INCIDENCE COLORING—COLD CASES

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Abstract

An incidence in a graph $G$ is a pair $(v, e)$ where $v$ is a vertex of $G$ and $e$ is an edge of $G$ incident to $v$. Two incidences $(v, e)$ and $(u, f)$ are adjacent if at least one of the following holds: (i) $v = u$, (ii) $e = f$, or (iii) edge $vu$ is from the set \{e, f\}. An incidence coloring of $G$ is a coloring of its incidences assigning distinct colors to adjacent incidences. The minimum number of colors needed for incidence coloring of a graph is called the incidence chromatic number.

It was proved that at most $\Delta(G) + 5$ colors are enough for an incidence coloring of any planar graph $G$ except for $\Delta(G) = 6$, in which case at most 12 colors are needed. It is also known that every planar graph $G$ with girth at least 6 and $\Delta(G) \geq 5$ has incidence chromatic number at most $\Delta(G) + 2$.

In this paper we present some results on graphs regarding their maximum degree and maximum average degree. We improve the bound for planar graphs with $\Delta(G) = 6$. We show that the incidence chromatic number is at
most $\Delta(G) + 2$ for any graph $G$ with $\text{mad}(G) < 3$ and $\Delta(G) = 4$, and for any graph with $\text{mad}(G) < \frac{\Delta}{3}$ and $\Delta(G) \geq 8$.

**Keywords:** incidence coloring, incidence chromatic number, planar graph, maximum average degree.

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1. **Introduction**

Incidence coloring was defined by Brualdi and Massey [2] as a tool to study strong edge colorings of bipartite graphs. However, soon after its definition, the coloring itself attracted the attention of several researchers from different points of view.

An *incidence* in a graph $G$ is a pair $(v, e)$ where $v$ is a vertex of $G$ and $e$ is an edge of $G$ incident to $v$. Two incidences $(v, e)$ and $(u, f)$ are adjacent if at least one of the following holds: (i) $v = u$, (ii) $e = f$, or (iii) edge $vu$ is from the set $\{e, f\}$. An *incidence coloring* of $G$ is a coloring of its incidences assigning distinct colors to adjacent incidences. The minimum number of colors needed for incidence coloring of a graph is called the *incidence chromatic number* of $G$, denoted by $\chi_i(G)$.

Brualdi and Massey [2] conjectured that $\chi_i(G) \leq \Delta(G) + 2$ for any graph $G$, where $\Delta(G)$ denotes the maximum degree of $G$. The conjecture was disproved by Guiduli [3], who showed that Paley graphs with maximum degree $\Delta$ have incidence chromatic number at least $\Delta + \Omega(\log \Delta)$. However, for many of the commonly considered graph classes the incidence chromatic number is bounded by $\Delta + c$ for some constant $c$, and several papers are devoted to the proof of this type of result, including the following one.

**Theorem 1** (Maydanskiy, 2005). *Five colors suffice for an incidence coloring of any subcubic graph.*

In order to obtain upper bounds on the incidence chromatic number, in many cases, stronger statements concerning incidence colorings with further local constraints are proved, allowing to apply induction in a more efficient way.

An incidence coloring of a graph $G$ using $k$ colors is an *incidence $(k, p)$-coloring* of $G$ if for every vertex $v$ of $G$, the number of colors used for coloring the incidences of the form $(u, uv)$ is at most $p$.

Hosseini Dolama, Sopena and Zhu [5] proved that every planar graph with maximum degree $\Delta$ admits an incidence $(\Delta + 7, 7)$-coloring and, thus, has incidence chromatic number at most $\Delta + 7$. This bound was further improved to $\Delta + 4$ for triangle-free planar graphs [6], to $\Delta + 3$ (respectively, $\Delta + 2$, $\Delta + 1$) for planar graphs of girth at least 6 (respectively, 11, 16) [6]. The last result was further improved to girth 14 [1].
Some of these results were proved for more general graph classes, namely graphs with bounded maximum average degree. The average degree of a graph $G$ is the mean value of the degrees of its vertices. The maximum average degree $\text{mad}(G)$ of a graph $G$ is then defined as the maximum value of the average degrees of its subgraphs. When $G$ is a planar graph with girth $g$, it is folklore to establish the inequality $\text{mad}(G) < \frac{2g}{g-2}$.

