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INCIDENCE COLORING—COLD CASES

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Abstract

An *incidence* in a graph G is a pair (v, e) where v is a vertex of G and e is an edge of G incident to v. Two incidences (v, e) and (u, f) are adjacent if at least one of the following holds: (i) $v = u$, (ii) $e = f$, or (iii) edge vu is from the set $\{e, f\}$. An *incidence coloring* of G is a coloring of its incidences assigning distinct colors to adjacent incidences. The minimum number of colors needed for incidence coloring of a graph is called the *incidence chromatic number*.

It was proved that at most $\Delta(G) + 5$ colors are enough for an incidence coloring of any planar graph G except for $\Delta(G) = 6$, in which case at most 12 colors are needed. It is also known that every planar graph G with girth at least 6 and $\Delta(G) \geq 5$ has incidence chromatic number at most $\Delta(G) + 2$.

In this paper we present some results on graphs regarding their maximum degree and maximum average degree. We improve the bound for planar graphs with $\Delta(G) = 6$. We show that the incidence chromatic number is at most $\Delta(G)+2$ for any graph G with $\mathrm{mad}(G) < 3$ and $\Delta(G) = 4$, and for any graph with $\mathrm{mad}(G) < \frac{10}{3}$ and $\Delta(G) \geq 8$.

Keywords: incidence coloring, incidence chromatic number, planar graph, maximum average degree.

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1. Introduction

Incidence coloring was defined by Brualdi and Massey [2] as a tool to study strong edge colorings of bipartite graphs. However, soon after its definition, the coloring itself attracted the attention of several researchers from different points of view.

An *incidence* in a graph G is a pair (v,e) where v is a vertex of G and e is an edge of G incident to v. Two incidences (v,e) and (u,f) are *adjacent* if at least one of the following holds: (i) $v = u$, (ii) $e = f$, or (iii) edge vu is from the set $\{e,f\}$. An *incidence coloring* of G is a coloring of its incidences assigning distinct colors to adjacent incidences. The minimum number of colors needed for incidence coloring of a graph is called the *incidence chromatic number* of G, denoted by $\chi_i(G)$.

Brualdi and Massey [2] conjectured that $\chi_i(G) \leq \Delta(G)+2$ for any graph G, where $\Delta(G)$ denotes the maximum degree of G. The conjecture was disproved by Guiduli [3], who showed that Paley graphs with maximum degree Δ have incidence chromatic number at least $\Delta + \Omega\big(\log \Delta\big)$. However, for many of the commonly considered graph classes the incidence chromatic number is bounded by $\Delta + c$ for some constant c, and several papers are devoted to the proof of this type of result, including the following one.

Theorem 1 (Maydanskiy, 2005)**.** *Five colors suffice for an incidence coloring of any subcubic graph.*

In order to obtain upper bounds on the incidence chromatic number, in many cases, stronger statements concerning incidence colorings with further local constraints are proved, allowing to apply induction in a more efficient way.

An incidence coloring of a graph G using k colors is an *incidence* (k,p)*-coloring* of G if for every vertex v of G, the number of colors used for coloring the incidences of the form (u,uv) is at most p.

Hosseini Dolama, Sopena and Zhu [5] proved that every planar graph with maximum degree Δ admits an incidence $(\Delta+7,7)$-coloring and, thus, has incidence chromatic number at most $\Delta+7$. This bound was further improved to $\Delta+4$ for triangle-free planar graphs [6], to $\Delta+3$ (respectively, $\Delta+2$, $\Delta+1$) for planar graphs of girth at least 6 (respectively, 11, 16) [6]. The last result was further improved to girth 14 [1].

Some of these results were proved for more general graph classes, namely graphs with bounded maximum average degree. The *average degree* of a graph G is the mean value of the degrees of its vertices. The *maximum average degree* $\text{mad}(G)$ of a graph G is then defined as the maximum value of the average degrees of its subgraphs. When G is a planar graph with girth g, it is folklore to establish the inequality $\text{mad}(G) < \frac{2g}{g-2}$.

In [6] the authors proved the following result.

