A SHORT PROOF FOR A LOWER BOUND ON THE ZERO FORCING NUMBER

Maximilian Füüst

and

Dieter Rautenbach

Institute of Optimization and Operations Research
Ulm University, Germany

e-mail: maximilian.fuerst@uni-ulm.de
dieter.rautenbach@uni-ulm.de

Abstract

We provide a short proof of a conjecture of Davila and Kenter concerning a lower bound on the zero forcing number $Z(G)$ of a graph $G$. More specifically, we show that $Z(G) \geq (g - 2)(\delta - 2) + 2$ for every graph $G$ of girth $g$ at least 3 and minimum degree $\delta$ at least 2.

Keywords: zero forcing, girth, Moore bound.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

We consider finite, simple, and undirected graphs and use standard terminology.

For an integer $n$, let $[n]$ denote the set of positive integers at most $n$. For a graph $G$, a set $Z$ of vertices of $G$ is a zero forcing set of $G$ if the elements of $V(G) \setminus Z$ have a linear order $u_1, \ldots, u_k$ such that, for every $i$ in $[k]$, there is some vertex $v_i$ in $Z \cup \{u_j : j \in [i - 1]\}$ such that $u_i$ is the only neighbor of $v_i$ outside of $Z \cup \{u_j : j \in [i - 1]\}$; in particular, $N_G[v_i] \setminus (Z \cup N_G[u_1] \cup \cdots \cup N_G[u_{i-1}]) = \{u_i\}$ for $i \in [k]$. The zero forcing number $Z(G)$ of $G$, defined as the minimum order of a zero forcing set of $G$, was proposed by the AIM Minimum Rank - Special Graphs Work Group [1] as an upper bound on the nullity of matrices associated with a given graph. The same parameter was also considered in connection with quantum physics [5, 7, 14] and logic circuits [6].
In [11] Davila and Kenter conjectured that
\[ Z(G) \geq (g - 2)(\delta - 2) + 2 \]
for every graph \( G \) of girth \( g \) at least 3 and minimum degree \( \delta \) at least 2. They observe that, for \( g > 6 \) and sufficiently large \( \delta \) in terms of \( g \), the conjectured bound follows by combining results from [3] and [8]. For \( g \leq 6 \), it was shown in [12, 13], Davila and Henning [9] showed it for \( 7 \leq g \leq 10 \), and, eventually, Davila, Kalinowski, and Stephen [10] completed the proof. The proof in [10] is rather short itself but relies on [12, 13, 9]. While the cases \( g \leq 6 \) have rather short proofs, the proof in [9] for \( 7 \leq g \leq 10 \) extends over more than eleven pages and requires a detailed case analysis. Therefore, the complete proof of (1) obtained by combining [9, 10, 12, 13] is rather long.

In the present note we propose a considerably shorter and simpler proof. Our approach only requires a special treatment for the triangle-free case \( g = 4 \) [12], involves a new lower bound on the zero forcing number, and an application of the Moore bound [2].

2. Proof of (1)

Our first result is a natural generalization of the well known fact \( Z(G) \geq \delta(G) \) [4], where \( \delta(G) \) is the minimum degree of a graph \( G \). For a set \( X \) of vertices of a graph \( G \) of order \( n \), let \( N_G(X) = (\bigcup_{u \in X} N_G(u)) \setminus X \), \( N_G[X] = X \cup N_G(X) \), and \( \delta_p(G) = \min \{|N_G(X)| : X \subseteq V(G) \text{ and } |X| = p\} \) for \( p \in [n] \). Note that \( \delta_1(G) \) equals \( \delta(G) \).

**Lemma 1.** If \( G \) is a graph of order \( n \), then \( Z(G) \geq \delta_p(G) \) for every \( p \in [n] \).

**Proof.** Let \( Z \) be a zero forcing set of minimum order. Let \( u_1, \ldots, u_k \) and \( v_1, \ldots, v_k \) be as in the introduction. Since, by definition, \( \delta_p(G) \leq n - p \), the result is trivial for \( p \geq k = n - |Z| \), and we may assume that \( p < k \). As noted above, we have \( N_G[v_i] \setminus (Z \cup N_G[v_1] \cup \cdots \cup N_G[v_{i-1}]) = \{u_i\} \) for \( i \in [k] \), which implies that \( X = \{v_1, \ldots, v_p\} \) is a set of \( p \) distinct vertices of \( G \). Furthermore, it implies that \( |N_G[X]| \leq |Z| + p \), and, hence, \( \delta_p(G) \leq |N_G(X)| = |N_G[X]| - p \leq |Z| \) as required.