In [6] the authors proved the following result.

**Theorem 2** (Hosseini Dolama, Sopena, 2005). Let $G$ be a graph with $\text{mad}(G) < 3$ and $\Delta(G) \geq 5$. Then $G$ admits a $(\Delta(G) + 2, 2)$-incidence coloring. Therefore, $\chi_i(G) \leq \Delta(G) + 2$.

In Section 2 we extend this result to $\text{mad}(G) < 3$ and $\Delta(G) \geq 4$ (Theorem 4). Moreover, we present another result for graphs with larger maximum average degree (Theorem 5).

Recall that the star arboricity of an undirected graph $G$ is the smallest number of star forests needed to cover $G$. Yang [8] observed the following: let $G$ be an undirected graph with star arboricity $\text{st}(G)$, let $s : E(G) \to \{1, \ldots, \text{st}(G)\}$ be a mapping such that $s^{-1}(i)$ is a forest of stars for every $i$, $1 \leq i \leq \text{st}(G)$, and let $\lambda$ be a proper edge coloring of $G$. Now define the mapping $f$ by $f(u, uv) = s(uv)$ if $v$ is the center of a star in some forest $s^{-1}(i)$ (if some star is reduced to one edge, we arbitrarily choose one of its end vertices as the center) and $f(u, uv) = \lambda(uv)$ otherwise. It is not difficult to check that $f$ is indeed an incidence coloring of $G$. Therefore, thanks to the classical result of Vizing, the relation $\chi_i(G) \leq \Delta(G) + \text{st}(G)$ (respectively, $\chi_i(G) \leq \Delta(G) + \text{st}(G) + 1$) holds for every graph of class 1 (respectively, of class 2). (Recall that the chromatic index $\chi'(G)$ of any graph $G$ is either $\Delta(G)$—such graphs are said to be of class 1—or $\Delta(G) + 1$—such graphs are said to be of class 2.) The facts that planar graphs with $\Delta \geq 7$ are class 1 [7] and that the star arboricity of any planar graph is at most 5 [4] led to the following result.

**Theorem 3** (Yang, 2007). If $G$ is a planar graph with $\Delta(G) \neq 6$, then $\chi_i(G) \leq \Delta(G) + 5$. If $\Delta(G) = 6$, then $\chi_i(G) \leq \Delta(G) + 6$.

Yang [8] proposed the following question: Are $\Delta(G) + 5$ colors enough for graphs with maximum degree 6? We give a positive answer to this question (in a stronger form) in Section 3.

2. Graphs with Bounded Maximum Average Degree

In this section we present two results: one of them extends Theorem 2, the other one concerns graphs with larger maximum average degree.
Theorem 4. Let $G$ be a graph with $\text{mad}(G) < 3$ and $\Delta(G) \geq 4$. Then $G$ admits a $(\Delta(G) + 2, 2)$-incidence coloring. Therefore, $\chi_i(G) \leq \Delta(G) + 2$.

Theorem 5. Let $G$ be a graph with $\text{mad}(G) < \frac{10}{3}$ and $\Delta(G) \geq 8$. Then $G$ admits a $(\Delta(G) + 2, 2)$-incidence coloring. Therefore, $\chi_i(G) \leq \Delta(G) + 2$.

2.1. Reducible configurations

We first introduce some additional notation used in the proofs of both results. We denote by $\text{deg}_G(v)$ the degree of a vertex $v$ in a graph $G$. By a $k$-vertex, a $k^+$-vertex and a $k^-$-vertex, we mean a vertex of degree $k$, at least $k$ and at most $k$, respectively. A $(k_1, k_2)$-edge is an edge $v_1v_2$ such that for every $i \in \{1, 2\}$, $v_i$ is a $k_i$-vertex. More generally, a $(k_1, k_2, \ldots, k_\ell)$-path (respectively, a $(k_1, k_2, \ldots, k_\ell)$-cycle), $\ell \geq 3$, is a path (respectively, a cycle) $v_1v_2\cdots v_\ell$ such that for every $i$, $1 \leq i \leq \ell$, $v_i$ is a $k_i$-vertex.