Theorem 2 (Hosseini Dolama, Sopena, 2005)**.** *Let G be a graph with* $\text{mad}(G) < 3$ *and $\Delta(G) \geq 5$. Then G admits a $\big(\Delta(G)+2, 2\big)$-incidence coloring. Therefore, $\chi_i(G) \leq \Delta(G) + 2$.*

In Section 2 we extend this result to $\text{mad}(G) < 3$ and $\Delta(G) \geq 4$ (Theorem 4). Moreover, we present another result for graphs with larger maximum average degree (Theorem 5).

Recall that the *star arboricity* of an undirected graph G is the smallest number of star forests needed to cover G. Yang [8] observed the following: let G be an undirected graph with star arboricity $st(G)$, let $s : E(G) \to \{1, \ldots, st(G)\}$ be a mapping such that $s^{-1}(i)$ is a forest of stars for every i, $1 \leq i \leq st(G)$, and let λ be a proper edge coloring of G. Now define the mapping f by $f(u, uv) = s(uv)$ if v is the center of a star in some forest $s^{-1}(i)$ (if some star is reduced to one edge, we arbitrarily choose one of its end vertices as the center) and $f(u, uv) = \lambda(uv)$ otherwise. It is not difficult to check that f is indeed an incidence coloring of G. Therefore, thanks to the classical result of Vizing, the relation $\chi_i(G) \leq \Delta(G) + st(G)$ (respectively, $\chi_i(G) \leq \Delta(G) + st(G) + 1$) holds for every graph of class 1 (respectively, of class 2). (Recall that the chromatic index $\chi'(G)$ of any graph G is either $\Delta(G)$—such graphs are said to be of class 1—or $\Delta(G)+1$—such graphs are said to be of class 2.) The facts that planar graphs with $\Delta \geq 7$ are class 1 [7] and that the star arboricity of any planar graph is at most 5 [4] led to the following result.

Theorem 3 (Yang, 2007)**.** *If G is a planar graph with $\Delta(G) \neq 6$, then $\chi_i(G) \leq \Delta(G) + 5$. If $\Delta(G) = 6$, then $\chi_i(G) \leq \Delta(G) + 6$.*

Yang [8] proposed the following question: Are $\Delta(G) + 5$ colors enough for graphs with maximum degree 6? We give a positive answer to this question (in a stronger form) in Section 3.

2. Graphs with Bounded Maximum Average Degree

In this section we present two results: one of them extends Theorem 2, the other one concerns graphs with larger maximum average degree.

Theorem 4. *Let G be a graph with* $\mathrm{mad}(G) < 3$ *and* $\Delta(G) \geq 4$*. Then G admits a* $(\Delta(G)+2, 2)$*-incidence coloring. Therefore,* $\chi_i(G) \leq \Delta(G) + 2$*.*

Theorem 5. *Let G be a graph with* $\mathrm{mad}(G) < \frac{10}{3}$ *and* $\Delta(G) \geq 8$*. Then G admits a* $(\Delta(G)+2, 2)$*-incidence coloring. Therefore,* $\chi_i(G) \leq \Delta(G) + 2$*.*

2.1. Reducible configurations

We first introduce some additional notation used in the proofs of both results. We denote by $\deg_G(v)$ the degree of a vertex v in a graph G. By a k-vertex, a k^+-vertex and a k^--vertex, we mean a vertex of degree k, at least k and at most k, respectively. A (k_1, k_2)-edge is an edge v_1v_2 such that for every $i \in \{1, 2\}$, v_i is a k_i-vertex. More generally, a $(k_1, k_2, \ldots, k_\ell)$-path (respectively, a $(k_1, k_2, \ldots, k_\ell)$-cycle), $\ell \geq 3$, is a path (respectively, a cycle) $v_1v_2 \cdots v_\ell$ such that for every i, $1 \leq i \leq \ell$, v_i is a k_i-vertex.

Let c be a partial incidence coloring of a graph G. We say that a color a is *admissible* for an (uncolored) incidence (v, e) in G if there is no incidence colored by a adjacent to (v, e); otherwise the color a is *forbidden.* We denote $F^c(v, e)$ the set of forbidden colors for the incidence (v, e).

