D)$. 
Lemma 4. Let $D$ be a strong digraph with $\delta^+(D) \geq 2k - 1$ or $\delta^-(D) \geq 2k - 1$, and let $S = \partial^+(X)$ be a $\lambda^k$-cut of $D$, where $X$ is a subset of $V(D)$. If $D[X]$ contains a connected subdigraph $B$ with order $k$ such that $|N^+(x) \cap X| \geq k$ for any $x \in X \setminus V(B)$ or $D[\overline{X}]$ contains a connected subdigraph $C$ with order $k$ such that $|N^-(y) \cap \overline{X}| \geq k$ for any $y \in \overline{X} \setminus V(C)$, then $D$ is $\lambda^k$-optimal.

Proof. By Lemma 3, $D$ is $\lambda^k$-connected and $\lambda^k(D) \leq \xi^k(D)$. By reason of symmetry, we only prove that $D[X]$ contains a connected subdigraph $B$ with order $k$ such that $|N^+(x) \cap X| \geq k$ for any $x \in X \setminus V(B)$. The hypotheses imply that

$$
\xi^k(D) \leq |\partial^+(V(B))| = |V(B), X \setminus V(B)| + |V(B), \overline{X}|
$$

$$
\leq k|X \setminus V(B)| + |V(B), \overline{X}| \leq \sum_{x \in X \setminus V(B)} |N^+(x) \cap \overline{X}| + |V(B), \overline{X}|
$$

$$
= |X \setminus V(B), \overline{X}| + |V(B), \overline{X}| = |X, \overline{X}| = |S| = \lambda^k(D).
$$

Thus $\lambda^k(D) = \xi^k(D)$ and $D$ is $\lambda^k$-optimal. $\square$

Lemma 5 (Lemma 4.1 in [10]). Let $D$ be a strong digraph with $|V(D)| \geq 6$ and $\delta(D) \geq 4$, and let $S$ be a $\lambda^3$-cut of $D$. If $D$ is not $\lambda^3$-optimal, then there exists a subset of vertices $X \subset V(D)$ such that $S = \partial^+(X)$ and both induced subdigraphs $D[X]$ and $D[\overline{X}]$ contain a connected subdigraph with order 3.

Proof of Theorem 1. Clearly, $D$ is a strong digraph with $\delta(D) \geq 5$. By Lemma 3, $D$ is $\lambda^3$-connected and $\lambda^3(D) \leq \xi^3(D)$. Suppose, on the contrary, that $D$ is not $\lambda^3$-optimal, that is, $\lambda^3(D) < \xi^3(D)$. Let $S$ be a $\lambda^3$-cut of $D$. By Lemma 5, there exists a subset of vertices $X \subset V(D)$ such that $S = \partial^+(X)$ and both induced subdigraphs $D[X]$ and $D[\overline{X}]$ contain a connected subdigraph with order 3.

Let $Y = \overline{X}$, and let $X_i = \{x \in X : |N^+(x) \cap X| = i\}$, $Y_i = \{y \in Y : |N^-(y) \cap X| = i\}$, $i = 0, 1, 2$, and let $X_3 = \{x \in X : |N^+(x) \cap Y| \geq 3\}$, $Y_3 = \{y \in Y : |N^-(y) \cap X| \geq 3\}$.

Claim 1. $\min\{|X|, |Y|\} \geq 4$.

Proof. Suppose that $|X| = 3$. Then $\lambda^3(D) = |S| = |\partial^+(X)| \geq \xi(X) \geq \xi^3(D)$, contrary to the assumption. Suppose that $|Y| = 3$. Then $\lambda^3(D) = |S| = |\partial^-(Y)| \geq \xi(Y) \geq \xi^3(D)$, contrary to the assumption. Claim 1 follows. $\square$

Claim 2. $X_0 = Y_0 = \emptyset$.

Proof. For the reason of symmetry, we only prove that $X_0 = \emptyset$ by contradiction. Suppose $X_0 \neq \emptyset$ and let $x \in X_0$. Then for any $\overline{x} \in Y$, $x\overline{x} \notin A(D)$ and we have that $5 \leq |N^+(x) \cap N^-(\overline{x})| = |N^+(x) \cap N^-(\overline{x}) \cap X| + |N^+(x) \cap N^-(\overline{x}) \cap Y| \leq$
\[|N^-(x) \cap X| + |N^+(x) \cap Y| = |N^-(x) \cap X|.\] It implies that \(|N^-(x) \cap X| \geq 5\). Therefore \(Y \subseteq Y_3\). So \(D\) is \(\lambda^3\)-optimal by Lemma 4, a contradiction to our assumption. \[
\]
Combining Claim 2 with Lemma 4, we have that \(Y_1 \cup Y_2 \neq \emptyset\) and \(X_1 \cup X_2 \neq \emptyset\). Otherwise we will obtain that \(D\) is \(\lambda^3\)-optimal, which is a contradiction. Next, we consider two cases.

