A SHORT PROOF FOR A LOWER BOUND ON THE ZERO FORCING NUMBER

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Abstract

We provide a short proof of a conjecture of Davila and Kenter concerning a lower bound on the zero forcing number \( Z(G) \) of a graph \( G \). More specifically, we show that \( Z(G) \geq (g - 2)(\delta - 2) + 2 \) for every graph \( G \) of girth \( g \) at least 3 and minimum degree \( \delta \) at least 2.

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1. Introduction

We consider finite, simple, and undirected graphs and use standard terminology. For an integer \( n \), let \([n]\) denote the set of positive integers at most \( n \). For a graph \( G \), a set \( Z \) of vertices of \( G \) is a zero forcing set of \( G \) if the elements of \( V(G) \setminus Z \) have a linear order \( u_1, \ldots, u_k \) such that, for every \( i \) in \([k]\), there is some vertex \( v_i \) in \( Z \cup \{ u_j : j \in [i-1] \} \) such that \( u_i \) is the only neighbor of \( v_i \) outside of \( Z \cup \{ u_j : j \in [i-1] \} \); in particular, \( N_G[v_i] \setminus (Z \cup N_G[v_i] \cup \cdots \cup N_G[v_{i-1}]) = \{ u_i \} \) for \( i \in [k] \). The zero forcing number \( Z(G) \) of \( G \), defined as the minimum order of a zero forcing set of \( G \), was proposed by the AIM Minimum Rank - Special Graphs Work Group [1] as an upper bound on the nullity of matrices associated with a given graph. The same parameter was also considered in connection with quantum physics [5, 7, 14] and logic circuits [6].
In [11] Davila and Kenter conjectured that

\[ Z(G) \geq (g - 2)(\delta - 2) + 2 \]

for every graph \( G \) of girth \( g \) at least 3 and minimum degree \( \delta \) at least 2. They observe that, for \( g > 6 \) and sufficiently large \( \delta \) in terms of \( g \), the conjectured bound follows by combining results from [3] and [8]. For \( g \leq 6 \), it was shown in [12, 13], Davila and Henning [9] showed it for \( 7 \leq g \leq 10 \), and, eventually, Davila, Kalinowski, and Stephen [10] completed the proof. The proof in [10] is rather short itself but relies on [12, 13, 9]. While the cases \( g \leq 6 \) have rather short proofs, the proof in [9] for \( 7 \leq g \leq 10 \) extends over more than eleven pages and requires a detailed case analysis. Therefore, the complete proof of (1) obtained by combining [9, 10, 12, 13] is rather long.

In the present note we propose a considerably shorter and simpler proof. Our approach only requires a special treatment for the triangle-free case \( g = 4 \) [12], involves a new lower bound on the zero forcing number, and an application of the Moore bound [2].

2. Proof of (1)

Our first result is a natural generalization of the well known fact \( Z(G) \geq \delta(G) \) [4], where \( \delta(G) \) is the minimum degree of a graph \( G \). For a set \( X \) of vertices of a graph \( G \) of order \( n \), let \( N_G(X) = \bigcup_{u \in X} N_G(u) \setminus X \), \( N_G[X] = X \cup N_G(X) \), and \( \delta_p(G) = \min \{|N_G(X)| : X \subseteq V(G) \text{ and } |X| = p\} \) for \( p \in [n] \). Note that \( \delta_1(G) \) equals \( \delta(G) \).

**Lemma 1.** If \( G \) is a graph of order \( n \), then \( Z(G) \geq \delta_p(G) \) for every \( p \in [n] \).

**Proof.** Let \( Z \) be a zero forcing set of minimum order. Let \( u_1, \ldots, u_k \) and \( v_1, \ldots, v_k \) be as in the introduction. Since, by definition, \( \delta_p(G) \leq n - p \), the result is trivial for \( p \geq k = n - |Z| \), and we may assume that \( p < k \). As noted above, we have \( N_G[v_i] \setminus (Z \cup N_G[v_1] \cup \cdots \cup N_G[v_{i-1}]) = \{u_i\} \) for \( i \in [k] \), which implies that \( X = \{v_1, \ldots, v_p\} \) is a set of \( p \) distinct vertices of \( G \). Furthermore, it implies that \( |N_G[X]| \leq |Z| + p \), and, hence, \( \delta_p(G) \leq |N_G(X)| = |N_G[X]| - p \leq |Z| \) as required.