Let v be a vertex of G. We set $I_v := \{(v, uv) : uv \in E(G)\}$ and $A_v := \{(u, uv) : uv \in E(G)\}$. If c is a partial incidence coloring of G, we necessarily have $c(I_v) \cap c(A_v) = \emptyset$ for each vertex v of G. Moreover, if c is a partial $(k, 2)$-incidence coloring of G, then $|c(A_v)| \leq 2$. By $A^c(v)$ we will denote a set of exactly two colors such that $A^c(v) \supseteq c(A_v)$ and $A^c(v) \cap c(I_v) = \emptyset$.

We now prove a series of lemmas.

Lemma 6. *Let G be a graph, v be a 1-vertex in G and $k \geq \Delta(G)+2$ be an integer. If $G - v$ admits a $(k, 2)$-incidence coloring, then G also admits a $(k, 2)$-incidence coloring.*

Proof. Let c be a $(k, 2)$-incidence coloring of $G - v$, and w denote the unique neighbor of v in G. We will extend c to a $(k, 2)$-incidence coloring of G. Since $|F^c(w, wv)| = |c(I_w) \cup c(A_w)| \leq \Delta(G) - 1 + 2 = \Delta(G) + 1$, there is an admissible color a for (w, wv). We then set $c(w, wv) = a$ and $c(v, vw) = b$ for any color b in $A^c(w)$. Clearly, c is a $(k, 2)$-incidence coloring of G. ∎

Lemma 7. *Let G be a graph, $k \geq \Delta(G)+2$ be an integer, and uv be a $(2, (k-3)^-)$-edge in G. If $G - uv$ admits a $(k, 2)$-incidence coloring, then G also admits a $(k, 2)$-incidence coloring.*

Proof. Let w be the other neighbor of u in G and c be a $(k, 2)$-incidence coloring of $G - e$; $e = uv$. We extend c to a $(k, 2)$-incidence coloring of G in the following way. We first uncolor (u, uw). We then set $c(u, e) = a$, for some color $a \in A^c(v) - c(w, uw)$, and $c(u, uw) = b$ for some color $b \in A^c(w) - c(u, e)$. Finally, since $\left|F^c(v,e)\right| = \left|c(I_v) \cup c(A_v) \cup \{c(u,uw)\}\right| \leq (k-4)+2+1 = k-1 < k$, there is an admissible color for (v,e), so that we can complete the coloring. ■

Lemma 8. *Let G be a graph with no 1-vertices and $k \geq \Delta(G)+2$ be an integer. Let v be an s-vertex in G, $s \geq 3$, adjacent to at most one 3^+-vertex, and let u_i, $1 \leq i \leq s-1$, denote the 2-neighbors of v. If the graph $G - \{vu_i, 1 \leq i \leq s-1\}$ admits a $(k,2)$-incidence coloring, then G also admits a $(k,2)$-incidence coloring.*

Proof. Let $e_i = vu_i$, $f_i = u_iw_i$ be the other edge incident to u_i for every i, $1 \leq i \leq s-1$, and u_s be the last neighbor of v and $e_s = vu_s$. Let c be a $(k,2)$-incidence coloring of $G - \{e_i, 1 \leq i \leq s-1\}$. We extend c to a $(k,2)$-incidence coloring of G as follows.

We first uncolor (v,e_s) and all incidences (u_i,f_i), $1 \leq i \leq s-1$. Let $a_i = c(w_i,f_i)$, $1 \leq i \leq s-1$. Since we have k colors and $k \geq \Delta(G)+2$, there is a color t not in $\{a_i, 1 \leq i \leq s-1\}$; moreover, we can choose t such that $t \notin A^c(w_1)$. We then set $c(u_i,u_iv) = t$, $1 \leq i \leq s-1$.

Next, for every i, $2 \leq i \leq s-1$, we set $c(u_i,f_i) = t_i$ with $t_i \in A^c(w_i) - \{t\}$, $c(v,e_s) = t_s$ with $t_s \in A^c(u_s) - \{t\}$, and $c(u_1,f_1) = t_1$ with $t_1 \in A^c(w_1) - \{t_2\}$.