Let $c$ be a partial incidence coloring of a graph $G$. We say that a color $a$ is admissible for an (uncolored) incidence $(v, e)$ in $G$ if there is no incidence colored by $a$ adjacent to $(v, e)$; otherwise the color $a$ is forbidden. We denote $F^c(v, e)$ the set of forbidden colors for the incidence $(v, e)$.

Let $v$ be a vertex of $G$. We set $I_v := \{(v, w) : w \in E(G)\}$ and $A_v := \{(u, w) : w \in E(G)\}$. If $c$ is a partial incidence coloring of $G$, we necessarily have $c(I_v) \cap c(A_v) = \emptyset$ for each vertex $v$ of $G$. Moreover, if $c$ is a partial $(k, 2)$-incidence coloring of $G$, then $|c(I_v)| \leq 2$. By $A^c(v)$ we will denote a set of exactly two colors such that $A^c(v) \supseteq c(A_v)$ and $A^c(v) \cap c(I_v) = \emptyset$.

We now prove a series of lemmas.

Lemma 6. Let $G$ be a graph, $v$ be a 1-vertex in $G$ and $k \geq \Delta(G) + 2$ be an integer. If $G - v$ admits a $(k, 2)$-incidence coloring, then $G$ also admits a $(k, 2)$-incidence coloring.

Proof. Let $c$ be a $(k, 2)$-incidence coloring of $G - v$, and $w$ denote the unique neighbor of $v$ in $G$. We will extend $c$ to a $(k, 2)$-incidence coloring of $G$. Since $|F^c(w, vw)| = |c(I_w) \cup c(A_w)| \leq \Delta(G) - 1 + 2 = \Delta(G) + 1$, there is an admissible color $a$ for $(w, vw)$. We then set $c(w, vw) = a$ and $c(v, wv) = b$ for any color $b$ in $A^c(w)$. Clearly, $c$ is a $(k, 2)$-incidence coloring of $G$.

Lemma 7. Let $G$ be a graph, $k \geq \Delta(G) + 2$ be an integer, and $vw$ be a $(2, (k-3)^-)$-edge in $G$. If $G - uv$ admits a $(k, 2)$-incidence coloring, then $G$ also admits a $(k, 2)$-incidence coloring.

Proof. Let $w$ be the other neighbor of $u$ in $G$ and $c$ be a $(k, 2)$-incidence coloring of $G - e; e = uv$. We extend $c$ to a $(k, 2)$-incidence coloring of $G$ in the following way. We first uncolor $(u, uv)$. We then set $c(u, e) = a$, for some color $a \in A^c(v) - c(w, uw)$, and $c(u, uw) = b$ for some color $b \in A^c(w) - c(u, e)$. Finally,
since $|F^c(v, e)| = |c(I_v) \cup c(A_v) \cup \{c(u, uw)\}| \leq (k - 4) + 2 + 1 = k - 1 < k$, there is an admissible color for $(v, e)$, so that we can complete the coloring.

Lemma 8. Let $G$ be a graph with no 1-vertices and $k \geq \Delta(G) + 2$ be an integer. Let $v$ be an $s$-vertex in $G$, $s \geq 3$, adjacent to at most one $3^+$-vertex, and let $u_i$, $1 \leq i \leq s - 1$, denote the 2-neighbors of $v$. If the graph $G - \{vu_i, 1 \leq i \leq s - 1\}$ admits a $(k, 2)$-incidence coloring, then $G$ also admits a $(k, 2)$-incidence coloring.

Proof. Let $e_i = vu_i$, $f_i = u_iw_i$ be the other edge incident to $u_i$ for every $i$, $1 \leq i \leq s - 1$, and $u_s$ be the last neighbor of $v$ and $e_s = vu_s$. Let $c$ be a $(k, 2)$-incidence coloring of $G - \{e_i, 1 \leq i \leq s - 1\}$. We extend $c$ to a $(k, 2)$-incidence coloring of $G$ as follows.