For later reference, we recall the Moore bound for irregular graphs.

**Theorem 2** (Alon, Hoory and Linial [2]). If \( G \) is a graph of order \( n \), girth at least \( 2r \) for some integer \( r \), and average degree \( d \) at least 2, then \( n \geq 2 \sum_{i=0}^{r-1} (d-1)^i \).

We also need the following numerical fact.
Lemma 3. For positive integers \( p \) and \( q \) with \( p \geq 5 \) and \( 2p - 1 \leq q \leq \binom{p}{2} \),
\[
1 + \frac{2(q-p)}{q+p} \geq q - p + 1.
\]

Proof. For \( p \geq 17 \), it follows from \( q \geq 2p - 1 \) that \( 1 + \frac{2(q-p)}{q+p} \geq 1.64 \), and, since \( 1.64[\frac{p}{2}] + 1 > \binom{p}{2} - p + 1 \), the desired inequality follows for these values of \( p \). For the finitely many pairs \( (p,q) \) with \( 5 \leq p \leq 16 \) and \( 2p - 1 \leq q \leq \binom{p}{2} \), we verified it using a computer.

We proceed to the proof of (1).

Theorem 4. If \( G \) is a graph of girth \( g \) at least 3 and minimum degree \( \delta \) at least 2, then \( Z(G) \geq (g-2)(\delta-2) + 2 \).

Proof. For \( g = 3 \), the inequality simplifies to the known fact \( Z(G) \geq \delta(G) \), and, for \( g = 4 \), it has been shown in [12]. Now, let \( g \geq 5 \). Let \( G \) be a set of \( g - 2 \) vertices of \( G \) with \( |N_G(X)| = \delta_{g-2}(G) \), and, let \( N = N_G(X) \). By the girth condition, the components of \( G[X] \) are trees, and no vertex in \( N \) has more than one neighbor in any component of \( G[X] \).

Let \( K_1, \ldots, K_p \) be the vertex sets of the components of \( G[X] \).

If \( p \geq 3 \) and there are two vertices in \( N \) that both have neighbors in the same two distinct components of \( G[X] \), then \( G \) contains a cycle of order at most \( 2 + |K_i| + |K_j| \leq 2 + (g-2) - (p-2) < g \) which is a contradiction. Thus, \( 0 \leq |N_G(K_i) \cap N_G(K_j)| \leq 1 \) for \( 1 \leq i < j \leq n \). Similarly, if \( p = 2 \), and there are three vertices \( u, v, \) and \( w \) in \( N \) that all three have neighbors in \( K_1 \) and \( K_2 \), then let \( u_1, v_1, \) and \( w_1 \) denote the corresponding neighbors in \( K_i \) for \( i \in \{1,2\} \), respectively. If any of \( u_1, v_1, \) and \( w_1 \) are distinct, then \( G[K_1] \) contains a path between two of the vertices \( u_1, v_1, \) and \( w_1 \) avoiding the third, and \( G \) contains a cycle of order at most \( 2 + (|K_1| - 1) + |K_2| = g - 1 \), which is a contradiction. By symmetry, this implies \( u_1 = v_1 = w_1 \) and \( u_2 = v_2 = w_2 \), and \( G \) contains the cycle \( u_1u_2v_1u_1 \) of order 4, which is a contradiction. Thus, \( 0 \leq |N_G(K_1) \cap N_G(K_2)| \leq 2 \).