Now $F^c(v,e_i) = \{t, c(u_i,f_i), c(u_s,e_s), c(v,e_s)\}$. Therefore we have at least $k-4 \geq s-2$ admissible colors for every uncolored incidence. As $c(u_1,f_1) \neq c(u_2,f_2)$, we can choose at least $s-1$ distinct colors b_i such that $b_i \notin F^c(v,e_i)$, and we set $c(v,e_i) = b_i$ for every i, $1 \leq i \leq s-1$. ■

Lemma 9. *Let G be a graph with $\Delta(G) \geq 7$, $k \geq \Delta(G)+2$ be an integer, and $C = v_1v_2v_3$ be a $(3,3,3)$-cycle in G. If the graph $G - \{v_1v_2, v_2v_3, v_3v_1\}$ admits a $(k,2)$-incidence coloring, then G also admits a $(k,2)$-incidence coloring.*

Proof. Let c be a $(k,2)$-incidence coloring of $G - \{v_1v_2, v_2v_3, v_3v_1\}$. Let u_i be the neighbor of v_i not included in C, $1 \leq i \leq 3$. We extend c to a $(k,2)$-incidence coloring of G as follows. Let $a_i = c(u_i,u_iv_i)$, $b_i = c(v_i,v_iu_i)$, $1 \leq i \leq 3$. Since $k \geq 9$, there are three colors $c_1, c_2, c_3 \notin \{a_i,\ 1 \leq i \leq 3\} \cup \{b_i,\ 1 \leq i \leq 3\}$. We then color the six incidences of C, cyclically, with colors $c_1, c_2, c_3, c_1, c_2, c_3$. ■

Lemma 10. *Let G be a graph with $\Delta(G) \geq 8$, $k \geq \Delta(G)+2$ be an integer, and $P = u_1v_1v_2u_2$ be a $(4^-,3,3,4^-)$-path in G. If the graph $G - \{u_1v_1, v_1v_2, v_2u_2\}$ admits a $(k,2)$-incidence coloring, then G also admits a $(k,2)$-incidence coloring.*

Proof. Let c be a $(k,2)$-incidence coloring of $G - \{u_1v_1, v_1v_2, v_2u_2\}$ and w_i be the third neighbor of v_i, $i = 1,2$. We will extend c to a $(k,2)$-incidence coloring of G.

We can assume that $\{c(w_i,w_iv_i), c(v_i,v_iw_i)\} \neq A^c(u_i)$, $i = 1,2$ (otherwise we recolor (v_i,v_iw_i) using the other color from $A^c(w_i)$). Thus we can set $c(v_i,v_iu_i) = t_i$ with $t_i \in A^c(u_i) - \{c(w_i,w_iv_i), c(v_i,v_iw_i)\}$, $i = 1,2$.

We now consider three cases:

Case 1. $c(w_2, w_2v_2) \notin c(I_{v_1}) \cup c(A_{v_1})$. We first set $c(v_1, v_1v_2) = c(w_2, w_2v_2)$. Since $k \geq 10$, there exists a color $c_1 \notin c(I_{u_1}) \cup c(A_{u_1}) \cup \{c(v_1, v_1w_1), c(v_2, v_2w_2), c(w_2, w_2v_2), c(v_2, v_2u_2)\}$. We then set $c(u_1, u_1v_1) = c(v_2, v_2v_1) = c_1$. Since the incidence (u_2, u_2v_2) is adjacent to at most nine other incidences, it can be colored.

Case 2. $c(w_1, w_1v_1) \notin c(I_{v_2}) \cup c(A_{v_2})$. We proceed similarly as in the previous case.