We first uncolor $(v, e_s)$ and all incidences $(u_i, f_i)$, $1 \leq i \leq s - 1$. Let $a_i = c(w_i, f_i)$, $1 \leq i \leq s - 1$. Since we have $k$ colors and $k \geq \Delta(G) + 2$, there is a color $t$ not in $\{a_i, 1 \leq i \leq s - 1\}$; moreover, we can choose $t$ such that $t \not\in A^c(w_i)$.

We then set $c(u_i, u_j) = t$, $1 \leq i \leq s - 1$.

Next, for every $i$, $2 \leq i \leq s - 1$, we set $c(u_i, f_i) = t_i$ with $t_i \in A^c(w_i) - \{t\}$, $c(u, e_s) = t_s$ with $t_s \in A^c(u_s) - \{t\}$, and $c(u_i, f_i) = t_i$ with $t_i \in A^c(w_i) - \{t_2\}$.

Now $F^c(v, e_i) = \{t, c(u_i, f_i), c(u_s, e_s), c(v, e_s)\}$. Therefore we have at least $k - 4 \geq s - 2$ admissible colors for every uncolored incidence. As $c(u_i, f_i) \neq c(u_2, f_2)$, we can choose at least $s - 1$ distinct colors $b_i$ such that $b_i \not\in F^c(v, e_i)$, and we set $c(v, e_i) = b_i$ for every $i$, $1 \leq i \leq s - 1$.

Lemma 9. Let $G$ be a graph with $\Delta(G) \geq 7$, $k \geq \Delta(G) + 2$ be an integer, and $C = v_1v_2v_3$ be a $(3, 3, 3)$-cycle in $G$. If the graph $G - \{v_1v_2, v_2v_3, v_3v_1\}$ admits a $(k, 2)$-incidence coloring, then $G$ also admits a $(k, 2)$-incidence coloring.

Proof. Let $c$ be a $(k, 2)$-incidence coloring of $G - \{v_1v_2, v_2v_3, v_3v_1\}$. Let $u_i$ be the neighbor of $v_i$ not included in $C$, $1 \leq i \leq 3$. We extend $c$ to a $(k, 2)$-incidence coloring of $G$ as follows. Let $a_i = c(u_i, u_iw_i)$, $b_i = c(v_i, v_iu_i)$, $1 \leq i \leq 3$. Since $k \geq 9$, there are three colors $c_1, c_2, c_3 \not\in \{a_i, 1 \leq i \leq 3\} \cup \{b_i, 1 \leq i \leq 3\}$. We then color the six incidences of $C$, cyclically, with colors $c_1, c_2, c_3, c_1, c_2, c_3$.

Lemma 10. Let $G$ be a graph with $\Delta(G) \geq 8$, $k \geq \Delta(G) + 2$ be an integer, and $P = u_1v_1v_2u_2$ be a $(4^-, 3, 3, 4^-)$-path in $G$. If the graph $G - \{u_1v_1, v_1v_2, v_2u_2\}$ admits a $(k, 2)$-incidence coloring, then $G$ also admits a $(k, 2)$-incidence coloring.

Proof. Let $c$ be a $(k, 2)$-incidence coloring of $G - \{u_1v_1, v_1v_2, v_2u_2\}$ and $w_i$ be the third neighbor of $v_i$, $i = 1, 2$. We will extend $c$ to a $(k, 2)$-incidence coloring of $G$.

We can assume that $\{c(w_1, w_1v_1), c(v_1, v_1w_1)\} \neq A^c(u_i)$, $i = 1, 2$ (otherwise we recolor $(v_1, v_1w_1)$ using the other color from $A^c(w_i)$). Thus we can set $c(v_1, v_1u_i) = t_i$ with $t_i \in A^c(u_i) - \{c(w_1, w_1v_1), c(v_1, v_1w_1)\}$, $i = 1, 2$. 