Combining these observations, we obtain
\[
(2) \quad \sum_{1 \leq i < j \leq p} |N_G(K_i) \cap N_G(K_j)| \leq \begin{cases} \binom{p}{2}, & \text{for } p \geq 3, \text{ and} \\ 2p - 2, & \text{for } p \in \{1,2\}. \end{cases}
\]

Let the bipartite graph \( H \) arise from \( G[X \cup N] \) by contracting the component \( K_i \) of \( G[X] \) to a single vertex \( u_i \) for every \( i \in [p] \), and removing all edges of \( G[N] \). Note that \( \sum_{i \in [p]} d_H(u_i) - \sum_{v \in N} d_H(v) = 0 \) in the bipartite graph \( H \) with partite sets \( \{u_1, \ldots, u_p\} \) and \( N \). By the girth condition, no vertex in \( N \) has two neighbors in \( K_i \), and \( K_i \) induces a tree, which implies \( d_H(u_i) = \sum_{v \in K_i} d_G(v) - 2(|K_i| - 1) \geq 0 \).
\[ \delta(K_i) - 2(|K_i| - 1) \] for every \( i \in [p] \). Let \( q = \sum_{v \in N}(d_H(v) - 1) \). Now, Lemma 1 implies

\[
Z(G) \geq \delta_{g-2}(G) = |N| = \sum_{v \in N} 1 + \left( \sum_{i \in [p]} d_H(u_i) - \sum_{v \in N} d_H(v) \right) \\
= \sum_{i \in [p]} d_H(u_i) - q \geq \sum_{i=1}^{p} \left( \delta|K_i| - 2(|K_i| - 1) \right) - q \\
= (g - 2)(\delta - 2) + 2 + ((2p - 2) - q).
\]

If \( q \leq 2p - 2 \), then this implies (1). Hence, we may assume \( q \geq 2p - 1 \).

Note that

\[
2p - 1 \leq q = \sum_{v \in N}(d_H(v) - 1) \leq \sum_{v \in N} \left( \frac{d_H(v)}{2} \right) = \sum_{1 \leq i < j \leq p} |N_G(K_i) \cap N_G(K_j)|,
\]

where the last equality follows, because every vertex \( v \) in \( N \) contributes exactly \( \left( \frac{d_H(v)}{2} \right) \) to the right hand side. Now, (2) implies \( p \geq 5 \).

Let \( H' \) arise by removing all vertices of degree 1 from \( H \). Since, for every \( i \in [p] \), we have \( d_H(u_i) \geq \delta|K_i| - 2(|K_i| - 1) \geq 2 \), the graph \( H' \) contains all \( p \) vertices \( u_1, \ldots, u_p \). Let \( H' \) contain \( r \) vertices of \( N \). Since \( H' \) has order \( p + r \) and size

\[
\sum_{v \in N \cap V(H')} d_H(v) = r + \sum_{v \in N} (d_H(v) - 1) = r + q,
\]

its average degree is at least \( \frac{2(r+q)}{p+r} \), which is at least 2, because \( q \geq 2p - 1 \geq 2 \).

If \( H' \) contains a cycle of order \( 2\ell \), then \( G \) contains a cycle that alternates between \( X \) and \( N \), contains \( \ell \) vertices from \( N \), and avoids \( p - \ell \) of the components of \( G[X] \), which implies that this cycle has order at most \( \ell + (|X| - (p - \ell)) = \ell + (g - 2) - (p - \ell) \). By the girth condition, this implies that the bipartite graph \( H' \) has girth at least \( \ell + 2 \), if \( p \) is even, and \( p + 3 \), if \( p \) is odd.

Using Theorem 2 and \( q \geq r \), we obtain

\[
p + r \geq 2 \sum_{i=0}^{\left\lceil \frac{q}{2} \right\rceil} \left( \frac{2(r + q)}{p + r} - 1 \right)^i = \frac{2(p + r)}{2(q - p)} \left( 1 + \frac{2(q - p)}{p + r} \right) \left( \left\lceil \frac{q}{2} \right\rceil + 1 \right) - 1 \\
\geq \frac{2p + r}{2(q - p)} \left( 1 + \frac{2(q - p)}{p + q} \right) \left( \left\lceil \frac{q}{2} \right\rceil + 1 \right) - 1,
\]

which implies \( \left( 1 + \frac{2(q - p)}{q + p} \right) \left( \left\lceil \frac{q}{2} \right\rceil + 1 \right) \leq q - p + 1 \). Since \( q \geq 2p - 1 \), and, by (2), \( q \leq \left( \frac{p}{2} \right) \), this contradicts Lemma 3, which completes the proof. 
\[ \blacksquare \]
References


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