Case 3. $c(w_1, w_1v_1) \in c(I_{v_2}) \cup c(A_{v_2})$ and $c(w_2, w_2v_2) \in c(I_{v_1}) \cup c(A_{v_1})$. We will color the incidences (u_1, u_1v_1) and (v_2, v_2v_1) with a common color c_1, and the incidences (u_2, u_2v_2) and (v_1, v_1v_2) with a common color c_2. Note that we have at most nine forbidden colors for each of c_1 and c_2. If we can choose $c_1 \neq c_2$, we are done. If not, we necessarily have $k = 10$, the sets of forbidden colors for c_1 and c_2 are the same, and both contain nine distinct colors. Since in this case we have $c(w_1, w_1v_1) \in c(I_{v_2}) \cup c(A_{v_2})$ and $c(w_1, w_1v_1), c(v_2, v_2u_2), c(v_2, v_2w_2)$ are different (they are different forbidden colors for c_2), we get $c(w_1, w_1v_1) = c(w_2, w_2v_2)$. Without loss of generality, we may assume that $c(w_1, w_1v_1) = c(w_2, w_2v_2) = 9$, $c(v_1, v_1w_1) = 8$, $c(v_1, v_1u_1) = 7$, $c(v_2, v_2w_2) = 6$, and $c(v_2, v_2u_2) = 5$ (see Figure 1). Then $c(I_{u_2}) \cup c(A_{u_2}) = \{1, 2, 3, 4, 5\}$ and $c(I_{u_1}) \cup c(A_{u_1}) = \{1, 2, 3, 4, 7\}$. We can replace $c(v_1, v_1u_1)$ with the other color from $c(A_{u_1})$. Now, 7 is no more forbidden for c_2, so we have only eight forbidden colors for c_2. Therefore, we can now choose $c_1 \neq c_2$ to obtain the desired coloring. ■

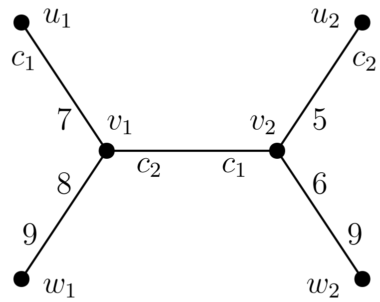

Figure 1. A partial incidence coloring of a $(4^-, 3, 3, 4^-)$-path.

2.2. Discharging rules

2.2.1. Proof of Theorem 4

We prove Theorem 4 by contradiction. Let $\Delta_0 \geq 4$ and G be a minimal counterexample (with respect to the number of vertices) with $\text{mad}(G) < 3$, $\Delta(G) \leq \Delta_0$ and with no $(\Delta_0 + 2, 2)$-incidence coloring. From Theorem 1 and Lemmas 6, 7 and 8 it follows that $\delta(G) \geq 2$, every 2-vertex in G is adjacent to two Δ_0-vertices and every 3^+-vertex is adjacent to at least two 3^+-vertices. Moreover, $\Delta_0 = \Delta(G)$. We will reach a contradiction by using the discharging method.

We assign an initial charge $\omega(v) = \deg_G(v)$ to each vertex v of G, and we use the following discharging rule: each 4^+-vertex gives $\frac{1}{2}$ to each of its 2-neighbors. We shall prove that the new charge $\omega'(v)$ of each vertex v of G is at least 3, which contradicts our assumption $\text{mad}(G) < 3$ (since $\sum_{v \in G} \omega'(v) = \sum_{v \in G} \omega(v)$).

Let v be a vertex of G. We consider three cases, according to $\deg_G(v)$.

Case 1. $\deg_G(v) = 2$. Every 2-vertex in G is adjacent to two $\Delta(G)$-vertices. Therefore, since $\Delta(G) \geq 4$, $\omega'(v) = 2 + 2 \times \frac{1}{2} = 3$.

Case 2. $\deg_G(v) = 3$. The discharging rule does not involve 3-vertices, thus $\omega'(v) = \omega(v) = 3$.

Case 3. $\deg_G(v) = d \geq 4$. Since every d-vertex is adjacent to at most $(d-2)$ 2-vertices, $\omega'(v) \geq d - \frac{1}{2}(d-2) = \frac{d+2}{2} \geq 3$.

2.2.2. Proof of Theorem 5

We prove Theorem 5 by contradiction. Let $\Delta_0 \geq 8$ and G be a minimal counterexample (with respect to the number of vertices) with $\text{mad}(G) < \frac{10}{3}$, $\Delta(G) \leq \Delta_0$ and no $\left(\Delta_0 + 2, 2\right)$-incidence coloring. From Lemmas 6, 7, 8, 9 and 10 it follows that $\delta(G) \geq 2$, every 2-vertex in G is adjacent to two Δ_0-vertices, every 3^+-vertex is adjacent to at least two 3^+-vertices, G does not contain any 3-cycle only on 3-vertices as a subgraph and G contains no $\left(4^-, 3, 3, 4^-\right)$-path as a subgraph.