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We now consider three cases:

**Case 1.** $c(w_2, w_2v_2) \notin c(I_{v_1}) \cup c(A_{v_1})$. We first set $c(v_1, v_1v_2) = c(w_2, w_2v_2)$. Since $k \geq 10$, there exists a color $c_1 \notin c(I_{u_1}) \cup c(A_{u_1}) \cup \{c(v_1, v_1v_1), c(v_2, v_2w_2), c(w_2, w_2v_2), c(v_2, v_2u_2)\}$. We then set $c(u_1, u_1v_1) = c(v_2, v_2v_1) = c_1$. Since the incidence $(w_2, w_2v_2)$ is adjacent to at most nine other incidences, it can be colored.

**Case 2.** $c(w_1, w_1v_1) \notin c(I_{v_2}) \cup c(A_{v_2})$. We proceed similarly as in the previous case.

**Case 3.** $c(w_1, w_1v_1) \in c(I_{v_2}) \cup c(A_{v_2})$ and $c(w_2, w_2v_2) \in c(I_{v_1}) \cup c(A_{v_1})$. We will color the incidences $(u_1, u_1v_1)$ and $(v_2, v_2v_1)$ with a common color $c_1$, and the incidences $(w_2, w_2v_2)$ and $(v_1, v_1v_2)$ with a common color $c_2$. Note that we have at most nine forbidden colors for each of $c_1$ and $c_2$. If we can choose $c_1 \neq c_2$, we are done. If not, we necessarily have $k = 10$, the sets of forbidden colors for $c_1$ and $c_2$ are the same, and both contain nine distinct colors. Since in this case we have $c(w_1, w_1v_1) \in c(I_{v_2}) \cup c(A_{v_2})$ and $c(w_1, w_1v_1)$, $c(v_2, v_2w_2)$, $c(v_2, v_2u_2)$ are different (they are different forbidden colors for $c_2$), we get $c(w_1, w_1v_1) = c(w_2, w_2v_2)$. Without loss of generality, we may assume that $c(w_1, w_1v_1) = c(w_2, w_2v_2) = 9$, $c(v_1, v_1w_1) = 8$, $c(v_1, v_1u_1) = 7$, $c(v_2, v_2w_2) = 6$, and $c(v_2, v_2u_2) = 5$ (see Figure 1). Then $c(I_{u_2}) \cup c(A_{u_2}) = \{1, 2, 3, 4, 5\}$ and $c(I_{u_1}) \cup c(A_{u_1}) = \{1, 2, 3, 4, 7\}$. We can replace $c(v_1, v_1u_1)$ with the other color from $c(A_{u_1})$. Now, 7 is no more forbidden for $c_2$, so we have only eight forbidden colors for $c_2$. Therefore, we can now choose $c_1 \neq c_2$ to obtain the desired coloring.

![Figure 1. A partial incidence coloring of a $(4^-, 3, 3, 4^-)$-path.](image-url)

2.2. Discharging rules

2.2.1. Proof of Theorem 4

We prove Theorem 4 by contradiction. Let $\Delta_0 \geq 4$ and $G$ be a minimal counterexample (with respect to the number of vertices) with $\text{mad}(G) < 3$, $\Delta(G) \leq \Delta_0$ and...
with no \((\Delta_0 + 2, 2)\)-incidence coloring. From Theorem 1 and Lemmas 6, 7 and 8 it follows that \(\delta(G) \geq 2\), every 2-vertex in \(G\) is adjacent to two \(\Delta_0\)-vertices and every \(3^+\)-vertex is adjacent to at least two \(3^+\)-vertices. Moreover, \(\Delta_0 = \Delta(G)\).

We will reach a contradiction by using the discharging method.

We assign an initial charge \(\omega(v) = \deg_G(v)\) to each vertex \(v\) of \(G\), and we use the following discharging rule: each \(4^+\)-vertex gives \(\frac{1}{2}\) to each of its 2-neighbors.

We shall prove that the new charge \(\omega'(v)\) of each vertex \(v\) of \(G\) is at least 3, which contradicts our assumption \(\text{mad}(G) < \frac{10}{3}\) (since \(\sum_{v \in G} \omega'(v) = \sum_{v \in G} \omega(v)\)).

Let \(v\) be a vertex of \(G\). We consider three cases, according to \(\deg_G(v)\).