**Case 1.** \(X_1 \neq \emptyset\). Let \(x' \in X_1\) and suppose \(N^+(x') \cap Y = \{y'\}\). Then for any \(y \in Y \setminus \{y'\}\), \(x'y \notin A(D)\), so we have that \(5 \leq |N^+(x') \cap N^-(y)| = |N^+(x') \cap N^-(y) \cap X| + |N^+(x') \cap N^-(y) \cap Y| \leq |N^-(y) \cap X| + |N^+(x') \cap Y| = |N^-(y) \cap X| + 1\). So \(|N^-(y) \cap X| \geq 4\) and \(Y \setminus \{y'\} \subseteq Y_3\). On the other hand, since \(Y_1 \cup Y_2 \neq \emptyset\), so \(y' \in Y_1 \cup Y_2\). Besides, \(5 \leq \delta(D) \leq \delta^-(y') = |N^-(y')| = |N^-(y') \cap Y| + |N^-(y') \cap X| \leq |N^-(y') \cap Y| + 2\), thus \(|N^-(y') \cap Y| \geq 3\). Let \(y_1, y_2 \in N^-(y') \cap Y\), then \(D[y', y_1, y_2]\) is connected and \(|N^-(y) \cap X| \geq 4\) for any \(y \in Y \setminus \{y', y_1, y_2\}\). By Lemma 4, we have that \(D\) is \(\lambda^3\)-optimal, which is a contradiction.

**Case 2.** \(X_2 \neq \emptyset\). Let \(x' \in X_2\) and suppose \(N^+(x') \cap Y = \{y', y''\}\). Then for any \(y \in Y \setminus \{y', y''\}\), \(x'y \notin A(D)\), thus \(5 \leq |N^+(x') \cap N^-(y)| = |N^+(x') \cap N^-(y) \cap X| + |N^+(x') \cap N^-(y) \cap Y| \leq |N^-(y) \cap X| + |N^+(x') \cap Y| = |N^-(y) \cap X| + 2\). So \(|N^-(y) \cap X| \geq 3\) and \(Y \setminus \{y', y''\} \subseteq Y_3\). On the other hand, since \(Y_1 \cup Y_2 \neq \emptyset\), \(y' \in Y_1 \cup Y_2\) or \(y'' \in Y_1 \cup Y_2\). If \(|Y_1 \cup Y_2| = 1\), then we can prove that \(D\) is \(\lambda^3\)-optimal by a proof similar to Case 1, which is a contradiction. If \(Y_1 \cup Y_2 = \{y', y''\}\), then we consider two subcases.

**Subcase 2.1.** \(y'y'' \in A(D)\) or \(y''y' \in A(D)\). Since \(y'' \in Y_1 \cup Y_2\) and \(\delta(D) \geq 5\), then there exists \(y_1 \in N^-(y'') \cap Y\) such that \(y_1 \neq y'\). Therefore \(D[y', y'', y_1]\) is connected and \(|N^-(y) \cap X| \geq 3\) for any \(y \in Y \setminus \{y', y'', y_1\}\). By Lemma 4, we have that \(D\) is \(\lambda^3\)-optimal, a contradiction.

**Subcase 2.2.** \(y'y'' \notin A(D)\) and \(y''y' \notin A(D)\). Since \(y'y'' \notin A(D)\) and \(y''y' \notin A(D)\), then \(5 \leq |N^+(y') \cap N^-(y'') \cap X| + |N^+(y') \cap N^-(y'') \cap Y| \leq |N^+(y') \cap N^-(y'') \cap X| + |N^+(y') \cap N^-(y'') \cap Y| \leq 3\). \(D[y', y'', y_1]\) is connected and \(|N^-(y) \cap X| \geq 3\) for any \(y \in Y \setminus \{y', y'', y_1\}\). By Lemma 4, we have that \(D\) is \(\lambda^3\)-optimal, a contradiction.