For later reference, we recall the Moore bound for irregular graphs.

**Theorem 2** (Alon, Hoory and Linial [2]). If \( G \) is a graph of order \( n \), girth at least \( 2r \) for some integer \( r \), and average degree \( d \) at least 2, then \( n \geq 2 \sum_{i=0}^{r-1} (d-1)^i \).

We also need the following numerical fact.
Lemma 3. For positive integers $p$ and $q$ with $p \geq 5$ and $2p - 1 \leq q \leq \binom{p}{2}$,
\[
\left(1 + \frac{2(q - p)}{q + p}\right)^{\left\lceil \frac{q}{2} \right\rceil + 1} > q - p + 1.
\]

Proof. For $p \geq 17$, it follows from $q \geq 2p - 1$ that $1 + \frac{2(q - p)}{q + p} \geq 1.64$, and, since $1.64^{\left\lceil \frac{q}{2} \right\rceil + 1} > \binom{p}{2} - p + 1$, the desired inequality follows for these values of $p$. For the finitely many pairs $(p, q)$ with $5 \leq p \leq 16$ and $2p - 1 \leq q \leq \left\lceil \frac{p}{2} \right\rceil$, we verified it using a computer.

We proceed to the proof of (1).

Theorem 4. If $G$ is a graph of girth $g$ at least 3 and minimum degree $\delta$ at least 2, then $Z(G) \geq (g - 2)(\delta - 2) + 2$.

Proof. For $g = 3$, the inequality simplifies to the known fact $Z(G) \geq \delta(G)$, and, for $g = 4$, it has been shown in [12]. Now, let $g \geq 5$. Let $X$ be a set of $g - 2$ vertices of $G$ with $|N_G(X)| = \delta_{g-2}(G)$, and, let $N = N_G(X)$. By the girth condition, the components of $G[X]$ are trees, and no vertex in $N$ has more than one neighbor in any component of $G[X]$.

Let $K_1, \ldots, K_p$ be the vertex sets of the components of $G[X]$.

If $p \geq 3$ and there are two vertices in $N$ that both have neighbors in the same two distinct components of $G[X]$, then $G$ contains a cycle of order at most $2 + |K_i| + |K_j| \leq 2 + (g - 2) - (p - 2) < g$ which is a contradiction. Thus, $0 \leq |N_G(K_i) \cap N_G(K_j)| \leq 1$ for $1 \leq i < j \leq n$. Similarly, if $p = 2$, and there are three vertices $u$, $v$, and $w$ in $N$ that all three have neighbors in $K_1$ and $K_2$, then let $u_1$, $v_1$, and $w_1$ denote the corresponding neighbors in $K_i$ for $i \in \{1, 2\}$, respectively. If any of $u_1$, $v_1$, and $w_1$ are distinct, then $G[K_1]$ contains a path between two of the vertices $u_1$, $v_1$, and $w_1$ avoiding the third, and $G$ contains a cycle of order at most $2 + (|K_1| - 1) + |K_2| = g - 1$, which is a contradiction. By symmetry, this implies $u_1 = v_1 = w_1$ and $u_2 = v_2 = w_2$, and $G$ contains the cycle $u_1w_2u_1$ of order 4, which is a contradiction. Thus, $0 \leq |N_G(K_1) \cap N_G(K_2)| \leq 2$.