**Case 1.** \(\deg_G(v) = 2\). Every 2-vertex in \(G\) is adjacent to two \(\Delta_0\)-vertices. Therefore, since \(\Delta(G) \geq 4\), \(\omega'(v) = 2 + 2 \times \frac{1}{2} = 3\) by R1.

**Case 2.** \(\deg_G(v) = 3\). The discharging rule does not involve 3-vertices, thus \(\omega'(v) = \omega(v) = 3\).

**Case 3.** \(\deg_G(v) = d \geq 4\). Since every \(d\)-vertex is adjacent to at most \((d - 2)\) 2-vertices, \(\omega'(v) \geq d - \frac{1}{2}(d - 2) = \frac{d + 2}{2} \geq 3\).

### 2.2.2. Proof of Theorem 5

We prove Theorem 5 by contradiction. Let \(\Delta_0 \geq 8\) and \(G\) be a minimal counterexample (with respect to the number of vertices) with \(\text{mad}(G) < \frac{10}{3}\), \(\Delta(G) \leq \Delta_0\) and no \((\Delta_0 + 2, 2)\)-incidence coloring. From Lemmas 6, 7, 8, 9 and 10 it follows that \(\delta(G) \geq 2\), every 2-vertex in \(G\) is adjacent to two \(\Delta_0\)-vertices, every \(3^+\)-vertex is adjacent to at least two \(3^+\)-vertices, \(G\) does not contain any 3-cycle only on 3-vertices as a subgraph and \(G\) contains no \((4^-, 3, 3, 4^-)\)-path as a subgraph.

Let us define a *cluster* as a maximal connected subgraph of \(G\) induced on 3-vertices.

We will reach a contradiction by using the discharging method.

We assign an initial charge \(\omega(v) = \deg_G(v)\) to each vertex \(v\) of \(G\), and we use the following discharging rules:

(R1) Each \(\Delta_0\)-vertex gives \(\frac{2}{3}\) to each of its 2-neighbors.

(R2) Each 4-vertex gives \(\frac{1}{3}\) to each of its 3-neighbors.

(R3) Each \(5^+\)-vertex gives \(\frac{2}{3}\) to each of its 3-neighbors.

We shall prove that the new charge \(\omega'(v)\) of each \(k\)-vertex \(v\) of \(G\), \(k = 2\) or \(k \geq 4\), is at least \(\frac{10}{3}\) and that each cluster has average charge at least \(\frac{10}{3}\) too, which contradicts our assumption \(\text{mad}(G) < \frac{10}{3}\).

Let \(v\) be a vertex of \(G\). We consider four cases, according to \(\deg_G(v)\).

**Case 1.** \(\deg_G(v) = 2\). Every 2-vertex in \(G\) is adjacent to two \(\Delta_0\)-vertices. Therefore, \(\omega'(v) = 2 + 2 \times \frac{2}{3} = \frac{10}{3}\) by R1.
Case 2. \( \deg_G(v) = 4 \). Due to R2, we have \( \omega'(v) \geq 4 - 4 \times \frac{1}{9} = \frac{32}{9} > \frac{10}{3} \).

Case 3. \( \deg_G(v) = d \), with \( 5 \leq d < \Delta_0 \). According to R3, vertex \( v \) sends a charge at most \( \frac{2}{9} \) to each of its neighbors. Hence, \( \omega'(v) \geq d - \frac{2}{9}d = \frac{7}{9}d \geq \frac{35}{9} > \frac{10}{3} \).

Case 4. \( \deg_G(v) = \Delta_0 \). Each \( \Delta_0 \)-vertex sends \( \frac{2}{3} \) to each of its \( 2 \)-neighbors and at most \( \frac{7}{9} \) to its other neighbors. Moreover \( v \) is adjacent to at most \( (\Delta_0 - 2) \) \( 2 \)-vertices and, therefore, we have \( \omega'(v) \geq \Delta_0 - \frac{2}{3}(\Delta_0 - 2) - 2 \times \frac{2}{9} = \frac{10}{3} + \frac{3\Delta_0 - 22}{9} > \frac{10}{3} \).