The proof is complete.

From Theorem 1, we have the following corollaries.

**Corollary 6.** Let \(D\) be a digraph with \(|V(D)| \geq 6\). If \(d^+(u) + d^-(v) \geq |V(D)| + 3\) for any \(u, v \in V(D)\) with \(uv \notin A(D)\), then \(D\) is \(\lambda^3\)-optimal.

**Corollary 7.** (Theorem 1.7 in [10]). Let \(D\) be a digraph with \(|V(D)| \geq 6\). If \(\delta(D) \geq \frac{|V(D)|+3}{2}\), then \(D\) is \(\lambda^3\)-optimal.
Remark 8. To show the condition that “$|N^+(u) \cap N^-(v)| \geq 5$ for any $u, v \in V(D)$ with $uv \notin A(D)$” in Theorem 1 is sharp, we give a class of digraphs. Let $m, k$ be positive integers with $m \geq 3$, and let $D$ be a digraph with $|V(D)| = 4m + 4$. Define the vertex set of $D$ as $V(D) = B \cup C$, where $B = \{x_0, \ldots, x_m, w_0, \ldots, w_m\}$ and $C = \{y_0, \ldots, y_m, z_0, \ldots, z_m\}$. And define the arc set of $D$ as $A(D) = A(D[B]) \cup A(D[C]) \cup M_1 \cup M_2 \cup M_3 \cup M_4$, where $A(D[B]) \cup A(D[C]) = \{uv : \text{for any } u, v \in B \text{ or } C\}$, $M_1 = \{x_iy_k(mod\ m + 1) : 0 \leq i \leq m \text{ and } 0 \leq k - i \leq 1\}$, $M_2 = \{w_ix_k(mod\ m + 1) : 0 \leq i \leq m \text{ and } 0 \leq k - i \leq 2\}$, $M_3 = \{y_ix_k(mod\ m + 1) : 0 \leq i \leq m \text{ and } 0 \leq k - i \leq 2\}$ and $M_4 = \{z_ix_k(mod\ m + 1) : 0 \leq i \leq m \text{ and } 0 \leq k - i \leq 2\}$. Clearly, $D$ is strong and there exists $0 \leq i, j \leq m$ such that $|N^+(x_i) \cap N^-(y_j)| = 4$ and $x_iy_j \notin A(D)$. And $\delta^+(B)$ is a 3-restricted edge cut with $|\delta^+(B)| = (2 + 3) \cdot (m + 1) = 5m + 5$. On the other hand, $\xi^3(D) = \xi(\{x_1, x_p, x_q\}) = |\delta^+(\{x_1, x_p, x_q\})| = 3 \cdot (2m + 3) - 6 = 6m + 3$, where $0 \leq l, p, q \leq m$. So $\lambda_3(D) \leq |\delta^+(B)| = 5m + 5 < 6m + 3 = \xi^3(D)$ for $m \geq 3$. Thus $D$ is not $\lambda_3$-optimal.

Besides, in $D$, there exists $0 \leq i, j \leq m$ such that $x_iy_j \notin A(D)$ and $d^+(x_i) + d^-(y_j) = 2 \cdot (2m + 3) = |V(D)| + 2 < |V(D)| + 3$, and $\delta(D) = 2m + 3 = \frac{|V(D)| - 3}{2}$. So this example also shows that the conditions that “$d^+(u) + d^-(v) \geq |V(D)| + 3$ for any $u, v \in V(D)$ with $uv \notin A(D)$” in Corollary 6 and “$\delta(D) \geq \frac{|V(D)| - 3}{2}$” in Corollary 7 are sharp.

3. Proof of Theorem 2

We first introduce several useful lemmas.

Lemma 9 (Lemma 2.1 in [10]). Let $D$ be a strong digraph and $X_1, Y_1$ disjoint subsets of $V(D)$. If $D[X_1]$ contains a connected subdigraph with order at least $k$ and $D[Y_1]$ contains a strong subdigraph with order at least $k$, then $D$ is $\lambda^k$-connected and each arc set in $\{\delta^-(Y_1), \delta^+(Y_1)\} \cup \Omega(X_1)$ is a $k$-restricted arc cut of $D$.

Lemma 10. Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. Then $D$ is $\lambda^3$-connected and $\lambda^3(D) \leq \xi^3(D)$.