Let us define a *cluster* as a maximal connected subgraph of G induced on 3-vertices.

We will reach a contradiction by using the discharging method.

We assign an initial charge $\omega(v) = \deg_G(v)$ to each vertex v of G, and we use the following discharging rules:

(R1) Each Δ_0-vertex gives $\frac{2}{3}$ to each of its 2-neighbors.

(R2) Each 4-vertex gives $\frac{1}{9}$ to each of its 3-neighbors.

(R3) Each 5^+-vertex gives $\frac{2}{9}$ to each of its 3-neighbors.

We shall prove that the new charge $\omega'(v)$ of each k-vertex v of G, $k = 2$ or $k \geq 4$, is at least $\frac{10}{3}$ and that each cluster has average charge at least $\frac{10}{3}$ too, which contradicts our assumption $\text{mad}(G) < \frac{10}{3}$.

Let v be a vertex of G. We consider four cases, according to $\deg_G(v)$.

Case 1. $\deg_G(v) = 2$. Every 2-vertex in G is adjacent to two Δ_0-vertices. Therefore, $\omega'(v) = 2 + 2 \times \frac{2}{3} = \frac{10}{3}$ by R1.

Case 2. $\deg_G(v) = 4$. Due to R2, we have $\omega'(v) \geq 4 - 4 \times \frac{1}{9} = \frac{32}{9} > \frac{10}{3}$.

Case 3. $\deg_G(v) = d$, with $5 \leq d < \Delta_0$. According to R3, vertex v sends a charge at most $\frac{2}{9}$ to each of its neighbors. Hence, $\omega'(v) \geq d - \frac{2}{9}d = \frac{7}{9}d \geq \frac{35}{9} > \frac{10}{3}$.

Case 4. $\deg_G(v) = \Delta_0$. Each Δ_0-vertex sends $\frac{2}{3}$ to each of its 2-neighbors and at most $\frac{2}{9}$ to its other neighbors. Moreover v is adjacent to at most $\big(\Delta_0 - 2\big)$ 2-vertices and, therefore, we have $\omega'(v) \geq \Delta_0 - \frac{2}{3}\big(\Delta_0 - 2\big) - 2 \times \frac{2}{9} = \frac{10}{3} + \frac{3\Delta_0 - 22}{9} > \frac{10}{3}$.

Finally, we consider a cluster K. The initial charge of K is $3|K|$. We will show that the final charge $\omega'(K) = \sum_{v \in K} \omega'(v)$ is at least $\frac{10}{3}|K|$. As G contains no $(3,3,3)$-cycle and no $\big(4^-, 3, 3, 4^-\big)$-path, we have only four possibilities for K.

- K is a single 3-vertex v. In this case $\omega'(K) = \omega'(v) \geq 3 + 3 \times \frac{1}{9} = \frac{10}{3}$.
- K is a $(3,3)$-edge. By Lemma 10, K is adjacent to at least two 5^+-vertices and we have $\omega'(K) \geq 2 \times 3 + 2 \times \frac{1}{9} + 2 \times \frac{2}{9} = 2 \times \frac{10}{3}$.
- K is a $(3,3,3)$-path. Again by Lemma 10, K has at least four 5^+-vertices in its neighborhood and $\omega'(K) \geq 3 \times 3 + 1 \times \frac{1}{9} + 4 \times \frac{2}{9} = 3 \times \frac{10}{3}$.
- K is a star on four 3-vertices. In this case each neighbor of K is a 5^+-vertex and $\omega'(K) = 4 \times 3 + 6 \times \frac{2}{9} = 4 \times \frac{10}{3}$.