Finally, we consider a cluster \( K \). The initial charge of \( K \) is \( 3|K| \). We will show that the final charge \( \omega'(K) = \sum_{v \in K} \omega'(v) \) is at least \( \frac{10}{3}|K| \). As \( G \) contains no \((3,3,3)\)-cycle and no \((4^-,3,3,4^-)\)-path, we have only four possibilities for \( K \).

- \( K \) is a single 3-vertex \( v \). In this case \( \omega'(K) = \omega'(v) \geq 3 + 3 \times \frac{1}{9} = \frac{10}{3} \).

- \( K \) is a \((3,3)\)-edge. By Lemma 10, \( K \) is adjacent to at least two \( 5^+ \)-vertices and we have \( \omega'(K) \geq 2 \times 3 + 2 \times \frac{1}{9} + 2 \times \frac{2}{9} = 2 \times \frac{10}{3} \).

- \( K \) is a \((3,3,3)\)-path. Again by Lemma 10, \( K \) has at least four \( 5^+ \)-vertices in its neighborhood and \( \omega'(K) \geq 3 \times 3 + 1 \times \frac{1}{9} + 4 \times \frac{2}{9} = 3 \times \frac{10}{3} \).

- \( K \) is a star on four 3-vertices. In this case each neighbor of \( K \) is a \( 5^+ \)-vertex and \( \omega'(K) = 4 \times 3 + 6 \times \frac{2}{9} = 4 \times \frac{10}{3} \).

3. Graphs with Maximum Degree 6

Yang [8] proved that \( \chi_i(G) \leq \Delta(G) + 5 \) for every planar graph \( G \) with \( \Delta(G) \neq 6 \), using the relation between the incidence chromatic number, the star arboricity and the chromatic index of a graph. For planar graphs with \( \Delta(G) = 6 \) he only proved \( \chi_i(G) \leq 12 \). We improve this bound and get the following result for a more general class of graphs.

![Figure 2. An Eulerian (multi)graph \( G' \) with an additional (multi)edge.](image)

**Theorem 11.** If \( G \) is a graph with \( \Delta(G) \leq 6 \) and with no 6-regular component on an odd number of edges, then \( \chi_i(G) \leq 10 \).
**Proof.** Let $G$ be a graph with $\Delta(G) \leq 6$ which has no 6-regular component on an odd number of edges. Without loss of generality we may assume that $G$ is connected, otherwise we consider each of its components separately. If $G$ is an Eulerian graph, then we color the edges of an Eulerian trail $T$ alternately with red and blue, starting at a vertex of degree less than 6 (if there exists one; otherwise we start at an arbitrary vertex). The subgraphs $R$ and $B$ of $G$ induced by the sets of red and blue edges, respectively, are subcubic. Hence, by Theorem 1, $\chi_i(R) \leq 5$ and $\chi_i(B) \leq 5$. Using two disjoint sets of colors for incidence coloring of the subgraphs $R$ and $B$, we obtain an incidence coloring of $G$ with (at most) 10 colors.

If $G$ is connected but not Eulerian, then we add edges joining pairs of vertices of odd degree in $G$ to obtain an Eulerian (multi)graph $G'$. Clearly, $\Delta(G') \leq 6$. We then assign colors red and blue alternately to edges of an Eulerian trail $T$ in $G'$. It is easily seen that the subgraphs $R$ and $B$ of $G$ obtained as before are subcubic, unless $G'$ is 6-regular and has an odd number of edges. We can avoid this by starting a trail $T$ at a vertex of degree less than 6 (if such a vertex exists) or by some added (multi)edge (see Figure 2). Therefore, we can ensure that $R$ and $B$ are subcubic. Again, using two disjoint sets of colors for incidence coloring the subgraphs $R$ and $B$, we obtain an incidence coloring of $G'$ (and of $G$) with (at most) 10 colors. Therefore, $\chi_i(G) \leq 10$.

As a consequence of the previous theorem, we positively answer Yang’s question about planar graphs with maximum degree 6, even improving the suggested bound.

**Corollary 12.** Every planar graph $G$ with $\Delta(G) = 6$ satisfies $\chi_i(G) \leq 10$.

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