Proof. By reason of symmetry, we only consider the case that $\delta^-(D) \geq 3$. Let $X'$ be a subset of $V(D)$ with $|X'| = 3$ such that $D[X']$ is connected and $\xi^3(D) = \xi(X')$. Without loss of generality, assume that $|X' \cap X| = 1$ and $|X' \cap Y| = 2$. Let $X' \cap X = \{x\}$ and $X' \cap Y = \{y, z\}$. Let $D_1, \ldots, D_p$ be an acyclic ordering of the strong components of $D[X']$.

First, we claim that $V(D_1) \cap Y \neq \emptyset$. Otherwise, we have that $V(D_1) \subseteq X$ and $|V(D_1)| = 1$. Let $V(D_1) = \{u\}$. Then $N^-(u) \subseteq \{y, z\}$. So $3 \leq \delta^-(D) \leq d^-(u) = |N^-(u)| \leq |\{y, z\}| = 2$, a contradiction. Next, we aim to prove $|V(D_1)| \geq 3$.  


Since $N^-(v) \subseteq \{x\} \cup (V(D_1) \cap X)$ for any $v \in V(D_1) \cap Y$, we have $3 \leq \delta^-(D) \leq d^-(v) = |N^-(v)| \leq |\{x\} \cup (V(D_1) \cap X)| = |\{x\}| + |V(D_1) \cap X| = 1 + |V(D_1) \cap X|$. Thus $|V(D_1) \cap X| \geq 2$ and $|V(D_1)| = |V(D_1) \cap X| + |V(D_1) \cap Y| \geq 2 + 1 = 3$. It follows that $|V(D_1)| \geq 3$. Since $D[X']$ is connected and $D[X'] \subseteq D \setminus V(D_1)$, by Lemma 9, each arc set in $\Omega(X')$ is a 3-restricted arc cut of $D$. Therefore, $D$ is $\lambda^3$-connected and $\lambda^3(D) \leq \xi(X') = \xi^3(D)$.  

**Lemma 11.** Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$, and let $S = \partial^+(X')$ be a $\lambda^3$-cut of $D$, where $X'$ is a subset of $V(D)$. If $D[X']$ contains a connected subdigraph $B$ with order 3 such that $|N^+(x) \cap \overline{X}'| \geq 2$ for any $x \in X' \setminus V(B)$ or $D[\overline{X}']$ contains a connected subdigraph $C$ with order 3 such that $|N^-(y) \cap X'| \geq 2$ for any $y \in \overline{X}' \setminus V(C)$, then $D$ is $\lambda^3$-optimal.  

**Proof.** By Lemma 10, $D$ is $\lambda^3$-connected and $\lambda^3(D) \leq \xi^3(D)$. By reason of symmetry, we only prove the case that $D[X']$ contains a connected subdigraph $B$ with order 3 such that $|N^+(x) \cap \overline{X}'| \geq 2$ for any $x \in X' \setminus V(B)$. The hypotheses imply that

$$\xi^3(D) \leq |\partial^+(V(B))| = ||V(B), X' \setminus V(B)|| + ||V(B), \overline{X}'|| \leq 2|X' \setminus V(B)| + ||V(B), \overline{X}'|| \leq \sum_{x \in X' \setminus V(B)} |N^+(x) \cap \overline{X}'| + ||V(B), \overline{X}'|| = |[X' \setminus V(B), \overline{X}']| + ||V(B), \overline{X}'|| = |X', \overline{X}'| = |S| = \lambda^3(D).$$

Thus $\lambda^3(D) = \xi^3(D)$ and $D$ is $\lambda^3$-optimal.  

By a proof similar to that of Lemma 4.1 shown in [10], we can get the following Lemma 12.

**Lemma 12.** Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta(D) \geq 3$, and let $S$ be a $\lambda^3$-cut of $D$. If $D$ is not $\lambda^3$-optimal, then there exists a subset of vertices $X' \subset V(D)$ such that $S = \partial^+(X')$ and both induced subdigraphs $D[X']$ and $D[\overline{X}']$ contain a connected subdigraph with order 3.

**Proof of Theorem 2.** Since $|V(D)| \geq 6$, for any $u, v \in V(D)$ in the same partite, $|N^+(u) \cap N^-(v)| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \geq 3$. Therefore $D$ is strong and $\delta(D) \geq 3$. By Lemma 10, $D$ is $\lambda^3$-connected and $\lambda^3(D) \leq \xi^3(D)$. Suppose, on the contrary, that $D$ is not $\lambda^3$-optimal, that is, $\lambda^3(D) < \xi^3(D)$. Let $S$ be a $\lambda^3$-cut of $D$. Then by Lemma 12, there exists a subset of vertices $X' \subset V(D)$ such that $S = \partial^+(X')$ and both induced subdigraphs $D[X']$ and $D[\overline{X}']$ contain a connected subdigraph with order 3.