3. Graphs with Maximum Degree 6

Yang [8] proved that $\chi_i(G) \leq \Delta(G) + 5$ for every planar graph G with $\Delta(G) \neq 6$, using the relation between the incidence chromatic number, the star arboricity and the chromatic index of a graph. For planar graphs with $\Delta(G) = 6$ he only proved $\chi_i(G) \leq 12$. We improve this bound and get the following result for a more general class of graphs.

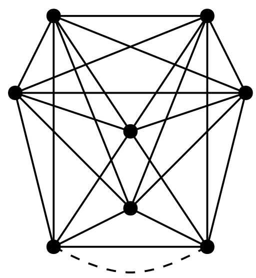

Figure 2. An Eulerian (multi)graph G' with an additional (multi)edge.

Theorem 11. *If G is a graph with $\Delta(G) \leq 6$ and with no 6-regular component on an odd number of edges, then $\chi_i(G) \leq 10$.*

Proof. Let G be a graph with $\Delta(G) \leq 6$ which has no 6-regular component on an odd number of edges. Without loss of generality we may assume that G is connected, otherwise we consider each of its components separately. If G is an Eulerian graph, then we color the edges of an Eulerian trail T alternately with red and blue, starting at a vertex of degree less than 6 (if there exists one; otherwise we start at an arbitrary vertex). The subgraphs R and B of G induced by the sets of red and blue edges, respectively, are subcubic. Hence, by Theorem 1, $\chi_i(R) \leq 5$ and $\chi_i(B) \leq 5$. Using two disjoint sets of colors for incidence coloring of the subgraphs R and B, we obtain an incidence coloring of G with (at most) 10 colors.

If G is connected but not Eulerian, then we add edges joining pairs of vertices of odd degree in G to obtain an Eulerian (multi)graph G'. Clearly, $\Delta(G') \leq 6$. We then assign colors red and blue alternately to edges of an Eulerian trail T in G'. It is easily seen that the subgraphs R and B of G obtained as before are subcubic, unless G' is 6-regular and has an odd number of edges. We can avoid this by starting a trail T at a vertex of degree less than 6 (if such a vertex exists) or by some added (multi)edge (see Figure 2). Therefore, we can ensure that R and B are subcubic. Again, using two disjoint sets of colors for incidence coloring the subgraphs R and B, we obtain an incidence coloring of G' (and of G) with (at most) 10 colors. Therefore, $\chi_i(G) \leq 10$. ■

As a consequence of the previous theorem, we positively answer Yang's question about planar graphs with maximum degree 6, even improving the suggested bound.

Corollary 12. *Every planar graph G with $\Delta(G) = 6$ satisfies $\chi_i(G) \leq 10$.*

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References

[1] M. Bonamy, B. Lévêque and A. Pinlou, 2-*distance coloring of sparse graphs*, J. Graph Theory **77** (2014) 190–218.
doi:10.1002/jgt.21782

[2] R.A. Brualdi and J.J. Quinn Massey, *Incidence and strong edge colorings of graphs*, Discrete Math. **122** (1993) 51–58.
doi:10.1016/0012-365X(93)90286-3

[3] B. Guiduli, *On incidence coloring and star arboricity of graphs*, Discrete Math. **163** (1997) 275–278.
doi:10.1016/0012-365X(95)00342-T

[4] S.L. Hakimi, J. Mitchem and E. Schmeichel, *Star arboricity of graphs*, Discrete Math. **149** (1996) 93–98.
doi:10.1016/0012-365X(94)00313-8

[5] M. Hosseini Dolama, É. Sopena and X. Zhu, *Incidence coloring of k-degenerated graphs*, Discrete Math. **283** (2004) 121–128.
doi:10.1016/j.disc.2004.01.015

[6] M. Hosseini Dolama and É. Sopena, *On the maximum average degree and the incidence chromatic number of a graph*, Discrete Math. Theor. Comput. Sci. **7** (2005) 203–216.

[7] D.P. Sanders and Y. Zhao, *Planar graphs of maximum degree seven are class* I, J. Combin. Theory Ser. B **83** (2001) 201–212.
doi:10.1006/jctb.2001.2047

[8] D. Yang, *Fractional incidence coloring and star arboricity of graphs*, Ars Combin. **105** (2012) 213–224.

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