Combining these observations, we obtain

\[
\sum_{1 \leq i < j \leq p} |N_G(K_i) \cap N_G(K_j)| \leq \begin{cases} \binom{p}{2}, & \text{for } p \geq 3, \\ 2p - 2, & \text{for } p \in \{1, 2\}. \end{cases}
\]

Let the bipartite graph $H$ arise from $G[X \cup N]$ by contracting the component $K_i$ of $G[X]$ to a single vertex $u_i$ for every $i \in [p]$, and removing all edges of $G[N]$. Note that $\sum_{i \in [p]} d_H(u_i) - \sum_{v \in N} d_H(v) = 0$ in the bipartite graph $H$ with partite sets $\{u_1, \ldots, u_p\}$ and $N$. By the girth condition, no vertex in $N$ has two neighbors in $K_i$, and $K_i$ induces a tree, which implies $d_H(u_i) = \sum_{v \in K_i} d_G(v) - 2(|K_i| - 1) \geq \frac{\delta_G(K_i)}{2}$.
δ|Ki| − 2(|Ki| − 1) for every \( i \in [p] \). Let \( q = \sum_{v \in N} (d_H(v) − 1) \). Now, Lemma 1 implies

\[
Z(G) \geq \delta_{g-2}(G) = |N| = \sum_{v \in N} 1 + \left( \sum_{i \in [p]} d_H(u_i) - \sum_{v \in N} d_H(v) \right)
\]

\[
= \sum_{i \in [p]} d_H(u_i) - q \geq \sum_{i=1}^{p} \left( \delta|K_i| - 2(|K_i| - 1) \right) - q
\]

\[
= (g - 2)(\delta - 2) + 2 + ((2p - 2) - q).
\]

If \( q \leq 2p - 2 \), then this implies (1). Hence, we may assume \( q \geq 2p - 1 \).

Note that

\[
2p - 1 \leq q = \sum_{v \in N} (d_H(v) - 1) \leq \sum_{v \in N} \left( \frac{d_H(v)}{2} \right) = \sum_{1 \leq i < j \leq p} |N_G(K_i) \cap N_G(K_j)|,
\]

where the last equality follows, because every vertex \( v \) in \( N \) contributes exactly \( \left( \frac{d_H(v)}{2} \right) \) to the right hand side. Now, (2) implies \( p \geq 5 \).

Let \( H' \) arise by removing all vertices of degree 1 from \( H \). Since, for every \( i \in [p] \), we have \( d_H(u_i) \geq \delta|K_i| - 2(|K_i| - 1) \geq 2 \), the graph \( H' \) contains all \( p \) vertices \( u_1, \ldots, u_p \). Let \( H' \) contain \( r \) vertices of \( N \). Since \( H' \) has order \( p + r \) and size

\[
\sum_{v \in N \cap V(H')} d_H(v) = r + \sum_{v \in N} (d_H(v) - 1) = r + q,
\]

its average degree is at least \( \frac{2(r + q)}{p + r} \), which is at least 2, because \( q \geq 2p - 1 \geq p \).

If \( H' \) contains a cycle of order \( 2\ell \), then \( G \) contains a cycle that alternates between \( X \) and \( N \), contains \( \ell \) vertices from \( N \), and avoids \( p - \ell \) of the components of \( G[X] \), which implies that this cycle has order at most \( \ell + (|X| - (p - \ell)) = \ell + (g - 2) - (p - \ell) \). By the girth condition, this implies that the bipartite graph \( H' \) has girth at least \( p + 2 \), if \( p \) is even, and \( p + 3 \), if \( p \) is odd.

Using Theorem 2 and \( q \geq r \), we obtain

\[
p + r \geq 2 \sum_{i=0}^{\left\lfloor \frac{p}{2} \right\rfloor} \left( \frac{2(r + q)}{p + r} - 1 \right)^i = \frac{2p + r}{2(q - p)} \left( 1 + \frac{2(q - p)}{p + r} \right)^{\left\lfloor \frac{p}{2} \right\rfloor + 1} - 1
\]

\[
\geq \frac{2p + r}{2(q - p)} \left( 1 + \frac{2(q - p)}{p + q} \right)^{\left\lfloor \frac{q}{2} \right\rfloor + 1} - 1,
\]

which implies \( \left( 1 + \frac{2(q - p)}{q + p} \right)^{\left\lfloor \frac{q}{2} \right\rfloor + 1} \leq q - p + 1 \). Since \( q \geq 2p - 1 \), and, by (2), \( q \leq \left( \frac{p}{2} \right) \), this contradicts Lemma 3, which completes the proof.
A Short Proof for a Lower Bound on the Zero Forcing Number

References