Let $\overline{X}' = X''$, and let $X'_X = X' \cap X$, $X'_Y = X' \cap Y$, $X''_X = X'' \cap X$ and $X''_Y = X'' \cap Y$. And let $X'_X = \{x \in X'_X : |N^+(x) \cap X'_X| = i\}$, $X'_Y = \{y \in X'_Y : |N^-(y) \cap X'_Y| = i\}$, $X''_X = \{x \in X''_X : |N^+(x) \cap X''_X| = i\}$, $X''_Y = \{y \in X''_Y : |N^-(y) \cap X''_Y| = i\}$.
\[|N^-(y) \cap X'_X| = i, \ i = 0, 1, \text{ and } X'_{X_2} = \{x \in X'_X : |N^+(x) \cap X'_Y| \geq 2\},\]
\[X'_{Y_2} = \{y \in X'_Y : |N^+(y) \cap X'_X| \geq 2\}, \ X''_{Y_2} = \{x \in X''_X : |N^-(x) \cap X'_Y| \geq 2\},\]

Claim 1. \[\min\{|X'_X|, |X'_Y|, |X''_X|, |X''_Y|\} \geq 2.\]

\textbf{Proof.} If, on the contrary \(|X'_X| = 1\), let \(X'_X = \{v\}.\) Then \(|N(v) \cap X'_Y| \geq 2\) for \(D[X']\) contains a connected subdigraph with order \(3.\) Let \(y_1, y_2 \in N(v) \cap X'_Y.\) Then \(D[v, y_1, y_2]\) is connected, and for any \(x' \in X' \setminus \{v, y_1, y_2\}\), \(N^+(x') \subseteq \{v\} \cup (N^+(x') \cap X''_Y),\) we have \(3 \leq \delta(D) \leq d^+(x') = |N^+(x')| \leq ||v|| + |N^+(x') \cap X''_Y| = 1 + |N^+(x') \cap X''_Y| .\) Therefore \(|N^+(x') \cap X''_Y| \geq 2.\) By Lemma 11, \(D\) is \(\lambda^3\)-optimal, a contradiction to our assumption. Thus \(|X'_X| \geq 2.\) Similarly, we can prove that \(\min\{|X'_X|, |X'_Y|, |X''_X|, |X''_Y|\} \geq 2.\)

Claim 2. Either \(X'_{X_0} = \emptyset \) or \(X''_{X_0} = \emptyset\) and either \(X'_{Y_0} = \emptyset \) or \(X''_{Y_0} = \emptyset.\)

\textbf{Proof.} If \(X'_{X_0} \neq \emptyset \) and \(X''_{X_0} \neq \emptyset,\) then there exists \(x \in X'_{X_0} \subseteq X \) and \(\overline{x} \in X''_{X_0} \subseteq X\) such that \(|N^+(x) \cap N^-(\overline{x})| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1.\) On the other hand, since \(x \in X'_{X_0} \) and \(\overline{x} \in X''_{X_0}, N^+(x) \subseteq X'_Y \) and \(N^-(\overline{x}) \subseteq X'_Y,\) which implies that \(N^+(x) \cap N^-(\overline{x}) = \emptyset,\) a contradiction. Thus either \(X'_{X_0} = \emptyset \) or \(X''_{X_0} = \emptyset.\) Similarly, we can obtain that either \(X'_{Y_0} = \emptyset \) or \(X''_{Y_0} = \emptyset.\)

We consider the following two cases.

Case 1. \(X'_{X_0} = X'_{Y_0} = \emptyset \) or \(X''_{X_0} = X''_{Y_0} = \emptyset.\) By reason of symmetry, we only prove the case that \(X'_{X_0} = X'_{Y_0} = \emptyset.\)

Claim 1.1. Either \(X'_{X_1} = \emptyset \) and \(X'_{Y_1} \neq \emptyset \) or \(X'_{X_1} \neq \emptyset \) and \(X'_{Y_1} = \emptyset.\)

\textbf{Proof.} Since \(D\) is not \(\lambda^3\)-optimal, by Lemma 11, we have that \(X'_{X_1} \cup X'_{Y_1} \neq \emptyset.\) Suppose \(X'_{X_1} \neq \emptyset \) and \(X'_{Y_1} \neq \emptyset.\) Take \(x_1 \in X'_{X_1}.\) Then for any \(\overline{x} \in X''_X,\) we have that \(\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \leq |N^+(x_1) \cap N^-(\overline{x})| = |N^+(x_1) \cap N^-(\overline{x}) \cap X'| + |N^+(x_1) \cap N^-(\overline{x}) \cap X'| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1.\) It implies that \(|N^-(\overline{x}) \cap X'| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil \geq 2.\) So \(X'_X \subseteq X'_{X_2}.\) By a similar proof, we can also prove that \(X''_X \subseteq X'_{Y_2}.\) Therefore \(D\) is \(\lambda^3\)-optimal by Lemma 11, a contradiction. The proof of Claim 1.1 is complete.

Without loss of generality, let \(X'_{X_1} \neq \emptyset \) and \(X'_{Y_1} = \emptyset.\)

Case 1.1. \(|X'_{X_1}| = 1.\) Let \(x_1 \in X'_{X_1}.\) Then \(3 \leq \delta(D) \leq d^+(x_1) = |N^+(x_1) \cap X'_Y| \geq 2.\) Let \(y_1, y_2 \in N^+(x_1) \cap X'_Y.\) Then \(D[x_1, y_1, y_2]\) is connected, and for any \(v \in X' \setminus \{x_1, y_1, y_2\}, \ |N^+(v) \cap X''_Y| \geq 2.\) By Lemma 11, \(D\) is \(\lambda^3\)-optimal, a contradiction.

Case 1.2. \(|X'_{X_1}| \geq 2.\) Let \(x_1, x_2 \in X'_{X_1}.\) Then \(\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \leq |N^+(x_1) \cap X'_{X_1}| \geq 2.\)
\[ N^{-}(x_{2}) \cap X'_{Y} + |N^{+}(x_{1}) \cap X''_{Y}| = |N^{+}(x_{1}) \cap N^{-}(x_{2}) \cap X'_{Y}| + 1. \] So \(|N^{+}(x_{1}) \cap N^{-}(x_{2}) \cap X'_{Y} \geq \left\lceil \frac{|V(D)|}{4} \right\rceil \geq 2. \] Let \( y_{1} \in N^{+}(x_{1}) \cap N^{-}(x_{2}) \cap X'_{Y} \). Then
\[
\xi^{3}(D) \leq \xi(\{x_{1}, x_{2}, y_{1}\}) \leq |\partial^{+}(\{x_{1}, x_{2}, y_{1}\})|
\]
\[ = |[\{x_{1}\}, X'_{Y}\backslash\{y_{1}\}]| + |[\{x_{1}\}, X''_{Y}]| + |[\{x_{2}\}, X'_{Y}]| + |[\{y_{1}\}, X''_{Y}]| + 2 \cdot (|X'_{Y}| - 1) + 2 + |X''_{Y}| - 2 + |[\{y_{1}\}, X''_{Y}]| \leq |S| = \lambda^{3}(D). \]

Thus \( D \) is \( \lambda^{3} \)-optimal, a contradiction.

Case 2. \( X'_{X_{0}} = X''_{X_{0}} = X_{Y_{0}} = \emptyset \) or \( X''_{X_{0}} = X'_{Y_{0}} = \emptyset \). By reason of symmetry, we only prove the case that \( X'_{X_{0}} = X''_{X_{0}} = \emptyset \). Without loss of generality, we may assume that \( X'_{X_{0}} \neq \emptyset \) and \( X''_{X_{0}} \neq \emptyset \). Otherwise, by Case 1, \( D \) is \( \lambda^{3} \)-optimal, a contradiction. On the other hand, since for any \( u \in X'_{Y_{0}}, N^{+}(u) \subseteq X'_{Y}, \) we have \( \left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \leq \delta(D) \leq d^{+}(u) = |N^{+}(u)| \leq |X'_{Y}| \). Therefore \( |X'_{Y}| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \).

Similarly, we can also prove that \( |X''_{Y}| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \). Thus
\[
|X'_{Y}| + |X''_{Y}| = |V(D)| - |X'_{Y}| - |X''_{Y}|
\]
\[
\leq |V(D)| - 2 \cdot \left( \left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \right) \leq \frac{|V(D)|}{2} - 2.
\]

Claim 2.1. \(|X'_{X}| \geq |X'_{Y}| + 1\) or \(|X''_{X}| \geq |X''_{Y}| + 1\).

**Proof.** Otherwise, we have that \( |X'_{Y}| + |X''_{Y}| \geq |X'_{X}| + |X''_{X}| \geq 2 \cdot \left( \left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \right) \geq \frac{|V(D)|}{2} + 2, \) a contradiction to (1). \( \square \)

Without loss of generality, we assume that \( |X'_{X}| \geq |X'_{Y}| + 1 \) in the following discussion.

Claim 2.2. \(|N^{+}(x) \cap X''_{Y}| \geq 3\) and \(|N^{-}(y) \cap X'_{X}| \geq 3\) for any \( x \in X'_{X} \) and \( y \in X'_{Y} \).

**Proof.** By reason of symmetry, we only prove that for any \( x \in X'_{X}, |N^{+}(x) \cap X''_{Y}| \geq 3 \). Since \( X''_{X_{0}} \neq \emptyset \), for any \( x \in X'_{X} \) and \( \pi \in X''_{X_{0}}, \left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \geq |N^{+}(x) \cap N^{-}(\pi)| = |N^{+}(x) \cap N^{-}(\pi) \cap X'_{Y}| + |N^{+}(x) \cap N^{-}(\pi) \cap X''_{Y}| \leq |N^{-}(\pi) \cap X'_{Y}| + |N^{+}(x) \cap X''_{Y}| = |N^{+}(x) \cap X''_{Y}|, \) so \( |N^{+}(x) \cap X''_{Y}| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \geq 3. \) \( \square \)

Claim 2.3. \( X'_{Y_{2}} \cap X''_{Y_{2}} = \emptyset \).

**Proof.** Here, we only prove that \( X'_{Y_{2}} = \emptyset \). The proof of the statement that \( X''_{Y_{2}} = \emptyset \) is similar. Suppose, by a contradiction, there exists \( y \in X'_{Y_{2}} \). Let
\(x_1, x_2 \in N^+(y) \cap X'_X\). Then

\[
\xi^3(D) \leq \xi(\{x_1, x_2, y\}) \leq |\partial^+(\{y\}) \cup \partial^-(\{x_1, x_2\})| = |\partial^+(\{y\})| + |\partial^-(\{x_1, x_2\})| - 2 = |\{y\}, X'_X| + |\{y\}, X''_X| + |\{x_1\}, X'_X| + |\{x_2\}, X'_X| - 2 \\
\leq |X'_X| + |\{y\}, X'_X| + |\{X'_Y, \{x_1\}\}| + 2|X'_Y| + |\{X'_Y, \{x_2\}\}| - 2 \\
\leq 3 \max\{|X'_X|, |X''_X|\} + |\{y\}, X'_X| + |\{X'_Y, \{x_1\}\}| + |\{X'_Y, \{x_2\}\}| - 2 \\
\leq |S| = \lambda^3(D).
\]

So \(D\) is \(\lambda^3\)-optimal, a contradiction.

The proof is complete.

\[\square\]

**Claim 2.4.** For any \(x \in X'_X\), \(|N(X) \cap X'_X| \geq 2\).

**Proof.** Let \(X'_Y = \{y_1, y_2, \ldots, y_p\}\) and let \(S^* = \{s^* : s^* \in N^+(y_i) \cap N^-(y_j) \cap X'_X\}\), where \(i, j \in \{1, \ldots, p\}\) and \(i \neq j\). Then \(D[S^* \cup X'_X]\) is strong. Besides, by Claim 2.3, we have that for any \(i, j \in \{1, \ldots, p\}\) and \(i \neq j\), \(y_i, y_j \in X'_Y \cap X'_X\).

Therefore \(|V(D)| = 1 \leq |N^+(y_i) \cap N^-(y_j)| = |N^+(y_i) \cap N^-(y_j) \cap X'_X| + |N^+(y_i) \cap N^-(y_j) \cap X'_X| + |N^+(y_i) \cap N^-(y_j) \cap X'_X| + 1\). So \(|N^+(y_i) \cap N^-(y_j) \cap X'_X| \geq \left|\frac{|V(D)|}{4}\right| \geq 2\). Similarly, we can prove that \(|N^+(y_j) \cap N^-(y_i) \cap X'_X| \geq 2\). On the other hand, since \(|X'_X| \geq 2\), we have \(|S^* \cup X'_X| \geq 3\). For any \(x \in S^*\), clearly, \(|N(x) \cap X'_X| \geq 2\). Next, we claim that for any \(x \in X'_X \setminus S^*\), \(|N(X) \cap X'_X| \leq |X'_Y, \{x\}|\). Suppose there exists \(x^* \in X'_X \setminus S^*\) such that \(|N^+(x^*) \cap X'_X| > |X'_Y, \{x^*\}|\).

Since \(D[S^* \cup X'_X]\) is strong and \(|S^* \cup X'_X| \geq 3\), we have \(X'_Y \setminus \{x^*\}\) is a 3-restricted edge cut. Therefore \(|\partial^+(X'_Y \setminus \{x^*\})| = |S| - |N^+(x^*) \cap X'_X| + |X'_Y, \{x^*\}| < |S|\), a contradiction to the minimality of \(S\). Thus \(|X'_Y, \{x\}| \geq |N^+(x) \cap X'_X|\). By Claim 2.2, we have that \(|X'_Y, \{x\}| \geq 3\). The proof of Claim 2.4 is complete. \(\square\)

Let \(x_1 \in X'_X\) such that \(|N^+(x_1) \cap X''_X| \leq |N^+(u) \cap X''_X|\) for any \(u \in X'_X\), and let \(y_1, y_2 \in N(x_1) \cap X'_X\). Then

\[
\xi^3(D) \leq |\partial^+(\{x_1, y_1, y_2\})| = |\{x_1, y_1, y_2\}, X'_X \setminus \{x_1, y_1, y_2\}| + |\{x_1, y_1, y_2\}, X''_X| \\
\leq 2|X'_X| - 1 + |X'_X| - 2 + |\{x_1\}, X''_X| + |\{y_1\}, X''_X| + |\{y_2\}, X''_X| \\
\leq 3|X'_X| - 5 + |\{x_1\}, X''_X| + |\{y_1\}, X''_X| \\
+ |\{y_2\}, X''_X|(|X'_X| \geq |X'_X| + 1) \leq |\{x_1\}, X''_X| \times |X'_X| + |\{y_1\}, X''_X| + |\{y_2\}, X''_X| \leq |S| = \lambda^3(D).
\]

So \(D\) is \(\lambda^3\)-optimal, a contradiction.

The proof is complete.
From Theorem 2, we have following corollaries.

**Corollary 13.** Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta(D) \geq 3$. If for any $u, v \in V(D)$ in the same partite, $d^+(u) + d^-(v) \geq |V(D)| - 1$, then $D$ is $\lambda^3$-optimal.

**Corollary 14.** Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $|V(D)| \geq 6$. If $\delta(D) \geq \left\lceil \frac{|V(D)|}{2} \right\rceil$, then $D$ is $\lambda^3$-optimal.

(Unordered edges represent two arcs with the same end-vertices and opposite directions.)

**Figure 1.** The example from Remark 15.

**Remark 15.** To show that the condition “$|N^+(u) \cap N^-(v)| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1$” for any $u, v \in V(D)$ in the same partite” in Theorem 2 is sharp, we consider the digraph $T$ shown in Figure 1. Clearly, $|V(D)| \geq 6$ and $D$ is strong. There exists $x_1, y_1$ in the same partite such that $|N^+(x_1) \cap N^-(y_1)| = 2 < 3 = \left\lceil \frac{|V(D)|}{4} \right\rceil + 1$. Clearly, $\partial^+\{x_1, x_2, x_3, x_4\}$ is a 3-restricted edge cut and $\xi^3(T) = |\partial^+\{x_1, x_2, x_3\}| = 5.$ Therefore, $\lambda^3(T) \leq |\partial^+\{x_1, x_2, x_3, x_4\}| = 4 < 5 = \xi^3(T)$ and $T$ is not $\lambda^3$-optimal.

Besides, since $d^+(x_3) + d^-(y_4) = 6 < 7 = |V(T)| - 1$ and $\delta(T) = 3 < 4 = \left\lceil \frac{|V(D)|}{2} \right\rceil$, this example also shows that the conditions “$d^+(u) + d^-(v) \geq |V(D)| - 1$ for any $u, v \in V(D)$ in the same partite” in Corollary 13 and “$\delta(D) \geq \left\lceil \frac{|V(D)|}{2} \right\rceil$” in Corollary 14 are sharp.

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**References**


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